What mathematics does everyone need to learn?

My interest in this article is to explore this question from the standpoint of universal education: what is the mathematics that we want all our citizens to know. Although curriculum framers recognize this to be the basic question they need to answer while formulating a universal curriculum, the considerations actually applied are different. The approach adopted is generally one of preparing students for the subsequent study of subjects founded on mathematics such as the sciences or engineering or economics. Indeed, the application of mathematics in these disciplines and in many professions and occupations is growing impressively. With these changes, the task of identifying the common mathematical foundation needed for various disciplines needs to be taken up anew. However, the approach to framing a universal mathematics curriculum responsive to peoples' needs may need to accommodate as well as go beyond the perspective of preparing students for further study of specific subjects.

One of the justifications for including mathematics in a universal curriculum is an aesthetic and cultural one – mathematics as an organized body of knowledge is beautiful, full of hidden connections and surprises. It is also one of the great accomplishments of human beings, and is part of a shared, universal culture. However, this is still insufficient reason to make mathematics a mandatory part of what everyone should know. The primary justification for the place of mathematics in universal education, in my view, is the power of its application. Through elementary education, we want our citizens to be able to use mathematics when it matters to them and also to get a feel for the broader and more advanced applications of mathematics, because these may be potentially important.

Thus an approach to a universal mathematics curriculum must aim at showing its varied application. If application is emphasized, then learners must see examples where mathematics is applied in situations that are both familiar and seem important. A common response to this need is to seek out examples from the world of finance or commerce, which affect a large number of people, at least those living in cities. However, this may unduly restrict the mathematics that learners actually acquire. So although I emphasize the application of mathematics, I do not advocate a narrow sense of utility. One must find a way of choosing situations that concern peoples' lives but also those that build bridges to important mathematics.

The considerations that have influenced curriculum planning traditionally also need to be accomodated. As mentioned earlier, mathematical knowledge is in many cases a pre-requisite for other knowledge that is relevant to understand important aspects of our lives. Further, since knowledge of mathematics acquired during schooling determines access to some branches of higher education and to certain careers, it is important for social and occupational mobility. Finally, we need to keep in mind the emancipatory potential of knowledge, its power to free individuals and small (or large, for that matter) collectives from dependence on others in making decisions and judgments.

The core part of a universal mathematics curriculum would include both mathematics that is necessary for life in modern society and mathematics as part of culture that would commonly be acknowledged as worth knowing for its own sake. The need for emphasizing this perspective in mathematics education is now recognized in many countries, which has found expression through the coining of the phrase 'mathematical literacy' (or 'quantitative literacy'). In the modern world, where quantitative information is critical to many aspects of living, mathematical literacy may be as important functionally as 'reading literacy'. Recent efforts have led to a clarification in detail of what components and skills make up mathematical literacy.[1] Some of the elements of mathematical literacy, besides the concepts and tools of mathematics, are seeking and understanding quantitative information, interpreting data, reading charts, tables and other presentations of data, logical thinking, number sense, symbol sense, being confident of mathematics, appreciating the role of mathematics in disciplines of knowledge. However, even in the countries which have accepted mathematical literacy as a part of the curriculum, there is a lack of consensus on how these different elements may be built into the curriculum, and especially about how this may be integrated with the traditional mathematics curriculum. In this article, I take the position that developing mathematical literacy does not need one to teach a different mathematics in school. I argue that many of the core strands of the traditional mathematics curriculum are those parts of the subject that have the greatest power of

1 Two books on mathematical literacy: Steen, L.A. (ed.) (2001) Mathematics and Democracy, The Case for Quantitative Literacy. National Council on Education and the Disciplines. Madison, B. L. and Steen, L. A. (eds.) (2003) Quantitative Literacy : Why Numeracy Matters for Schools and Colleges, Natonal Council on Education and the Disciplines , Princeton, New Jersey. Available on http://www.maa.org/ql/qltoc.html

application. However, the manner of implementation of the curriculum does not make this transparent. In what follows I will attempt to briefly illuminate the main strands of school mathematics against the background of their application.

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Even the elementary application of mathematics to the real world involves the activity of measurement, and the correlation of measures. There are two phenomenological sources for the activity of measurement that have existed since early historical times, both of which need to be appreciated. The first and perhaps the earlier source is that of simple exchange or barter, where the value of different goods is equated by balancing the number or quantity of each good. One good functions as a unit, so to speak, to measure the value of another good. This is a notional value based on choices and preferences, not of individuals, but between individuals or groups. The other phenomenological source of measurement is from the physical or the engineering world of direct measurement. The simplest case is of the measurement of length, where a unit length is iterated a certain number of times to measure another length. Many direct measurements follow the model of length measurement – smaller units are iterated till a quantity equal to the one being measured is reached. Examples are the measurement of weight, volume and also time. There are important contexts of measurement that go beyond these phenomenological sources. Science deals largely with quantities that are measured indirectly, and here the mediation of theoretical knowledge is crucial.

There is an important conceptual change when we move from the world of counting to the world of measurement. Counting and the use of the natural numbers arose early and in many different cultures. Despite the sophisticated conceptual and notational structures that form the counting number system, there is something comparatively intuitive about counting and many children master this aspect in the early years of schooling. Measurement and the correlation of measures however requires us to work flexibly with units, requiring us to use different units and to understand the consequences of the change of unit on the activity of measurement. This is the core of what has been called 'multiplicative thinking' and is one of the main strands of the middle school curriculum. Conceptually and notationally, the rational numbers provide the foundation for multiplicative thinking.

The place of mathematics in understanding our world stems largely from the role of measurement in understanding reality. A large part of the mathematics of the everyday world – and this has been the case from historical times – from the domain of economics and simple physics is the mathematics of measurement and has to do, in the first instance, with ratio, proportion and linear functions. Hence one of the chief objectives of school mathematics education is to learn how to handle linear functions in different contexts: forward and inverse problems, multiple constraint problems, multiple proportions. This should also include an idea about the situations where linear functions are not applicable. Since rational numbers provide the conceptual and procedural basis for understanding linear functions, school education must have as an outcome the mastery of the use of rational numbers in suitable contexts.

Davydov, an influential and creative mathematics educator and psychologist in the Russian school which followed Vygotsky went a step further. He said that the goal of school mathematics education was knowledge of the real numbers.[2] Real numbers make it possible to include representations of infinite processes in calculations. It is important to remember that the basis for applying infinite processes is the fact that they can be approximated by rational numbers or decimal numbers and operations on them. It is not entirely clear whether the facility to understand and use real numbers should be a part of universal mathematics education, but I leave this question open for the moment.

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There are two broad areas of the application of mathematics that are important from the point of view of what people need to know. The first is the area that falls broadly under economics and includes both economic activity as well as its bureaucratic regulation. It involves understanding in some detail, aspects relating to personal finance such as wages, loans, taxes, grants, scholarships, and also public finance such as public funds, their collection, management, allocation, beneficiaries, etc. The present curriculum emphasizes only personal finance under the rubric of 'commercial mathematics'. This needs to be expanded to cover the important area of understanding public funds. Moreover the emphasis is on mechanical computational aspects.

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2 Schmittau, J. (2003). Cultural historical theory and mathematics education. In A. Kozulin, B. Gindis, S. Miller, & V. Ageyev (Eds.), *Vygotsky's educational theory in cultural context.* Cambridge, UK: Cambridge University Press.

Another serious omission is statistics and the capability to understand and analyse large aggregates of data is a serious omission. Some other aspects need to be studied for possible inclusion in a universal curriculum: for example, those that are relevant to a deep understanding of the principles of economic justice that underlie economic activity? It may also turn out that the mathematics of choice and combining or aggregating choices, which is the foundation for democratic functioning, may be important for a general mathematics education.

The second broad area of application is science. Science education has the overall goal of reshaping intuitive or culturally gained knowledge to align it with more powerful and general conceptual and knowledge structures. An important feature of science is the mathematization of concepts in a variety of domains through quantification and measurement, and the analysis of causal relations through mathematical functions. Geometry too finds a place here through the application of the powerful tools of spatial reasoning to abstract functional relationships. Apart from the mathematization of concepts, the approach in science includes a mathematization of situations through modelling. Students need to understand the various aspects of modelling such as idealization, approximation and interpretation of the model to draw inferences in the real world. From the point of view of areas that are important to people's knowledge, the two most important areas of science are health and environment, both requiring a deep understanding of biology besides physics and chemistry.

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Another major area of the traditional curriculum is geometry. There are at least three important goals of geometry education:

- The concept-enriched retraining of perception: Spatial understanding plays a large role in efficiently performing many everyday tasks. However, intuitive spatial understanding acquired through everyday perception and interaction is misleading and context bound, and hence limited, especially in a 'carpentered world'. In the theory of the development of geometrical understanding proposed by Van Hiele, one of the important achievements is to go beyond the dominance of perceptual and intuitive knowledge structures.

- The formalization of visual intuition as well as visual reasoning through developing a powerful conceptual system of geometry.
- Geometry also provides a basis for the real numbers which underlie measurement, and the model for a deep understanding functions.

The third goal listed above relates to the previous discussion about the application of mathematics to measurement. The first two goals belong solely to the domain of geometry. From this point of view, the geometry of the plane is the foundation for the rest of geometry, which is the reason why it forms the cornerstone of school geometry. It is interesting to note that one reason for this may be the role that drawing plays in reasoning in general and geometrical reasoning in particular. Drawing is essentially a matter of reducing three- (or n-) dimensional geometry to the geometry of the plane. Two dimensional graphical representations that aid thinking are easily accessible, and capitalize on the hand-eye-mind coordination that has a central place in human thinking. Another important function of geometry in school education is that it is a gateway to abstract and axiomatic mathematics. Geometry is the oldest branch of knowledge to be cast in axiomatic form. So it occupies a special place in mathematics education.

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Algebra is one fo the core topics of school mathematics education at the secondary level and is also difficult for most students. What is the role that algebra plays in school mathematics? It is the main tool for representing and working with generalizations. These include representing in a generalized manner functions or causal relationships between quantities in the real world. We can then derive further information about the relationship such as how one quantity changes as another does. We can also add specific information and compute values that we need to know. This is what we do when we solve a problems by setting up and solving equations. We can also work on the generalized representations to derive certain conclusions, as when we want to justify, prove or explain a mathematical property or fact.

The need to understand and apply generalizations in quantitative contexts is an important one from the point of view of education for all. Such contexts arise in the world of finance, in making optimal choices and in understanding scientific knowledge. So we need to address the problem of how to make algebra an accessible and interesting part of a unversal school curriculum. Recent approaches to algebra education strengthen the algebraic thinking inherent in students' understanding of arithmetic and use that as a basis to build algebraic knowledge.

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We turn now to another important question that must be addressed while framing a universal mathematics curriculum. This is about the nature of abstraction which is essential to mathematics. From the educational point of view, the question may be framed in two parts: 'Is abstraction necessary?' and 'Is abstraction too difficult?'

Clearly mathematics, even at the school level, is abstract. The first encounter with abstraction is the concept of number and its symbolization using a numeral system based on place value. Then students need to go beyond the natural numbers. The fraction notation and the rational number are used in different contexts with a variety of meanings. They need to master all this, which needs a high level of abstraction. Finally, as discussed above, to understand and represent processes of generalization and to handle the notion of a function conveniently they need algebra. Algebra is the strand that introduces students to the abstract aspects of mathematics. The notations of algebra stand for symbolic objects that may designate generalized entities or the result of a process that is only potentially realized. In this sense, as other mathematics educators have pointed out, the first encounter with algebra is actually the fraction notation standing for the quotient of a divided by b, where a and b are integers.

Symbols help consolidate abstraction processes. When procedures of operating with these symbols are made explicit and systematized, not only is application made easy and convenient, but further conceptualization becomes possible. This is the domain of algebra. The decimal positional numeral system is a good example of a system of notations that is convenient for calculation, which also facilitates the further building of concepts. This is one of the outstanding contributions made by Indian mathematicians.

What is the method of abstraction in mathematics? The chief tool here is the axiomatic method which has a triple function.

- The first of these is to lay out mathematical statements in the form of deductive systems, so that the warrant for these is established.

- Secondly, the axioms have to be chosen in an economical manner – leads to an attempt to sharpen and purify intuitions in a manner that yields the most power.
- Once the axioms are formulated, the possibility of further generalization and abstraction is created.

It is remarkable that one method contains so much power and that it is the central method for extending mathematical knowledge. This makes a strong case for all students to be given an opportunity to understand and possibly apply the axiomatic method.

Moreover, when forms of classroom teaching and learning become more dialogic and less authoritative, children are bound to ask for justification of the propositions that they encounter. As a teacher attempts to answer them, such demands for justification can regress rapidly to the simpler propositions that underlie others, but which are harder to prove or justify. If the process of inquiry is not halted by dogmatic injunction, it will only be satisfied by a properly developed axiomatic structure. The opportunity to experience this movement of thought is offered by the deductive organization of mathematics. The axiomatic structure also provides a sequencing of the elements of knowledge that has a reasonable basis and can hence be reconstructed by a learner. This makes it possible for the piece of knowledge to be recalled and rescanned without the aid of external representations, an activity that is important if learning is to result in genuine understanding.

From the pedagogical point of view, we need a better understanding of abstraction. Are processes of abstraction natural? We do find examples of abstraction in the domain of language and semiotics. These abstractions have to do not with the laws of linguistics, but with the play of language in ascending spirals of abstraction – in the use of metaphors, for example. A higher level of organization of abstraction is found in poetry and in mythology. There may be similar examples from other sign systems (film, dance, music) – so the study of semiotics is important in clarifying how we create and use abstractions.

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In this paper, I have attempted to show the importance of core elements of the school mathematics curriculum from the viewpoint of universal mathematics education. This viewpoint must give primacy to mathematics as having wide and powerful application, and to the points where mathematics intersects with reality as it is experienced. At the same time the approach must not interpret use or application too narrowly. The points outlined above are intended as points for discussion, rather than as constituting a fully worked out position. The discussion would I hope help identify more sharply what mathematics we want all our students to know.

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