Naming practices that support reasoning about and with expressions

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Introduction

Several researchers have attempted a characterisation of algebraic thinking. A frequently cited framework proposed by Kaput (1998) identifies five forms of algebraic thinking: generalizing and formalizing, manipulation of formalism, study of abstract systems and structures, study of functions and co-variation, and modeling. In this paper, I deal with the issue of how the development of the second of these five forms – 'algebra as the syntactically guided manipulation of formalisms' – may be supported. My focus is on understanding the initial forms of students' reasoning with symbolic expressions and on how instructional practices could shape and strengthen these initial steps in the development of the symbolic mathematical ability. Specifically, I highlight the importance of including an emphasis on syntactic understanding in beginning algebra instruction and suggest ways in which semantic and syntactic understanding may be mutually supported.

The first section of the paper will include analyses of examples of students' reasoning in the primary and middle grades where symbolic expressions are involved. I follow several authors in distinguishing between two modes of reasoning involving symbolic expressions, for which I shall use the names 'referential' and 'syntactic-structural'. The referential mode of symbolic reasoning is founded on 'number sense' or 'operational sense', that is, an understanding of the relation between numbers and of the effects of arithmetic operations. In referential reasoning, the warrant for the judgements that subjects make and the justification for their actions is drawn from their knowledge about numbers and operations. The essential difference in the syntactic-structural mode of reasoning is that such warrant flows from the subjects' knowledge of how to transform expressions and equations into other equivalent expressions and equations. Thus syntactic-structural reasoning goes beyond referential reasoning in requiring knowledge of the rules for transforming expressions and equations, and further, the notion that expressions can be derived from and substituted for other expressions.

The distinction between referential and syntactic-structural thinking with regard to symbolic expressions is not new. Many readers would readily concede the distinction and agree that reasoning on the basis of syntactic-structural transformations is harder and develops later among students. Most would also agree with the fact that actual instances of reasoning involve a back-and-forth movement between the two modes (Kaput, 1998; Arcavi, 1994). So what I shall argue for is the importance of this distinction in planning and designing teaching interventions, that is, the importance of allowing for and supporting both modes of reasoning during instruction. Syntactic considerations play a part even in the elementary instances of reasoning with expressions. Without taking due account of the syntactic side of reasoning, students' reasoning processes become impoverished and do not run their full course.

An act of reasoning may be classified as such only if sufficient warrant exists for each step of the reasoning process, both in the logical and in the psychological sense. When the reasoning involves symbolic expressions, for the referential moments in the reasoning process, students' experiences of arithmetic provides the necessary warrant. The warrant for the syntactic-structural moments in their reasoning must be built up and secured through an extended process of learning that is different from the experience they have gained in arithmetic. The nature of this process has been inadequately conceptualized. It stands to reason that the first steps in acquiring capability in syntactic thinking must use knowledge of arithmetic as a springboard. However, what these first steps would be, what the nature of the contribution of arithmetic knowledge is, what additional cognitive processes are at play are questions that need to be explored. Hence my interest is not so much on syntactic aspects of mature or expert reasoning with symbolic expressions, as on the preliminary forms of syntactic reasoning, on how these may provide a foundation for the development of the syntactic-structural sense, and on how these in turn, draw upon students' knowledge about numbers and operations.

In the second section of the paper, I discuss some teaching intervention studies in algebra that have focused on the structure of expressions. Many of these studies receive their impulse from the research on students' errors and difficulties in algebra. Most of these studies have attempted to build on students' knowledge of arithmetic. However, recently doubts have been raised about whether the approach to teaching algebra through arithmetic is viable (Linchevski and Livneh, 1999). Kirshner (2001) takes issue with the perspective that syntactic algebra must follow and build on the basis offered by referential understanding. I revisit this issue and discuss an alternative to building the connection between arithmetic and syntactic algebra through a restructuring of naming practices.

The third section of the paper outlines a structural approach to teaching arithmetic and algebraic expressions that is based on this alternative. Here I draw upon work done, in association with my colleagues, on developing the initial instructional materials of such an approach. The instructional materials and the principles embodied in the approach itself, have evolved over a few teaching cycles in out-of-school vacation programs conducted at the Homi Bhabha Centre for Science Education. The approach bears some resemblance to that proposed by Kirshner (2001). We concur with Bell's (1995) recommendation of using a combination of 'focused teaching' of syntactic rules and of tasks requiring reasoning on the basis of arithmetic knowledge in order to build syntactic understanding. Reading this from a slightly different angle, we propose in our teaching approach, a combination of tasks requiring reasoning *about* expressions and tasks requiring reasoning *with* expressions as a way of building students' abilities with symbolic expressions. We take a further step in the direction of a structural approach by proposing a set of naming practices for teaching arithmetic and algebraic expressions and by adding details that have emerged through our experience of teaching.

In the final section I discuss some instances of reasoning by students in the context of comparing arithmetic expressions by attending to their structure. I include this in order to show the spontaneous tendency among students to extract rules at the syntactic level and to indicate how they take account of semantic constraints while extending and applying what are essentially syntactic rules. This is a preliminary analysis of phenomena observed in an ongoing teaching intervention research program, but I hope that the discussion will aid in a more adequate conceptualization of the relation between referential, meaning-extracting processes and the generation of procedural rules that guide action at an essentially syntactic level.

Reasoning involving symbolic expressions

In this section, I discuss examples drawn from the research literature on algebra learning that involve students' reasoning with symbols. The aim of the discussion is to show how syntactic-structural reasoning plays a critical role in reasoning with symbolic expressions. This aspect of reasoning is often underemphasized in discussions of students' reasoning. The examples discussed will also serve to draw attention to important pedagogical principles that must be borne in mind while designing instruction aimed at fostering the understanding of symbolic expressions.

The first example that I will discuss is an instance of reasoning that involves symbols but not symbolic expressions. This will serve to highlight the difference between symbols that stand for objects, and symbols that stand for procepts (in the sense of Gray and Tall, 1994) where the reference is relatively opaque. The example is taken from Yackel (2002). First-graders in a teacher led classroom are discussing how to share a giant cookie (shown in the form of a circle) among four people. Josephina, a student, draws two lines to show how the cookie may be cut. The pieces of the cookie don't look like fair shares to many children. One of them, Armon, walks up to the drawing and draws a little circle in a portion of the cookie. When the teacher looks puzzled, he asks the teacher to draw more circles in the different portions. The argument that Armon is trying to make to show that the portions are unequal is now clear to the teacher, and she completes the argument by discussing it together with Armon and the class (see figure 1).

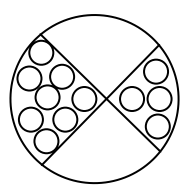


Figure 1: A rough sketch of Armon's argument (Yackel, 2002)

This wonderful example of reasoning by a young child belongs to the growing number of 'happy stories' (Kaput, 2004) about the capabilities of young children that recent studies have uncovered. The reasoning involved is sophisticated and shows that when the reference is clear, even young children can deploy symbols to reason and to communicate their reasoning to others. Such studies reinforce the view that students' difficulties in learning algebra may be the result of not engaging their thinking during the teaching-learning process, rather than because of the limitations imposed by their stage of cognitive development or individual cognitive abilities. These findings propel one to look for ways of engaging students' thinking and reasoning while learning difficult material including the handling of symbolic mathematics.

In this example, although Armon imaginatively uses a drawing, which is indeed a kind of symbol, it is quite different from the kinds of symbolic reasoning that we wish to focus upon. Our concern is narrower: it is with symbolic expressions, that is, strings of symbols containing numerals, operation symbols, variables or letter numerals and grouping symbols. Armon's reasoning involves the creation and use of symbols (small circles) probably preceded by the creation and manipulation of mental images. These symbols and images however, have a more direct link with physical objects. The reference of the symbols is relatively transparent in comparison with the symbols that make up an expression. For this reason, we restrict our discussion to symbolic expressions. As one might expect, and as is attested by numerous studies, students face enormous difficulties in dealing with symbols where the reference is opaque.

A second example that we will consider is a task from an instructional unit called 'building formulas' for seventh-graders developed at the Freudenthal Institute (van Reeuwijk, 1997). The task belongs to the general category of tasks where students are required to find a pattern in a sequence and to express this in the form of a function or a 'formula'. In the example, students are finding the relation between the length of a beam in the form of a truss composed of triangles (see figure 2) and the number of rods that are needed to build the beam. A recursive relation such as, 'when the length increases by 1, the number of rods increases by 4', is easier to find, but less useful, for finding the number of rods in very long beams, than a formula. The students, who are encouraged to find a general formula, come up with 'different' formulas such as

$$3 \times L + (L - 1)$$

$$L + 2 \times L + (L - 1)$$

$$4 \times L - 1$$

(van Reeuwijk, 1997, p. 233. I have introduced the '×' symbol in the expressions, while the article cited follows the standard algebraic convention of omitting it. Using brackets is also an important convention which needs to be secured over time. Assuming that students know how to use them in generating expressions, I have retained the brackets just as in the cited article.) Other similar examples are commonly encountered in teaching studies of algebra, where students are encouraged to find a pattern in a sequence of configurations and to express this in terms of a formula. The patterns include those made by matchsticks, dots or other geometrical shapes. I'll now pursue a hypothetical extension of this activity that elicits further thinking and reasoning concerning these formulas.

If a pattern finding task such as the above leads to the generation of different formulas, what would a group of students make of the formulas that are produced? They might wonder if all the formulas are correct. This could be done by checking the formulas one by one against specific cases where the length of the beam and the number of rods could be found from a diagram or from the recursive relation. A more sophisticated approach would be to verify by recomposing each formula, that is, by checking that it correctly represents a systematic method of counting the rods. For example, the formula $3 \times L + (L - 1)$ would represent counting all the rods that make up the upright triangles, and then the rods joining the top vertices of these triangles (see figure 2).

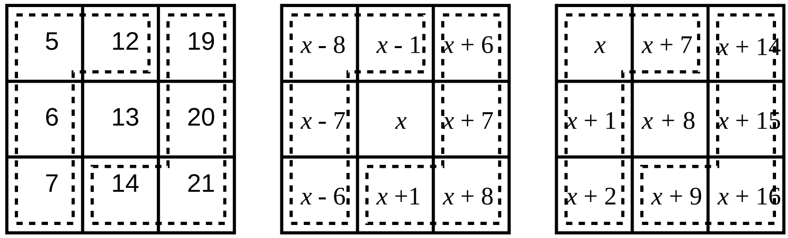
Asking whether all the three formulas are correct is different from asking whether they are the same. Students who are unfamiliar with this game, may be puzzled by the latter question. In an obvious sense, the formulas are different. If the question is about whether they give the same output when the same number is substituted for L, the obvious way to answer the question would be to check by calculation. However, this would be catching the wrong end of the stick; the point is to see the equivalence of the formulas without having to compute with actual numbers.

The teacher may induce the students away from such inductive reasoning by challenging them, 'Would you be able to tell me if the formulas are the same by just looking at them, without doing any calculation? If I were to just give you these formulas, without the diagrams or the problem, would you be able to say if they are the same without doing any calculation with numbers?'

To be able to see the relation between the expressions, the students will need to find parts of the expression that are common. The presence of the brackets may allow them to quickly identify the bracketed term *(L – 1)* as common to the first and second expressions. Further, if their knowledge that multiplication precedes addition in the order of operations is in place and is recalled, they may see that in the second expression *L* is added to twice *L*, which is the same as three times *L*. This line of reasoning involves two moments – the first a structural moment – seeing that a part of the expression is common and second, an operational moment, seeing that a composition of two operations is the same as a third operation. Reasoning about the composition of operations in this manner may require a reified understanding of operations as both process and object, as has been pointed out for example, by Sfard (1991) and Gray and Tall (1994). We will follow Gray and Tall in referring to this understanding as 'proceptual'.

We see, however, that the syntactic conventions (the order of operations in this case) must be in place before such proceptual knowledge can be applied. This is even more so in the third expression or formula, where the bracketed term is absent. In comparing the first and the third expressions, the cue provided by the bracket is misleading, suggesting a detachment of the '×' sign, and leads away from perceiving their equivalence of the two expressions. An intervening step in the reasoning must employ an understanding of syntax to break this cue, and then reason about the composition of operations.

While the activity of finding a formula for a pattern is common and widely reported in the literature, the extension of this activity to consider the equivalence of the formulas generated is not so common. The extension involves an aspect of algebraic thinking that is different from the one involved in generating the formulas, but that is nevertheless equally important. More studies are required on this aspect of algebraic thinking to develop an understanding of how students progress with respect to symbolic reasoning. A study by Carrahers and Earnest (2003) of third graders reflecting on their solutions of the 'guess my rule' task – where input and output numbers are given and students are asked to guess the function – shows the enormous difficulty they have in even comprehending the equivalence of expressions like $2 \times x$ and $x + x$. While this may be explained by claiming that such young students are yet to develop a proceptual understanding, this may not explain why older students continue to face difficulties with symbolic expressions. As our analysis above shows, knowledge of syntax mediates the application of proceptual knowledge, and is in general necessary to make sense of expressions. What is needed is a better understanding of the way syntactic and proceptual knowledge interact and support one another.



(a) (b) (c)

Figure 3: Difference between the sums formed by two L-shapes

We shall consider next some examples from Bell (1995) that involve the use of symbolic expressions in reasoning and justification. These are tasks set for 13 and 14 year olds and take the form of finding a pattern and using algebra to prove that the pattern is indeed general, or that it is not. In one of the tasks, students choose a 3 × 3 array of adjacent numbers on a calendar to study the patterns and relations between these numbers. Some patterns are found by adding and subtracting combinations of these numbers. A student, Julia, finds a pattern by adding the numbers inside two L-shapes and taking the difference of the sums (see figure 3). In her neat written work, she presents the first example with actual numbers from the calendar (figure 3a) and shows that the difference is 44. In the second example, she uses ‘*x*’ for the number in the middle of the array (figure 3b) and computes the difference ‘in algebra’. She again obtains the answer as 44, which is independent of ‘*x*’ leading her to conclude that the pattern is general and will hold for any 3 × 3 array on the calendar.

Let us carefully consider the steps in Julia’s reasoning that allow her to reach this conclusion. The first step is the use of expressions to denote the various numbers in the array. Presumably through prior supportive teaching, Julia has acquired this aspect of the notion of *substitutability*: that expressions can be used in place of the actual number. She takes the further step of operating with these ‘unclosed’ expressions by adding them to express the sum as

$$x - 6 + x - 7 + x - 8 + x - 1$$

Next Julia simplifies this expression to obtain $4x - 22$. This is an important step and it is interesting to ask what the basis for this action is. Is she thinking about the four different numbers that she is adding and the relation that they bear with each other, or is she reasoning syntactically? Thinking in the syntactic mode frees one to break up each expression standing for a number into ‘atoms’ that can freely move and combine with one another in new and convenient ways. (This point would be more obvious had Julia used brackets around each of the numbers.) Notice that Julia is using the algebraic convention of writing ‘$4x$’ instead of ‘$4 \times x$’. Does this suggest that she is merely gathering together the ‘*x*’s and not thinking of adding the four numbers with *x* forming a part of each? In other words, is she already thinking in a syntactic-structural mode?

After repeating a similar step for the other L-shape, Julia finds the difference between the two expressions as

$$(4x + 22) - (4x - 22).$$

Appropriately, Julia uses brackets to write the difference and evaluates it to 44. Bell remarks that she was successful in doing this by thinking of the ‘global meaning’ of the bracketed expressions (22 more than 4*x*, take 22 less than 4*x*, p.54). This would be a difficult move for most students, as Bell admits. In instructional sessions aimed at helping students learn to work with brackets, he reports the use of various strategies, besides the referential strategy of thinking of the meaning of the operations and the bracketed expressions, that are deployed to help students find the difference. One strategy is substituting with numbers and checking. Another is to evoke cognitive conflict through the application of a wrong rule such as 13 – (6 – 2) = 13 – 6 – 2?

Dealing successfully with brackets is an essential component of the reasoning process in many contexts where algebra is used to justify or prove a general property. It is, for example, a necessary step in completing the reasoning involved in ‘guess my number’ games. Thus it forms one of the core components of the ability to use symbolic expressions in reasoning. It is also one of the areas of difficulty for many students and even adults. The traditional approach is to teach students a set of bracket opening rules. However, the variability in the bracketed expressions that students may encounter, makes the learning of these rules particularly difficult. One can have the different operation signs: ‘+’, ‘–’, ‘×’ or ‘÷’, to the left or to the right of the brackets, or one can have any number or combination of numbers, letters and operation signs within brackets, or one can have embedded bracketed expressions. It is not surprising that many students fail to generate and apply the correct rules.

Given the variety of these possibilities and the ensuing complexity, a single instructional approach may not be enough to ensure that students learn to deal with brackets. For example, triggering the operational sense may work in simple cases with a majority of the students, but may impede progress in absorbing the more complex cases into the knowledge base. We have found that students can think more easily of the operational meaning of expressions of the form $a + (b \pm c)$, but find it difficult to do so with expressions of the form $a - (b \pm c)$. These difficulties increase when other operation symbols are used and with increasing number of terms within the brackets. This suggests that at some point students need to cross over into syntax based thinking. The learning process involved might be to secure a few simple syntax based rules and use this base to learn to deal with more complex examples. In general, the issue of how semantic processes, of associating symbols with the numbers or operations that they denote, aid the learning of syntactic rules is complex and needs to be carefully researched.

In discussing the example of Julia’s reasoning, I wished to draw attention to the syntactic-structural moments in her reasoning process. However, it would be injudicious to pass over the remarkable pedagogical innovation that the activity represents without comment. Despite being simple and familiar to students, the activity engages the power of algebra in generalizing and justifying. The sequence in Julia’s choice of the three presentations, the variations and the invariances between them together offer a rich opportunity for learning. In Julia’s third version of the array a subtle variation is introduced by changing the position of the number denoted by ‘*x*’. Julia then recomputes the difference of the sums in the two L-shapes and again finds it to be 44. The pedagogical insight at work here is important and noteworthy – a variation is introduced that does not make a difference to the final outcome. The student has a hunch that this must be so and uses this as a check and a guide while she carries out the steps of her reasoning and then makes the required syntactic transformations. We have found this didactic principle extremely useful in designing tasks that allow students to use their syntactic skills in a meaningful manner. We shall call this principle the 'many paths, single destination' principle and I shall make reference to it later on.

I hope that the points made above make the case for greater attention to be paid to algebra as the syntactically guided use of formalisms in innovative approaches to algebra teaching. This is not to contest the importance of algebra in other contexts such as generalizing or equation solving. These are important aspects of algebra but the power of algebra in dealing with formalisms gives access to a domain of reasoning and thinking that is otherwise not available to students. If students acquire this ability, it opens the door to reasoning about patterns that basically are extensions of students' knowledge of arithmetic. Therefore not only is this domain within the reach of students' understanding, but such reasoning is also a natural extension of arithmetic understanding. Experiencing the power of algebra in this domain, I suggest, is an essential component to taking students away from the notion that algebra is just a crank to be turned to produce answers. Just as research has illuminated other kinds of algebraic thinking that occur and can be strengthened through suitably designed instruction, research is needed to clarify how students come to acquire syntactic-structural reasoning abilities, what the preliminary forms of such reasoning are and what learning experiences and instructional practices support its development.

In the following section, I shall briefly mention certain approaches to teaching algebra that focus on structure. I shall use the opportunity to revisit the issue of how sense-making and learning abstract rules of symbolic transformations can be reconciled during teaching and learning.

Focusing on structure in algebra teaching

The power of algebra stems in large part from the use of symbolic expressions in problem solving and in justifying and proving. A recognition of the importance of acquiring facility with algebraic symbolism lies behind much of the emphasis on symbolic algebra in the traditional algebra curriculum. However, as several studies, many of them decades old, have shown, students' facility with symbolic algebra remains woefully inadequate even after instruction (Booth, 1984; Kieran 1992). Basic errors and misconceptions such as the conjoining error ($3 + x = 3x$) and misconceptions about what letters stand for are frequent and widespread. Clearly it is difficult to develop any ability in symbolic algebra on such a shaky basis. The early research inspired a succession of teaching intervention studies that have focused on enhancing students' understanding of different aspects of symbolic algebra: the concept of the variable (Booth, 1984), the solution of linear equations (Linchevski and Herscovics, 1996), the parsing of expressions (Thompson and Thompson, 1987) and the structure of expressions (Liebenberg et al, 1999).

A persistent concern in this thread of studies has been to leverage students' understanding of arithmetic in learning algebra. By the end of primary school many students develop a good understanding of operations with whole numbers. Despite this, students' ability to deal with arithmetic expressions remains limited. For example, Chaiklin and Lesgold (1984) found that students could not judge the equivalence of of expressions like 685 – 492 + 947 and 947 – 492 + 685 without recourse to computation. While some studies have addressed themselves to the issue of the 'cognitive gap' between arithmetic and algebra (Linchevski and Herscovics, 1996) others have attempted to show that errors found in students' work in algebra have their counterparts in arithmetic (Linchevski and Livneh, 1999), although with the interesting difference that in the case of arithmetic expressions, errors are also dependant on the specific numbers that appear in the expressions.

Most of the teaching studies mentioned above have attempted to build the basics of students' abilities to deal with symbols through strengthening their semantic connections. This perspective is captured in Booth's statement that 'the essential feature of algebraic representation and symbol manipulation, then, is that it should proceed from an understanding of the semantics or referential meanings that underlie it.' (Booth, 1989, p.58) However, the issue of the connection between arithmetic and algebra, especially in terms of symbolism, is not straight forward. One can legitimately ask whether the structure of an expression is more transparent in algebraic rather than in arithmetic expressions, since algebraic conventions are designed to enhance structure. Linchevski and Livneh (1999) have raised doubts about whether focusing on teaching structured arithmetic as a preparation for algebra is a good pedagogic strategy. In a similar vein, Kirshner (2001) has questioned the belief underlying Booth's statement that referential sense-making is the right preparation for dealing with algebraic symbolism. He advocates an approach that strengthens the syntactic-structural aspect of algebra together and in parallel with meaning-based instruction. He calls for a renewed effort to explore the possibilities of structural algebra teaching, that 'honors both structural and referential possibilities for meaning making in algebra'.

Our analysis of the examples in the earlier sections also point to the difficult and complex connection between referential sense-making and syntactic understanding of symbolic expressions. We may assume that the ability to understand and work with symbolisms proceeds from a knowledge base of rules for dealing with expressions. This assumption has not gone unchallenged. For example, Kirshner (2001) offers the interesting alternative suggestion that symbolic ability may stem from perception-based rather than rule-based abilities to interpret and match patterns in symbolic expressions. While this may be the case for those who have already reached expert level performance with regard to symbolic expressions, novices must still rely on rules. Studies of skill development show that the initial phase of learning is characterized by slow, deliberate actions where rules are recalled often and explicitly (Anderson, 1998). The issue of concern may then be reformulated as how conceptual understanding in the domain of arithmetic contributes to the learning of these rules. It may be necessary to go further than probing the connection between arithmetic and algebra and re-problematize the entire general issue of the connection between concepts and rules in learning.

A straight-forward connection between concepts and rules is found in the fact that rules are formulated in terms of concepts. The ability to choose concepts carefully so that rules are formulated clearly and economically is part of a good teacher's repertoire of instinctive pedagogical skills. We believe that much can be gained from a careful choice of concepts while introducing rules to children, and more importantly, from a careful choice of naming practices. The rules for dealing with expressions are traditionally formulated in terms of procedures. Thus we have the rules for the order of operations such as, 'do multiplication and division before addition and subtraction'. Stating rules in this way reinforces the procedural interpretation of expressions. We propose a structural recasting of these rules that can be introduced right at the beginning of students' work with expressions. The proposal finds support in studies of cognitive development concerning the effect of naming on categorization.

Markman (1989, 1990) in a series of experiments explored the effect of naming on categorization among young children. She points out that the enormous vocabulary that most children acquire by the age of six – between 9000 to 14000 words – could not have been acquired only through inductive processes, or even by forming hypotheses about the meanings of words and testing them. This phenomenon suggests the operation of cognitive constraints on how children interpret novel labels. Two important constraints that Markman identifies are the taxonomic assumption and the mutual exclusivity assumption. The taxonomic assumption takes labels to refer to objects of the same kind rather than objects that are thematically related. This is a particularly strong constraint since Piagetian and post-Piagetian studies have shown that thematic relations are more salient than taxonomic relations for young children. However, when children 'are learning a new *word*, they shift their attention from thematic to categorical organization' (Markman, 1990, p.60). The mutual exclusivity assumption rules out the interpretation of two different labels as denoting the same category. It is this assumption that allows the formation of super-ordinate or subordinate categories. Thus when children hear an object being labeled by two different words, they assume that one of them is at a different level of categorization (or in certain contexts that it refers to a salient part of the object rather than the whole object, etc.)

The relevance of Markman's work to our discussion is that it brings home the importance of naming practices in concept learning. A careful choice of names that are appropriate, and that reflect the hierarchical organization of the concepts is a powerful aid in concept learning. In the following section I outline a teaching approach that uses this principle in enhancing the salience of the structure of an expression. In this approach, the concept of 'term' forms a nucleus for the organization of the structure of both arithmetic and algebraic expressions.

A structural approach to teaching expressions

It is necessary to first clarify what we understand by the 'structure' of an expression. The pre-requisite to a structural understanding of arithmetic or algebraic expressions is the idea of the value of an expression, the idea that an expression stands for a number. Understanding the structure of an expression means that students should be able to correctly parse the component parts of the expression and to obtain a sense of the relation of these parts to each other and to the whole. The relation in question is how the parts contribute to determining the value of the expression as a whole. Understanding this relation would imply that students can recognize the equivalence of expressions on the basis of structure. Further they would be able to identify a neighbourhood of related expressions – either equal to the given expression or related to it in definite ways – into which the given expression may be transformed. Thus knowledge of permissible transformations is also a part of the structural understanding of expressions.

The structural approach to teaching arithmetic and algebraic expressions that I shall describe here was developed over teaching cycles involving different groups of students from schools that followed the traditional curriculum. Students who studied in the local language, Marathi, as well as students who studied in the English language (although English was not their mother-tongue) formed separate groups during instruction. Here we present only an outline of this approach with examples of activities that are included. More details and discussion of students' learning may be found in Subramaniam and Banerjee (2004) and Banerjee and Subramaniam (2004). Our approach includes an explicit initial focus on the 'product' and 'relation' aspect of expressions. It was felt that for students who were used to a traditional approach, an explicit focus on a different way of looking at expressions was necessary. At the beginning of the instructional module, students are told that while, in the earlier grades, they looked at expressions as describing a sequence of operations that needed to be carried out, they should now look at expressions differently. Each expression stands for a number, which is the value of the expression, and expresses some 'information' about the number. Thus the expression 5 + 3 stands for the number 8 and expresses the relation '(the number which is) 3 more than 5'. Students are encouraged to find other expressions for the number 8, and to verbalize the relation that each expression expresses. At this stage, they are also introduced to expressions that use a letter which stands for a number, which could be 'any number'.

The notion of equality is the pivotal concept in developing an understanding of expressions. Students are introduced to the idea that two expressions are equal if they stand for the same number, that is, if they have the same value. Expressing this symbolically using the '=' sign is a hurdle for many students because of a familiar misunderstanding of the '=' sign as an instruction to 'do something' and write down an answer. To obtain experience with using the '=' sign differently, students work at tasks involving the comparison of expressions, which are similar to the 'true' and 'false' number sentence tasks in Carpenter and Levi (2000). Other exercises along these lines require them to generate expressions equal to a given expression and to fill in terms to make expressions equal. Comparing expressions is also done using the metaphor of a balance, which is reinforced through the use of a diagram. Thus students get used to looking at expressions on both sides of the balance in order to compare them, initially by evaluating the expressions and later on increasingly, by focusing on their structure.

One of the most important features of the structural approach is the early and explicit focus on the concept of *term* (Banerjee and Subramaniam, 2004). This concept is introduced while working with arithmetic expressions and then reinforced as letters are introduced in the expression. In fact, as students work with expressions containing numbers, letters are introduced as early as possible and whenever the context allows, in order to reinforce the idea that letters stand for numbers. The concept of term is situated in the context of evaluating expressions and of comparing expressions. Terms are distinguished from numbers by attaching the preceding '+' or '–' sign. Thus the simplest expressions contain *simple terms* which may be positive or negative.

To get students to move closer to structural thinking, we explicitly contrast a 'new' method of evaluating expressions that is different from the method that students have been familiar with up until now. Instead of adding and subtracting the numbers in the simple expressions, they now learn to *combine terms*. Simple terms are easily combined. Simple positive terms combine by adding up. Similarly simple negative term also 'add up'. Positive and negative terms compensate one another. The idea of compensation is not difficult for students to understand and is reinforced through different exercises. The parsing of an expression into terms, segregating the positive and negative terms and combining terms, together form a set of connected actions that many students readily absorb. The main initial context for working with the concept of term are tasks of evaluating and comparing expressions and tasks of generating expressions equal to a given expression. In the process, students become familiar with the fact that changing the order of terms does not change the value of an expression.

The simple task of evaluating expressions can be made interesting by including terms that compensate or nearly compensate each other. During whole class teaching, students are encouraged to find easy ways of evaluating expressions in which terms have been deliberately and carefully chosen so that they compensate one another partially or wholly (figure 4). This reinforces the parsing of expressions into terms and the idea of compensation of terms. The experience with combining terms makes the concept of the expression with inverse value accessible to students. Students are asked to evaluate an expression like 24 – 8 – 7, and asked to guess the value of –24 + 8 + 7, or alternatively are asked to generate an expression that has the inverse value. The notion of inverse value is invoked again during instruction on bracket opening rules.

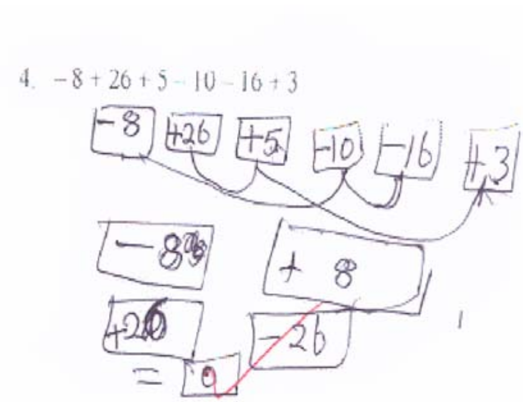
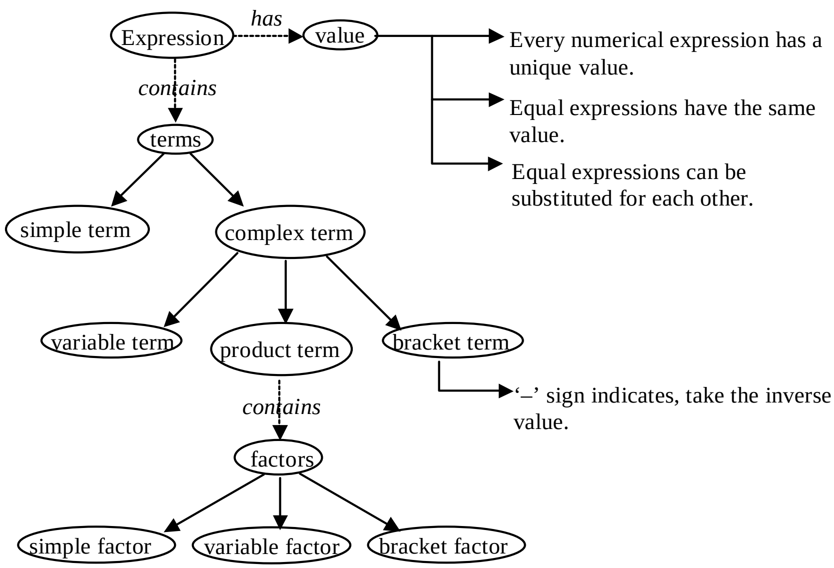


Figure 4: Finding compensating terms in an expression

The next step involves the introduction of the concept of the *product term*. When there is a '×' sign present, the factors together form a product term. An important restructuring element is introduced at this point: while simple terms are readily combined product terms cannot be. They must first be reduced to simple terms before they can be combined. (Later students learn the exception to the rule, namely that product terms may be combined if they have one or more common factors.) With this move we absorb the convention of the precedence of multiplication into a structural system of naming terms. Students again carry out tasks of identifying terms in given expressions and finding equal expressions, but now the expressions contain product terms. The idea that only simple terms can be combined and the concept of a product term helps to guard against the conjoining error while working with letters or *variable terms*. The conjoining error and the violation of the rule of order of operations are among the most common errors students make with regard to expressions. By taking advantage of the structural system of naming, we have recast these rules in the form of a *negative heuristic*, namely, that terms other than simple terms cannot be combined. We hypothesize that negative heuristics, especially those that are structural and related to patterns of visual parsing, are more effectively learned than positive heuristics, that state that such and such an action must be done when a certain condition is met, which are essentially procedural rules, such as the operation precedence rules. This hypothesis is in need of more intensive empirical testing. While the initial results of our teaching interventions are promising (reported in Subramaniam and Banerjee, 2004 and Banerjee and Subramaniam, 2004), more studies are needed to examine long-term effects in diverse settings.

Figure 5 shows the organization of the concepts, definitions, rules and procedures in the structural approach to teaching expressions in the form of a concept map. Rules and procedures that are invoked while dealing with expressions are shown separately. The other statements that appear in the concept map have the status of definitions that set up meanings and conventions.



Rules
1. Only simple terms can be combined.
2. Exceptionally, product terms can be combined if they have a common factor.

Procedures
1. Combining terms
2. Writing inverse expression
3. Applying distributive property

Figure 5. A concept map for teaching arithmetic and algebraic expressions
(Can be extended to include concepts of division, reciprocal and indices.)

Although the terminology, the concepts and the rules are new and seemingly artificial (in the sense of being formal and axiomatic), students do make connections with their arithmetic knowledge base. This is done at times explicitly on the initiative of the teacher, but more often occurs implicitly as children work with evaluating and comparing expressions and at generating equal expressions. We shall mention some of loci that connect students' arithmetic knowledge with the new syntactic concepts, rules and procedures. No explicit rules are laid down for combining terms and students' intuitive understanding of numbers is exploited, as for example in the idea of compensating positive and negative terms. Similarly the idea of an inverse expression whose value is the inverse of a given expression is founded on the arithmetic understanding of combining terms. A further reinforcement takes place when students are asked to guess the multiples of a given expression, that is to write down an expression that gives *n*-times the value of the given expression.

The network of concepts outlined allow students to reason about arithmetic and algebraic expressions in various tasks that we have just described. Students need to move on to using expressions in reasoning in the context of generalization or justification, in other words, they need to move on from reasoning *about* expressions to reasoning *with* expressions. The key notions that they need to master here are of *equality* or *equivalence, transformation* and *substitution*. Another set of contexts are now used in the instruction module to add a further dimension of meaning to work with expressions. The numberline forms one of these contexts around which a number of activities may be centred. A modification of the numberline that is very useful to work with is the letter numberline, which we have adapted from Carraher and others (2001). Students initially get used to seeing different parts of the numberline since the starting number is varied in different tasks. The letter numberline is then introduced as an unspecified or unknown part of the numberline around a number, say *n*. (One needs to anticipate a particularly robust 'origin-at-*n*' misconception here. Students often think that the numbers to the left of *n* are negative numbers, as if *n* were the origin. It helps therefore to say that the numberline shown with letters is an unspecified part of the numberline, rather than another kind of numberline.) In figure 5, the regular numberline and the letter numberline are shown. The letter numberline provides a meaningful context to familiarize students with the notion of *substitutability* – that expressions can stand in place of numbers. Building students' understanding of substitution is a critical part of the symbolic understanding (recall Julia's use of expressions to stand for numbers on the calendar). In the letter numberline, students verify that once the value of '*n*' is fixed, the expressions yield, on substitution, the correct numbers at their respective positions.

Brief descriptions of a sample of the tasks done with the numberline and with shapes built from rectangles are presented below. I also discuss ways in which these tasks allow students to apply their knowledge of symbolic expressions. The main design principle in generating these tasks is that students have alternative ways of arriving at the solution – the 'many paths, one destination' principle – and they use symbolic expressions in consolidating their understanding of the alternative ways.

Numberline journeys

Figure 6(a) below shows an example of a numberline journey with the expression describing the journey written next to it. Figure 6(b) describes an example of a journey on the letter numberline. In these tasks students find ways of combining terms to find the destination point of the journey. The visual map of the journey serves as a check while they work with expressions. Journeys can get as complicated (and as interesting) as the students wish, with the constraint that they must write down the expression to describe the journey!

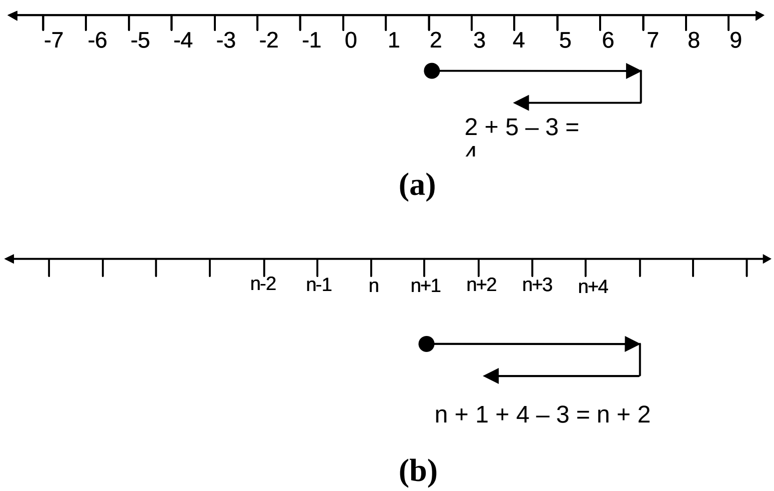


Figure 6: Numberline journeys

Distances on the numberline

Students work through a set of exercises of finding the distances on the numberline to generate the rule that the distance may be found by subtracting the smaller number from the bigger number. This is also verified for special cases involving the number zero or negative numbers. The activity is then extended to the letter numberline. Combining terms while working with numberline journeys is relatively simple. To find distances on the numberline, students have to operate with the unknown. They need to learn how to remove brackets and to subtract letter numbers in order to find the distances, say, between the numbers $n - 1$ and $n + 2$. Again the visual map of the distance serves as a check for the manipulation of symbolic expressions.

Numberline concordance

Students find a map from points on the numberline to a ‘co-numberline’, which is a copy of the numberline that may be displaced, shrunk or flipped, or may have undergone a combination of these transformations. Only vertical alignment lines (shown as dotted lines in figure 7) are used and so only linear functions are included. In describing the co-numberline, photographic metaphors of reduction and enlargement, and of left-right flipping in photo-negatives, provide an opportunity to engage students’ imagination. Besides being a powerful visual model of linear functions, this representation also offers opportunities to meaningfully use substitution. Having found an expression for the function, students check by substituting actual numbers, and also expressions, in place of the variable. For example, in the function $f: x \rightarrow 2 \times x$, substituting $x + 1$ in place of x should yield $2 \times x + 2$. This may be checked in two ways, (a) by looking at the map and seeing what corresponds to $x + 1$ and (b) by substituting in the expression for the function and using the distributive property to open brackets: $2 \times (x + 1) = 2 \times x + 2$.

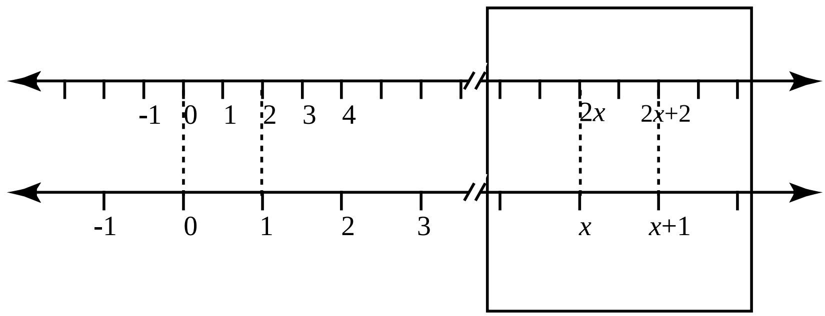


Figure 7: Numberline concordance

Area problems
Here students learn to find the area of rectangles such as those in figure 8(a) in two ways: by either finding the total length (6+2) or by dividing the given rectangle into two smaller rectangles. With support from the teacher, students write down an expression that describes the method that they followed. The two expressions for the rectangle in the figure are

$$5 \times (6 + 2) = 5 \times 8 = 40$$
and
$$5 \times 6 + 5 \times 2 = 30 + 10 = 40$$

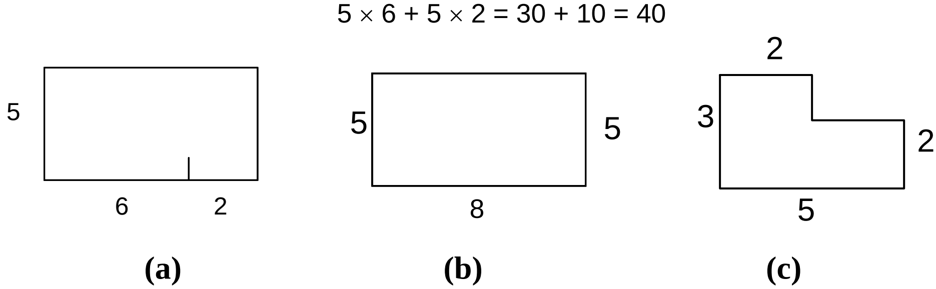


Figure 8: Areas of shapes made of rectangles

The important turn in the classroom discussion introduced by the teacher here is asking students to consider the two expressions written down for finding the area without referring to the figure. Can the students say without evaluating the expressions, by just looking at the expressions if they are equal? How can they do so? The discussion becomes a context for reinforcing the use of the distributive property in rewriting expressions without brackets.

In the group of 11 year olds who worked with the area task, some students had difficulties with the concept of area that we did not anticipate. Many did not have a clear understanding of why the area of a rectangle was a product of the length of the sides. Some students calculated the area of the rectangle in figure 8(b) as $5 \times 8 \times 5$! Other students did not immediately recognize that opposite sides of a rectangle have the same length. With such an insecure notion of area, it is not surprising that some of them did not recognize that the area of a shape was equal to the sum of the areas of its parts. For more complex shapes such as ‘L-shaped’ polygons as in figure 8(c), many students were unable to come up with a strategy to cut the shape into parts and find the area, while some students applied this strategy fairly easily.

In the following section I shall describe some examples of reasoning by students in the context of tasks requiring the comparison of expressions. I focus especially on the efforts that students make to bring to bear their arithmetic understanding while making sense of expressions and highlight instances of how students attempt to extract or extend rules at the syntactic level while struggling to make this consistent with what their operational sense tells them.

Comparing additive expressions: Students’ beginning attempts at syntactic-structural reasoning

I choose here for discussion a set of tasks adapted from earlier studies requiring the comparison of additive expressions without calculation. In a study by Chaiklin and Lesgold (1984) students were asked to compare expressions such as 685 – 492 + 947 and 947 – 492 + 685, but found the task difficult without resorting to computation. Van den Heuvel-Panhuizen and Gravemeijer (1994) have used additive expressions in assessment tasks for upper primary students. In these tasks, students are given the answer to an addition problem such as 238 + 487 = 725, and are asked to find the answer to 237 + 487 without actually computing. These latter assessment tasks formed the inspiration for the initial version of our own comparison tasks. After students had successfully found a related sum, we asked them to give a justification for what they had done. This was quite hard for the students to do both verbally and symbolically. One of the simple forms of justification adopted by the students was to compare the expressions term by term. For instance in the pair presented above, they would say ‘237 is one less than 238, 487 is the same. So the answer is 724.’ While the reasoning works perfectly well for this example, they would imitate its form for other examples where it was inappropriate and hence make an error. Such pairs either had a ‘–’ sign (487 – 238; 487 – 237) or had compensating terms (238 + 487; 237 + 488). In the initial instructional cycle, it appeared that the very act of eliciting verbal justification sometimes misled the students into making wrong judgments, which they might have avoided if they had to just compare the expressions without explaining their judgments.

One student in the group came up with an interesting justification for why the following expressions were equal: 27 + 32; 29 + 30.

Mitali: The two expressions are equal because we have taken 2 from 32 and given it to 27.

In subsequent teaching cycles this form of justification reappeared spontaneously with two groups of students studying in the local language Marathi. After one student had expressed this form of reasoning, it was readily picked up by other students for similar examples. Two forms of the comparison task were used in these and later teaching cycles. One required students to insert the correct sign (‘>’, ‘<’ or ‘=’) between two expressions. An alternative form required them to fill in a term in a blank in one of the expressions to complete an equation, for example, 28 + 13 ___ = 27 + 13.

Many students made mistakes while comparing expressions with a negative sign, such as 37 – 17; 37 – 18. Their justifications indicated that they had either generalized incorrectly from the previous examples with a ‘+’ sign, or were comparing only the

numbers involved without taking into account the operation sign. This was one of the reasons that initially led us to emphasize the concept of term as a way of parsing an expression. In the second teaching cycle, where students had learnt the concept of term and had used this concept in comparing expressions, one student made an interesting extension of the vocabulary of terms for comparing pairs of expressions with a '–' ve sign.
Arjun: The first term (in the two expressions) is the same, and –17 is greater than –18. So 37 – 17 is bigger.

Students are introduced to integers at the beginning of grade 6 in the curriculum followed in most schools. Arjun had used his recently acquired knowledge of negative numbers and the relatively novel idea that –17 is larger than –18, to make the comparison between the terms in the two expressions. This was readily picked up by many students in Arjun's group who produced this form of justification in subsequent written tasks. Learning the concept of term did lead the students to compare terms with a '–' sign more accurately. However, in the exercises where they had to fill in a missing term we found an interesting form of reversal error. In the blank in the question 28 + 13 ___ = 27 + 13, students would write '+1' instead of '–1'. This form of the error was one of the most frequent errors. We speculate that the reason for the error could be that students were comparing the values of the expressions and were filling in the blank a description of how much bigger or smaller the expression was. Thus the term that is filled in the blank is used in an 'adjectival' sense, to characterize the proximate expression, rather than as establishing an equality with the other expression. In the case of the example above, +1 indicates that the expression on the left is one more than the expression on the right. This error appears to be related to the well-known reversal error in writing equations. The most famous example of this is the equation 6S = P, that students wrote to show that there are six times as many students as there are professors with S standing for the number of students and P for the number of professors.

The most difficult pairs of additive expressions were those that had partially compensating terms. One of the instructional sessions dealt with comparing such expressions and finding their difference. We discuss an episode from one classroom lesson in the local language Marathi dealing with this topic. In this lesson, students worked with tasks such as the following:

> Compare the two expressions. If the expressions are unequal, find also the difference between the expressions: 54 + 27; 55 + 25

In the first part of the lesson, students worked with several pairs of expressions and found the difference. The teacher encouraged students to write a mathematical sentence to explain how they had obtained the difference between the expressions. At one point in the lesson, the teacher asks students to work with the pair of expressions presented above. After finding out which expression is larger, students write the following mathematical sentences: 2 – 1 = 1 and + 2 – 1 = 1 to find the difference between the expressions. At this point in the lesson, one of the students, Suraj starts to say something about these sentences but then checks himself and stops. Although the teacher asks him what he wanted to say, he is silent. The class then moves on to the next example which is to compare the expressions

67 + 38 and 65 + 37

Another student, Tanaji, offers the following symbolization and justification:

Tanaji: The difference between 67 and 65 is 2, the difference between 38 and 37 is 1. Take away 1 from 2, the difference is 1… So +2 – 1 = 1.

Tanaji's solution is wrong. Some students protest. The teacher points out that the difference between the expressions is not 1. Tanaji's error may be simply because of an over-generalization from previous examples where students found the difference of the differences between the expressions. In contrast, in this pair of expressions the differences add up, a point that Tanaji does not notice.

Pratibha then offers the expression +2+1 to determine the difference as +3. The teacher checks this with the students and finds that it is correct. At this point
Suraj asks,
Suraj: Bai (teacher), what is –2 – 1?

The question is significant because the students have not yet learnt to operate with negative numbers. On a few occasions, the teacher has explained addition and subtraction with the help of the numberline. So she explains how to find the value of –2 – 1 using the numberline. Stepping on an imaginary numberline, she shows that –2 – 1 is equal to –3. The teacher then turns to Suraj and asks him how he obtained the expression –2 – 1. Suraj explains how he got –2 by going from 67 to 65, and –1 by going from 38 to 37. The teacher completes the explanation by saying that –3 means that the expression becomes less while going from left to right.

Suraj's response is different from the other students' responses in that he is trying to use the symbols in a manner consistent with this sense of the relative magnitudes of the terms and the expression. This is different from a response where students 'adjust' the symbolism to obtain the answer that they think is right. It is also possible that other students were constrained by their lack of understanding of operations with negative numbers.

These examples from students' responses to the comparison tasks suggest several points. The first is that when students compare the expressions, they are looking at the relations and not specific numbers. This is suggested by their preference for referring to the number by indexing its order rather than by referring to its value, as for example, in the statement, 'the first term is less and the second term is more'. The specific numbers in the expression are relatively unimportant and function like quasi-variables (Fujii and Stephens, 2003). We see repeatedly a tendency among students to extract rules and to overgeneralize rules. We do not know whether this tendency is natural and develops spontaneously or whether this is a result of the approach to teaching currently prevalent in schools. Even if the latter is the case, learning and extending rules also have creative moments, as for example, when Arjun makes a connection between his knowledge of negative numbers and of terms. The manner in which students try out new rules in new and unfamiliar situations also suggests that a process of hypotheses formation and testing may be at work, akin to that in the development of lay or scientific theories. A written test may capture only one moment in the extended process of hypothesis-formation and testing, especially if test items are new and unfamiliar to students.

We need to consider the frequency with which students invoke syntactic rules, extend and generalize them, often erroneously, together with the relatively few but significant instances of successfully connecting these rules with the semantics underlying expressions. This pattern of response makes the case for a more systematic effort to find ways in which their understanding of symbolic expressions can be strengthened. Given the fact that mathematical symbolism is evolved over a considerable historical period, it is not likely to be spontaneously invented by students or easily absorbed. Hence from a research perspective, questions about how students acquire this form of algebraic thinking are bound together with questions about instructional practice. It appears therefore that the modality of teaching intervention research is appropriate to pursue these questions. In the course of instruction, naming practices can make a significant impact on concept learning and can guide the development of perceptual parsing skills and pattern recognition with regard to symbolic expressions. They also make a movement away from procedure based rules towards structural rules possible. While we are persuaded that carefully designed naming practices such as those outlined in this paper can make a significant impact on the learning of symbolic arithmetic and algebra, more evidence needs to be gathered before the case can be made for grafting this approach on to a curriculum. Similarly there is a need to identify more activities that allow students to engage in tasks along both the dimensions of reasoning about and with expressions and to calibrate these activities in terms of difficulty level and projected learning paths.

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