BRIDGING ARITHMEITC AND ALGEBRA: EVOLUTION OF A TEAHCING SEQUENCE[1]

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Abstract: *The aim of this paper is to describe the evolution of a teaching learning sequence for grade 6 students beginning algebra learning over a period of two years that included multiple trials. The teaching learning sequence was designed to enable the students to make a transition to algebra from arithmetic by connecting their prior knowledge of arithmetic and operations and exploiting the structure of arithmetic expressions. In the process, the study aimed to identify the concepts, rules and procedures which facilitate the connection between arithmetic and algebra and enable the transition. The repeated trials allowed us to see the potential of the two concepts 'term' and 'equality' identified during the study and the nature of tasks that help in making the connection between the two domains.*

INTRODUCTION

Researchers in algebra education have suggested a variety of approaches for introducing algebra. One set of approaches introduces symbolic algebra to students in the lower secondary grades through generalized arithmetic, emphasizing structure of arithmetic expressions and replacing the number by the letter to represent generalized rules and properties of operations in arithmetic (e.g. Thompson and Thompson, 1987, Liebenberg et al., 1999, Malara et al., 1999, Livneh and Linchevski, 2003). Much of this research has focused on building a sense of the structure of arithmetic and algebraic expressions among students. Earlier exploratory students (e.g. Chaiklin and Lesgold, 1984; Linchevski and Livneh, 1999) had shown that the lack of the understanding of structure was a major factor in not understanding the manipulation of algebraic expressions. Although the teaching studies just mentioned identified important elements of a beginning algebra curriculum, they have not yielded a well elaborated model of teaching and learning of algebra using arithmetic as the base. Some of these studies suggested the need to focus away from computation to be an important criterion for transiting to algebra from arithmetic (e.g. Liebenberg et al. 1999, Malara et al. 1999). Elsewhere, we have reported aspects of a teaching approach that aimed to develop a structural understanding of arithmetic expressions (Subramaniam and Banerjee, 2004, Banerjee and Subramaniam, 2005). In this paper, we describe the evolution of the teaching approach as part of a design experiment, highlighting the changes and the decisions made and the reasons for these decisions.

THE RESEARCH STUDY

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In a two year long study involving a design experiment methodology (Cobb et al., 2003), we developed a teaching approach to learning algebra using students' prior knowledge of arithmetic and operations. The approach aimed to build a strong structure and procedure sense of arithmetic and algebraic expressions by giving visual and conceptual support. In the process, we wanted to identify the nature of concepts, rules and procedures which would facilitate building the connection between the two domains. The study started with only a conjecture about the possibility of using the structure of arithmetic for teaching algebra and the many assumptions had to be progressively tested in order to build the sequence. The teaching learning sequence co-evolved with the developing understanding of the researchers about the phenomena under study as well as with the growing understanding of the students as evidenced from their performance and reasoning on various tasks. After each trial, the strengths and limitations of the concepts, ideas and tasks were identified leading to suitable modification of the sequence in the next trial of teaching.

The study was conducted with grade 6 students from nearby English and vernacular medium schools during vacation periods in summer and mid-year. Each trial had two to three student groups, with each group receiving 11-15 days of teaching, 1.5 hours per day. The teaching sequence, which included concepts and task that went well beyond those introduced in the school, was developed over five trials between 2003-2005 with the first two trials being exploratory in nature and considered pilot trials (PST-I and PST-II) and the last three trials forming the main study (MST-I, MST-II, MST-III). Different groups of students attended the pilot trials whereas the same students who attended MST-I were invited for MST-II and III. The data was collected through students' performance in pre and post tests, interviews, teachers' daily logs and video recordings of classroom discussions.

THE TEACHING CYCLE

The evolution of the teaching approach was similar to the 'mathematics teaching cycle' and the '*hypothetical learning trajectory*' described by Simon (1995). The approach was developed keeping in mind the insights from the literature using arithmetic as a 'template' to build the new algebraic symbolism. The main focus of the sequence was to move the students away from a sequential, procedural understanding of expressions to a relational, structural understanding, which is important for algebra. Besides learning to parse expressions correctly, developing understanding of structure of expressions requires students to turn the processes of computation into 'objects' (Sfard, 1991) or flexible 'procepts' (Tall et al., 2000). This would allow them to think mentally about operations, suspend computations, anticipate the outcome of actions and attend to the relations within components of the expressions as well as between two expressions. The sequence tried to achieve this gradually by creating appropriate learning tasks, and by identifying concepts, rules and procedures, together with visual and verbal support which could consolidate the reification of the processes of arithmetic.

A teaching sequence was constructed for the first trial which aimed at identifying instructional material as well as testing their efficacy, sequencing and identifying pre-requisite concepts or skills needed for developing structure sense among students. Tasks were chosen, adapted and modified from the existing literature for the trials. Students' intuitive as well as formal ideas about operations, symbols and procedures were given due importance in the classroom, allowing the students to articulate their reasoning, so as to be able to build on them. During the enactment of the teaching sequence in the classroom, the students were engaged in making sense of the tasks and the responses expected (e.g. that they have to explain their solution, that they have to understand the explanations given by others) and the teachers were engaged in observing and making sense of the students' responses and actions. This not only led to changes in the subsequent trials but also small immediate changes, with regard to examples and explanations in the same trial.

In the following paragraphs, we give an account of the processes that led to the evolution of the teaching approach and the rationale for emphasizing certain concepts/ ideas and choosing and changing some of the tasks.

THE PILOT TRIALS

The first two trials (PST-I and PST-II) explored how students' knowledge of arithmetic could be harnessed as a preparation for symbolic algebra. We began the trials with the understanding that procedure and structure sense are two separate pieces of knowledge and building the structure sense is enough to make the transition to algebra. But as we tried out the instructional sequence in the first trial, we found that building of the structure sense itself required adequate procedural understanding. This led us to include tasks which strengthened students' procedural knowledge, like working with brackets and later integer operations as well.

One of the goals of the first trial was to move the students away from a computational understanding of expressions towards a relational understanding. This was the first step towards attending to the structure of the expressions and appreciating the duality: that the expression stands for a number which is the value of the expression and that all the expressions for a number 'express' different information about the number, in the form of a relationship among two or more numbers. For example, students learnt that the expression $5 + 8$ stands for the number 13 and conveys the information that it is '8 more than 5'. Many other phrases like 'more than', 'sum', 'difference between', 'less than', 'product of', 'times' and 'quotient' were introduced. Rules of evaluating simple expressions, like $13 – 5 + 8$ and $6 + 2 \times 4$, were explained to them in the traditional fashion by explicating the precedence rules (giving precedence to '×' operation and computing from left to right) and strengthened using the meaning of the expressions. For example, $9 – 3 + 4$ is four more than the difference between nine and three whereas $9 – (3 + 4)$ is difference between nine and the sum of three and four, suggesting the difference in the way the computation is be carried out.

Another goal of the first trial was to develop among students an ability to judge equality of expressions without computation. However first, their understanding of the '=' sign needed to be broadened. They then compared expressions which could be seen as related such as 27+32 and 29+30. Although students were able to view expressions relationally, we saw overgeneralizations from the addition context to expressions with a negative sign. Such situations required students to keep track of the transformations on the number for which they did not have the resources, like the use of brackets. For expressions with brackets, simple bracket opening rules were introduced with the use of phrases like 'adding/ subtracting a sum or difference'. For example, 12 – (6 + 4) and 12 – 6 – 4 are equal and one can subtract the sum of 6 and 4 from 12 or subtract them one by one as in the expression 12 – 6 – 4. Other tasks included finding the value of an expression given the value of a related expression (find 228+149 if 227+148=375). The students were expected to explain their answers verbally. Attempts by the teacher to help students with symbolic justifications were not very successful. As students worked on these tasks, the concept of 'term' was introduced as a component of an expression (e.g. in 12 + 4 – 3 the terms are +12, +4, –3), and the students soon learnt by verification that the value of an expression remains the same on rearranging the terms. This concept not only helped the students parse an expression correctly but also allowed them to see relationships between the terms and with the expression as a whole, leading to the important idea of 'equal expressions'. Thus, 'terms' and 'equality' were the two key concepts identified during the first trial.

The second trial sought to build this sequence by extending it to include algebraic expressions. It had a two group design: students who were taught algebra together with the approach to arithmetic expressions as outlined above and a group who were taught algebra without any arithmetic beyond the instruction in school. The first group of students who worked on both arithmetic and algebra were taught the concept of term immediately after dealing with the procedures of evaluating arithmetic expressions. Terms were categorized into 'simple term' (e.g. +3, –4) and 'product term' (e.g. +3×4, +2×*y*). But the use of terms was restricted to tasks of comparison of expressions. In contrast to the group which had been exposed to only algebra, this group of students performed better in both procedures of evaluating expressions and using the surface structure of the expressions to identify and generate equal expressions, where terms and numbers were rearranged, in both arithmetic and algebra. However, the appreciation of surface structure did not allow abstraction of procedures to manipulate algebraic expressions, which needed a deeper understanding of rules and properties of operations. On retrospect, we realized that the procedures used with arithmetic expressions for evaluation and with algebraic expressions for simplification (by collecting like terms) were disparate, not allowing for transfer between the two, many students making the conjoining error ($3+5\times x=8\times x$) due to non-appreciation of the constraints on operation. Also, students were introduced to bracket opening rules by embedding them in story situations which could lead to two ways of representing and solving. This proved to be quite cumbersome and did not succeed in explicating the structure of the expressions.

At the end of these two trials, it was evident that although strengthening the understanding of arithmetic was helpful in making sense of algebra and rules of transformation of algebraic symbols, there was a need to make the sequence more coherent and bridge the gaps between procedure and structure and between arithmetic and algebra, so that the understanding developed in the context of arithmetic could be fruitfully used in the context of algebra (see Subramaniam and Banerjee, 2004). There was also a need to pay attention to negative numbers and bracket opening rules.

MAIN STUDY TRIALS

The three main study trials (MST-I, II and III) were carried out with two fresh groups of students. The students came soon after appearing for their grade 5 exams for MST-I (Summer, 2004), were in the middle of grade 6 during MST-II (mid-year vacations, 2004) and finished grade 6 during MST-III (Summer, 2005). These trials were aimed at achieving better coherence in the teaching learning sequence. In all the three trials, the concept of 'term' was introduced in the beginning and was used for both procedural and structural tasks in an increasingly integrated manner.

Students were introduced to the idea of 'terms' of an expression immediately after developing an understanding of expressions in MST-I. Terms were made visually salient by putting them in boxes (e.g. terms of 19 – 7 + 4 are [+19] [–7] [+4]) and were used to decide the precedence rule to be applied for evaluating arithmetic expressions and to identify like terms in the context of algebraic expressions. They were subsequently used to compare expressions, identify and generate equal expressions as earlier. Students again failed to make the connection between the simplification procedures of arithmetic and algebraic expressions due to the persisting disparity as in PST-II. Some efforts to make the connection explicit included evaluating algebraic expressions for given value of the letter (e.g. $5+4\times x$, $x=2$) and finding easy ways of evaluating expressions like 28-17-8+17, emphasizing non-sequential computation. These efforts were not entirely successful partly due to the rigidity of the rules of evaluation. Rules for transformation of expressions with brackets (+ and '–' to the left of the bracket) were connected to the idea of equal expressions verified through computations. Area of a rectangle model was used for distributive property. Number line and letter-number line (were used to give meaning to the integers and the letter. The letter-number line served the dual purpose of understanding expressions like x-1 as denoting a number by means of a relation (a number which is one less than x) and the process of decrementing 'x'. It could further be used in tasks like the journey on the letter-number line and finding the distance between two points on the letter-number line, both of which required the students to create a representation and manipulate it.

As we have noted, a strong connection between the procedural and the structural components of the expressions was still lacking in MST-I. The students also could not use their knowledge of rules of transformation in contexts where algebra was being used as a tool for justification (like, think-of-a-number game). It was clear that simply the presence of structural notions and explicating the surface structure is not sufficient to make the connection between procedure and structure and between arithmetic and algebra. The structural notions had to be used differently in such a manner that one could reflect on possibilities and constraints on operations, enhancing flexibility and anticipation with respect to results of carrying out operations.

In the second main study trial (MST-II), terms and equality were made more central to the teaching sequence and the approach was made radically structural. Terms were now classified as simple and complex terms (e.g. product term, bracket term). The procedures for combining terms for evaluating expressions were introduced as a structural reinterpretation of the precedence rules. The rules of evaluation were made flexible by including the idea of combining terms in any order, thus subsuming integer addition operations. Positive terms increased the value of the expression, while negative terms decreased the value. A product term needed to be converted into a simple term before combining with other simple terms. Two product terms with a common factor could be combined using the distributive property. This paved the way for integrating the transformation rules of arithmetic and algebraic expressions (where this flexibility and non-sequential computation is essential) as well as complement procedure sense with structure sense. Figure 1 illustrates the flexible ways in which students evaluated expressions as they learnt this approach. The complementary nature of procedure and structure was strengthened by the tasks of finding easy ways of evaluating expressions and generating expressions equal to a given expression (both arithmetic and algebra) using various transformations, requiring abilities to mentally operate in forward and reverse direction. Even the bracket opening rules were reformulated using ‘terms’ and ‘equality’ in conjunction with ideas of ‘inverse’ (taking care of the integer subtraction) and ‘multiple’. This evolved sequence was called the ‘terms approach’ and gave the students the vocabulary and visual and conceptual support to reason about the syntactic based transformations. The two structural concepts of ‘term’ and ‘equality’ and the reformulation of the rules of transformation enabled the students to consider the arithmetic processes as potential processes which could be suspended for a while and combined with other terms based on structural relations. Further, generating equal expressions separated the denotation from the meaning of the expressions, the transformations keeping the value same but changing the surface structure of the expressions.

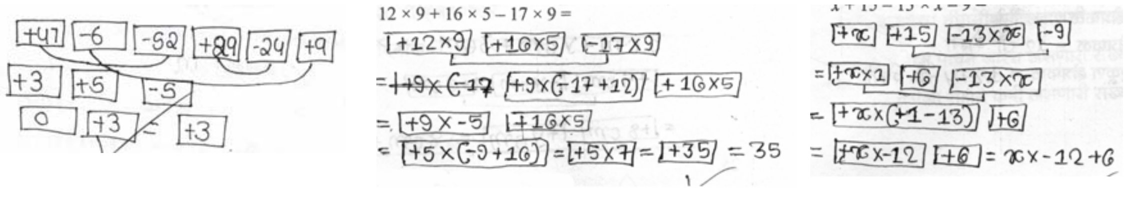

Figure 1: Students' solutions in evaluating/ simplifying
arithmetic and algebraic expressions

The last trial (MST-III) aimed to consolidate the teaching learning sequence focusing on students' verbalization and articulation of various concepts and rules and their use in different contexts. Evaluation of expressions with brackets (e.g. 23-(4+5×3)) together with understanding general principles of keeping the value of an expression invariant were the focus of this trial. Also a fair amount of time was spent on tasks which embedded the use of algebra in contexts requiring generalizing and proving/ justifying (Think-of-a-number game, pattern generalization from growing patterns). Building on our earlier observations of students' inability to use their knowledge of syntactic transformations in such contexts, students were engaged in verbalization of explanations of the answers before introducing symbolic justifications. These activities led to fruitful discussions about semantic and syntactic aspects of algebra: meaning of letters, correct representation and proper use of brackets and generalization from particular instances ('seeing the general in the particular') and goal directed manipulation of expressions. The study ended after this trial with indications that the transfer to 'reasoning *with* expressions' in context is not trivial but 'reasoning *about* expressions' in the course of working with syntax based transformations can play a part in predisposing students to think about situations with the help of expressions.

Interviews with a subset of students after MST-II (14) and III (17) revealed students' ability to appropriately articulate the reason for the incorrectness of the solution of an expression like 22-7+9 = 22-16 by pointing out the need for a bracket around 7 and 9 for the above solution to be correct or that -7+9=+2. Probing specific abilities of students with respect to simplification of algebraic expressions (e.g. 5×*a*+6-2×*a*+9) at the end of MST-III, almost all students were able to convincingly explain the procedure of simplification by drawing on their knowledge of evaluating arithmetic expressions. They stated the rules for combining terms for inability to simplify further expressions like 3+5×*x*. Also, eleven of the students understood that each step in the simplification process yields equivalent expressions. The remaining six students needed to calculate the simplified and the original expression to arrive at the above conclusion, generalizing their understanding from evaluating arithmetic expressions with similar structure.

CONCLUSIONS

The design experiment led to the development of a teaching learning sequence with the potential to bridge the gap between arithmetic and symbolic algebra for students beginning algebra learning. Through a long term engagement with the process and our own reflections on the assumptions and the tasks, the study helped us understand the nature of arithmetic and the tasks required to make the transition possible. The transition is not a trivial affair and the connection is not spontaneously seen by the students. Using arithmetic as a template, and enhancing both computational as well as non-computational reflective understanding of operations and their properties by the use of two structural concepts ‘term’ and ‘equality’ enabled the students to develop a new symbolic system of algebra and simple operations on them. The ‘radicalized’ structural treatment created meaning for the symbols in the context of syntactic transformation and allowed us to convert the processes of addition, subtraction, multiplication into ‘objects’ which could feed into the development of the algebraic symbols.

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