review

Pólya to the rescue

When you don't know the solution to a problem

Tips from a master

Is problem solving ability simply a gift, innate and inborn? A gift that some people have and others do not? Or is there something that can be learnt about problem solving? It is widely regarded as a gift, but the mathematician George Pólya thought differently. In this article we learn about the ways in which he looked at this important educational issue, and the approaches he recommended.

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Like most students, I too went through school learning solutions to problems and doing my best to remember them in an exam. We prepare for exams by 'practising' the solutions to problems – trying to remember a formula, a method, a trick, the steps, and so on. But, as my friend and fellow math educator Dr. Hridaykant Dewan puts it, *learning to solve* problems is different from *learning solutions* to problems. Learning to solve problems is learning how to tackle problems to which you don't know the solution already. You haven't just forgotten the solution; rather, it is a problem that you haven't solved before.

For most students, this may seem too difficult a task, even impossible. How do you solve the problem if you don't know the solution, if you have never been taught the solution? Many people try a mathematics problem for about 5 or 10 minutes at the most. If they cannot solve it in this short time they decide that they are incapable of solving the problem. But that's not really true. Good problems may take a long time to solve – an hour or more, sometimes even days. Mathematicians often think about a problem for a very long time trying to find a solution. So many people, who give up after a short time and think that they cannot solve a maths problem, are simply making a wrong judgement about their own capability.

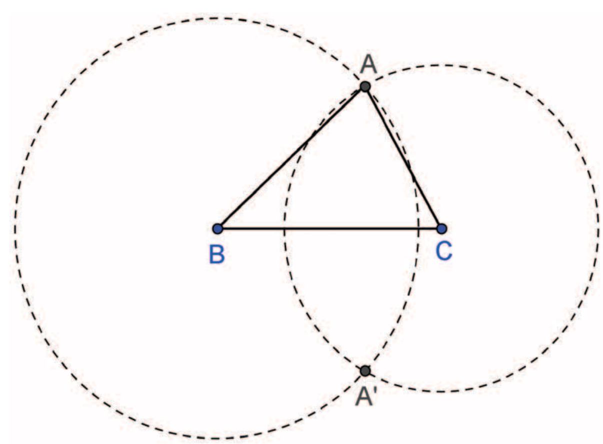
Why do some people – even some young students – keep at a problem for so long? Is it competition that drives them, refusing to be beaten by a problem, or wanting to prove a point? It may be that, partly. But often what keeps them going is the sheer pleasure that comes at the end of solving a problem and the joy at having discovered and learnt something worthwhile, purely by one's own effort. Often, when one struggles with a problem, not only does one learn the solution to the problem, but one makes other discoveries around the problem, and gets a glimpse into what doing mathematics is really like.

The master who wanted to bring the riches of problem solving to students, even those who have been turned away from mathematics, was the mathematician George Pólya. His most famous book is *How to solve it*, published in 1945 [1]. A number of problem solving books have been written over the years, but all of them trace their lineage back to this classic. In his book, Pólya gives a detailed account of how one could become a problem solver.

There are several elements which together make a person a good problem solver. As I said above, belief in one's own capability and the willingness to stick with a problem are important. Everybody knows, of course, that one must know some maths to be able to solve math problems. But just knowing the maths is not enough – many people know the maths, but they cannot apply it to find a solution. In his book, Pólya presented a stock of thumb rules useful when facing a new or unfamiliar situation, which he called *heuristics*. Heuristics are like approximate techniques: they suggest ways in which a problem can be solved. They don't guarantee a solution, but they are useful in pointing the way towards a possible solution. Generally all good problem solvers use heuristics consciously.

Pólya continued to write on problem solving, publishing other volumes. Twenty years later, Pólya published his last writings on the art of problem solving in two volumes titled *Mathematical Discovery* [2]. These volumes contain a more systematic presentation of heuristics than his earlier writings and an excellent collection of carefully chosen problems. They are a wonderful introduction to the world of problem solving for a high school student. Alan Schoenfeld, the Berkeley mathematician and maths educator, who used the book in his problem solving courses, has this to say: "... *Mathematical Discovery* is a classic.... It represents the capstone of Pólya's career..." [3].

Pólya begins the first volume of *Mathematical Discovery* with geometric construction problems. You are given some data about a geometric figure and you need to find a way of constructing the figure using straight edge and compasses. (You don't actually need to construct the figure, only find the procedure.) Pólya begins with one of the simplest of such problems: Given the three sides of a triangle ABC, construct the triangle. Nearly every high school student knows how to do this. Draw a line segment equal to the side BC. With B as centre and radius equal to AB, draw an arc, or better, draw a circle. Similarly, draw a circle with C as centre and radius equal to AC. The intersection of the two circles gives the point A. (A' is an alternative point; you get a congruent triangle if you use A'.)



Pólya shows how much there is to learn in this simple problem. The solution is an application of a heuristic he calls "the pattern of two loci", a heuristic that can be used for many other, more difficult, problems. "Locus" (plural "loci") means a path, a line on which a point with certain properties can lie. The steps in finding the pattern of the two loci are as follows:

1. Use some of the data in the problem to construct an initial geometric object. (This is the line segment BC in the problem above.)
2. Reduce the problem to finding a single point. (After drawing BC, once we find point A the problem is solved, all that remains is to join A to B and C, a mechanical task. But how do we find A?)
3. Of the data in the problem, use some and ignore the rest to obtain a locus for the missing point. (The circle with centre B and radius AB is the locus of all points that are at a distance AB from B. The point A lies somewhere on this circle. Note that we have ignored the information about the length AC.)
4. Use the remaining data to obtain another locus. (The circle with centre C and radius equal to AC is the locus of all points at a distance of AC from C. A is somewhere on this locus.)
5. The intersection of the two loci (in step 3 and step 4) is the required point.

If you think about it, this is an interesting way of looking at the problem. And the pattern generalizes to other problems. Before we look at more construction problems, here are some problems on finding the locus of a point given certain properties. Think of how you can construct the locus in each case with a straight edge and compass.

(Answers are given on page 66 of this issue)

1. A point moves so that it is always at a fixed distance *d* from a given point P. What is its locus?
2. A point moves so that it is at a fixed distance *d* from a given straight line *l*. What is its locus?
3. A moving point remains equally distant from two given points P and Q; what is its locus?
4. A moving point remains at equal distance from two given parallel straight lines *m* and *n*; what is its locus?
5. A moving point remains at equal distance from two given intersecting straight lines *l* and *m*; what is its locus?
6. Two vertices, A and B, of the triangle ABC are marked for you. Angle C, opposite to the side AB, is also given. The triangle is not determined, since the point C can vary. What is the locus of the point C?

A small sample of problems from Pólya's book is given below. These are from the exercise just after he discusses the pattern of the two loci. You can try them out. Take your time. Maybe one or more of the problems will need many hours to solve. Maybe you will need to return to them after a break, think about them the next day. But don't give up easily.

1. Construct a ΔABC given the length two sides BC = a and AC = b and the length of the median (mA) drawn from the vertex A to the side BC.
2. Construct a ΔABC given the length of BC, the length of the altitude (hA) from A to BC and the length of the median from A to BC (mA).
3. Construct a ΔABC given the length of BC, the length of the altitude (hA) from A to BC and angle A.
4. Two intersecting straight lines have been drawn on paper for you. Construct a circle with a given radius = r that touches the two given lines.

   (*See pages 44-48 for a discussion of this problem.)*
5. A straight line l is drawn on paper for you. A point P outside l is marked on the paper for you. Construct a circle with a given radius = r, such that the point P lies on its circumference and the line l is tangent to it. (Under what condition is it impossible to construct this circle?)

These are only a small selection of the many interesting problems that Pólya presents that can be solved using the pattern of the two loci. Notice that knowing the heuristic doesn't guarantee that you will find the solution. In fact, in each problem you will discover something interesting about the conditions of the problem. Perhaps you can come up with some new problems yourself.

Pólya describes other heuristics in the book and goes on to discuss more problems in geometry, algebra and combinatorics. In all, the book is a wonderful treat of problems both for a teacher and for a student, to be savoured slowly. Mulling over a problem even after one has found a solution is often a learning exercise: Are there any other solutions? What made it difficult (or easy) to find the solution? What happens if I vary some of the conditions in the problem? Are there similar problems that I can think of?

One of the problems related to the pattern of two loci has been a favourite in the problem solving sessions that I used to conduct some years ago for high school students. Again, it was Schoenfeld's writings which called my attention to this problem. This problem is surprisingly hard when students start on it. Often, we have spent the whole session discussing possible approaches to solving the problem, without solving the problem. Once the solution is found, it doesn't look so hard at all. So it is worth thinking about why it seemed hard in the first place. (Maybe thinking about this can also help in finding the solution.) Here is the problem:

- Construct a triangle ABC given the length of side BC, the measure of angle A, and the radius of the incircle of the triangle r.

An important rule in our problem solving sessions is that the teacher doesn't tell the solution even if he or she knows it. The teacher helps the students think aloud, notes down their ideas on the board for others to see and explore, and encourages students to think of alternative approaches. The session might end without the solution being found. But the students go back and think about it and often find the solution to the problem given above themselves.

Pólya devoted his life to describing many heuristics and making collections of problems that can be tackled using one or more of these heuristics. But there is no list of heuristics, memorizing which, will make one a problem solver. Heuristics are essentially approximate thumb rules, they help one to think more creatively, but don't guarantee solutions. As one gains experience in solving problems, one builds up a personal repertoire of what has been most useful in problem solving. Pólya only opened the doors, but one has to make the journey oneself.

(Solutions to some of the problems above will be given in the next issue. If you want to discuss solutions with the author, please send an email to subra@hbcse.tifr.res.in)

References

[1] Pólya, G. (1990). *How to solve it: A new aspect of mathematical method*. Penguin Books.

[2] Pólya, G. (1981). *Mathematical Discovery: on understanding, learning, and teaching problem solving*. (Combined paperback edition of both volumes), Wiley, New York.

[3] Schoenfeld, A. H. (1987). Pólya, problem solving, and education. *Mathematics magazine*, *60*(5), 283-291.

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