INTEGRATING THE MEASURE AND QUOTIENT INTERPRETATION OF FRACTIONS[1]

Shweta Naik   K. Subramaniam

Homi Bhabha Centre For Science Education, TIFR, Mumbai, India

Abstract: The study reported here is part of a longer study on developing students' fraction concepts as a preparation for solving ratio proportion problems. Here we investigate the impact of instruction aimed at developing the measure and quotient interpretation of fractions in an integrated manner. Grade 5 students, who participated in the study, had been previously exposed to the part-whole interpretation in school. The inadequacy of an exclusive emphasis on the part-whole interpretation, and the effectiveness of supplementary instruction as described are discussed. Student responses on pre- mid- and post tests, interviews and classroom discussion are analysed. The responses show that students' performance on representation and comparison tasks improved significantly after the instruction.

INTRODUCTION

A number of educators have stated that the learning of fractions is probably one of the most serious obstacles to the mathematical maturation of children (Behr et al., 1992; Kieren, 1988; Streefland, 1994). Many students have little conceptual understanding of fractions and are dependent on procedures that are learnt by rote, which are often incorrect. The fraction concept is also complex since it consists of multiple subconstructs, which are not adequately developed in traditional school curricula (Kieren, 1988. Charalambous and Pitta-Pantazi, 2007). Some researchers have located the source of many of these problems in the almost exclusive focus on the part-whole subconstruct of fraction in traditional instruction (Streefland, 1994).

In this article, we discuss the responses of grade 5 students (11 year olds), who have been exposed to traditional fraction curriculum based on the part-whole subconstruct in school, to supplementary instruction emphasizing the measure and quotient subconstructs. The instruction was a part of a teaching design experiment aimed at developing an understanding of fraction to prepare students to solve ratio and proportion problems. In this report, we restrict discussion to students' understanding of the magnitude of fractions as inferred from the tasks of pictorial representation and comparison of fractions.

BACKGROUND

The sub-construct theory of fraction understanding analyses the fraction concept as comprising the five sub-constructs of part-whole, measure, quotient, ratio and operator

(Kieren, 1988; Behr et al. 1992). It has been suggested that instruction must aim to develop an adequately integrated understanding of the multiple subconstructs (Post et al., 1993). Indian textbooks emphasize only the part-whole subconstruct in introductory fraction activity through the use of the area model. This model is used to introduce the fraction symbol and vocabulary, typically by comparing the number of shaded parts to the total number of parts. The approach then rapidly shifts to procedures for operating with fractions, drawing from time to time on the area model to explain the procedures.

There are several problems with this manner of introducing the part-whole interpretation. Although teachers often stress that the parts into which the whole is divided must be equal, students do not absorb this fact very well (as indicated also by student responses in our study, discussed below). An explanation of this may be found in the fact that the concept of unit is not highlighted. In situations where counting is used to arrive at the numerosity of a discrete set, the size (or even the nature) of each unit may be ignored, for example, when counting the number of people standing in a queue to enter a bus in order to find out if there are enough vacant seats. Thus children who operate on the basis of whole number thinking would only count the number of shaded parts, and the total number of parts, ignoring the size of the parts. In contrast, in measurement contexts the size of the unit is critically important. When a part of a whole is taken, we divide the whole into equal parts in order to obtain a subunit. This unit is then iterated to measure out the part that is taken. Traditional teaching typically emphasizes the counting aspect while missing out the measurement aspect. While the partitioning scheme is drawn upon and reinforced, the fact that partitioning involves formation of unit structures is underplayed (Lamon 1996).

In our approach, we brought the measurement aspect centrally into focus by emphasizing the notion of unit fraction. Although, many studies on various constructs of fractions have been reported, few have focused on exploiting the power of unit fractions in the construction of fraction understanding. Some approaches such as Lamon's (2002), which have emphasized the unit concept, and the strengthening of the unitizing schema within the part-whole interpretation, are indeed powerful. In Lamon's approach, the designated part is compared with the whole as a ratio by taking different, arbitrary units in a flexible manner. Thus 3/5 may be symbolized as both 6/10 and (1½)/(2½). A sense of the magnitude of the designated part is based on grasping the ratio of the part to the whole, while the unit recedes to the background. In contrast, in our approach, the role of the unit fraction is more central, and the magnitude of a part or a quantity is grasped in terms of the named unit fraction, which establishes the relation of the new subunit to the base unit. Since unit fractions are given prominence as objects in their own right, their properties become worthy of study. An important property that is easy for students to construct is that the unit fractions form a regular sequence ordered in terms of decreasing magnitude.

The milieu in which students learn in Indian schools is frequently multilingual. For the vast majority of students who study in the English language, English is not a language of comfort. So the noun forms that signify unit fractions such as 'fourths', 'fifths' are easily missed; indeed, they are often avoided and replaced by fraction names such as '3 out of 5' or '3 over 5'. Indian languages have fraction names similar to '3 over 5'. Hence the language support for calling attention to unit fractions is very weak, justifying the need for special emphasis on them to be built into instruction.

THE STUDY

The study reported here forms part of a larger ongoing study that uses a design experiment methodology (Cobb et al., 2003) and is aimed at developing an approach to teaching fractions as a preparation for understanding ratio and proportion. The study was conducted with grade 5 students from nearby English and vernacular medium schools during the summer vacation period between grade 5 and 6. An earlier cycle focusing on developing the operator interpretation of fractions as a preparation for understanding ratio, revealed the inadequate understanding of basic fraction concepts of students exposed to the traditional school curriculum. Accordingly, a unit emphasizing the measure and quotient interpretations of fractions in an integrated manner was developed and implemented in a vacation programme for students from schools in the summer of 2007. Students from two schools, one studying in the English language (N=41, Avg. age: 10.5 y) and one in the Marathi language (N=30, Avg. age: 11 y) volunteered for the programme. The English and Marathi groups received 16 and 14 days of instruction respectively, of approximately 1.5 hours per day. Each group was taught by a separate teacher (one of the authors). All the lessons were video recorded. Six students from the English group and five students from the Marathi group, whom the teachers judged to have a weak understanding of fractions, were chosen to be interviewed. The purpose of these interviews was to probe the nature of students' difficulties.

In the curriculum adopted in both the schools, fractions are introduced in grade 3. The whole gamut of fraction concepts and tasks: like and unlike fractions, equivalent fractions, improper fractions, mixed numbers and comparison of fractions is introduced in grade 4. All the four operations with fractions are introduced in grade 5. Thus students had received instruction on fractions over the previous three years before participating in our programme. Nearly all the conceptual work is done with the part-whole interpretation of fractions. Fraction as division is cursorily introduced without any explanation in grade 5.

The teaching unit in the programme consisted of two segments: the first dealing with interpretations of fractions in terms of the measure and quotient subconstructs, and the second dealing with equivalent fractions and the operator subconstruct. Representation and comparison tasks were the primary tasks used in the first segment. We include for

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analysis in this paper only the first segment of the programme that formed 9 days of instruction for the English language group and 6 days for the Marathi language group. Students were taught to write fractions in measure and share situations. All the other competencies such as representation and comparison of fractions were developed through reasoning about fractions. The main focus of the teaching was to move the students away from a sequential, procedural understanding of fractions to a conceptual and meaningful understanding of fractions.

Representing Fractions

The students were encouraged to represent fractions in two ways. The measure interpretation was developed through the notion of the unit fraction 1/n. We defined unit fraction as *when a whole is partitioned in to equal number of parts and then each such part represents a unit fraction which is 1/(number of equal parts).* This definition was further simplified by the students as *whenever one cake (or unit) is shared among a number of students equally the share of each child is a unit fraction.* The relation of the unit fraction to the basic unit as well as the relative sizes of unit fractions were highlighted, while also preserving the connection with the share interpretation. After practicing the representations for 1/7, 1/5 students represented the non-unit or composite fractions such as 2/5, 3/7, etc. Composite fractions were needed to specify quantities that could not be measured by whole units, and were introduced as built up of unit fractions.

The quotient interpretation was introduced as equal shares (Streefland, 1994). Students found the share of each person when m cakes were shared among n persons, and represented it as m/n of a cake. In the process, they linked this with the use of the traditional symbol for the division operation, as well as with the measure interpretation of *m/n* as *m '1/n'* units.

In a variety of situations such as sharing cakes equally, students solved the problem by actually drawing a cake and stick figures for students, followed by the act of equal partitioning. The video records show that students were usually careful about making exactly equal parts in trying to make fair shares. The measure interpretation was helpful in answering the question - *find how much of a cake is obtained by each child.*

The illustration in Figure 1 exemplifies the instructional approach adopted. It shows two representations for the fraction 2/7: '*share of each child*' on the left and as '*built up from unit fractions*' on the right. As the act of sharing invokes the idea of division in whole numbers, it is consistently extended in this situation which results in the division fact $2 \div 7 = 2/7$. (This understanding of students is used in the later part of the study dealing with the operator concept, but is not the focus of this paper.) The presentation on the right side, indicates more clearly the quantity of cake that each child has received: it shows formation of the unit 1/7 and construction of 2/7 as 1/7 + 1/7. This is

the portion of cake that each child has received. The discussion in the classroom consistently emphasized an integration of the measure and the quotient sub-constructs.
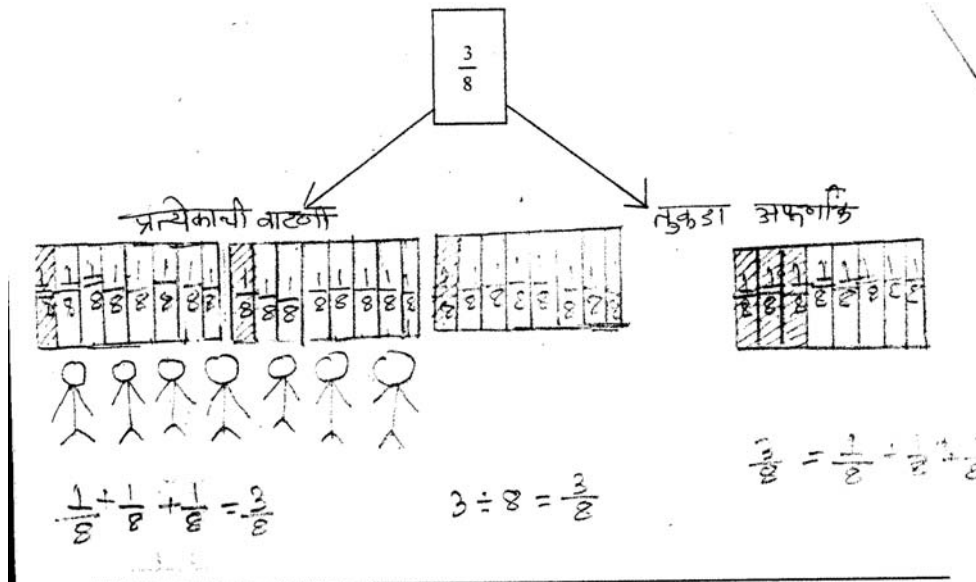

Fig.1: Representing fractions in two ways

RESULTS AND DISCUSSION

The table below presents the data on student performance with respect to two kinds of tasks: representation of fractions and comparison of fractions. The data is from three tests: pre-test conducted at the beginning of the camp, the mid-test conducted after the first segment on representation and comparison of fractions using the measure and quotient sub-constructs was completed, and the post-test taken after the entire programme was completed.

Task	*Percentage of correct responses*		
	Pre-test	*Mid-test*	*Post-test*
Writing a fraction for a shaded part (marked parts are unequal, need to be remarked)	8.7	31.5	70.1
Writing a fraction for a shaded part (more than a whole)	9.9	56.3	59.4
Pictorial representation of improper fraction	18	59.8	62
Pictorial representation of mixed number	24.3	73	66.3
Comparison of fractions (all items)	37.3	81.3	83.9
Comparison of unit fractions	21.2	84.5	83.2

The items in the test were items that students typically find difficult. In the area representation, unequal parts were marked and students needed to remark the parts. The representation tasks contained improper and mixed fractions, and the comparison tasks included tricky comparisons. Students showed considerable improvement over their pre-test performance on all these items. It was observed that a few students had used a sharing picture to represent the fraction either together with or without the measure picture. Another minor observation is that the performance marginally improved when students were asked to show the composition of a fraction in terms of unit fractions before drawing a picture representing the fraction.

Classroom discussion on comparing fractions

The comparison of quantities is based on an understanding of the magnitude of a quantity. Hence understanding how to compare quantities and understand how much a quantity is, are not two disjoint cognitive abilities. Students' performance in the pre-test comparison tasks showed evidence of whole number thinking, rather than an understanding of fraction magnitudes. For example, students judged that $1/2 < 1/3$ because 2 is smaller than 3. However they soon replaced their representation of fractions with the new interpretations of measuring and sharing that became available to them. Students were strongly encouraged to justify and give reasons for their answers. The classroom video recordings showed the variety of strategies that students used to solve these tasks. As we see below, the sharing and measure interpretations were both drawn upon.

Fractions with the same denominator:

As the number of children to share the cakes are same, children from the group with more number of cakes get bigger share.

As the unit piece is same in both the fractions more number of pieces represents the bigger fraction.

Fractions with the same numerator:

As the number of cakes to share are same, the group where more number of children are there will have a smaller share.

As the number of pieces are same what matters is the size of the unit.

Comparison with half:

The number of cakes are exactly half of the number of children to share.

The number of pieces picked are exactly half of the total number of pieces in the whole

Apart from these kinds of comparison, students reasoned about questions such as - which is the smallest unit fraction? Which is the biggest unit fraction?

In some of the open ended tasks students showed their ability to reason about fractions. Students understood every fraction as constructed from its unit. This constructive nature of unit fractions allowed students to explain how a improper fraction is constructed. They understood 5/4 as : 5/4 = 1/4 + 1/4 + 1/4 + 1/4 + 1/4 = 4/4 + 1/4 = 1 + 1/4 = 1 1/4. As the pre-test data reveal representing improper fractions was especially hard for the students. Some students were also comfortable in the idea of sharing and interpreted improper fraction as follows: '*as number of cakes are more than the number of children to share. Obviously each child will get at least one cake.*'

In one of the discussion tasks in the classroom, several fractions were written on the board and students were asked to compare them and give reasons for their responses. Many of the students first wrote all the fractions in the form of their unit fractions, and then made statements about the comparison of those fractions. While comparing the two fractions 4/5 and 6/7, students found the similarity that both of these need one piece to complete the whole. When asked which one of these is closer to one whole, many students could say that it is 6/7. When asked how, students came up with following reason -

Even though both the fractions 4/5 and 6/7 need one more piece to complete a whole. 4/5 needs one piece of 1/5 and 6/7 needs one piece of 1/7. but 1/5 is more than 1/7 as one cake is shared among 5 children only hence 4/5 is away from the whole.

Towards the end of the programme individual interviews were conducted of six students from the English language group and four students from the Marathi language group. The students interviewed were judged to have a weak understanding of fractions on the basis of their classroom work and the interviews were aimed at understanding the nature of their difficulty. Students were asked to represent the fraction 14/9 by drawing a picture. After they made a drawing, an alternative drawing was shown to them and they were asked if it was correct. For students who drew the correct representation, a representation of 9/14 was shown, and for those who drew a wrong picture, a picture with one whole and a 5/9 shaded was shown. Five of the ten students were confident about the correct response that they made initially. All of them used a measure picture, except one student who began with a sharing picture and changed to a measure picture. Of the remaining students, one could not complete the task. The remaining had difficulty in completing the task and changed their response midway, but eventually managed to do so. Two students who worked with the sharing picture had difficulty representing the share of each child, and both moved eventually towards a measure picture. Two students completed the task by rewriting the improper fraction as a mixed number, but were unsure about the representation of 14/9.

For the second task of representing and comparing the three fractions: 1/5, 5/5 and 5/1, all except one student completed the comparison task successfully. Eight of nine students could also represent the fraction either by a picture or by description. Seven

students used the share interpretation to justify their response. Two students, who used the measure interpretation, expressed themselves clearly and confidently. One of the students who reasoned on the basis of both the share and the measure interpretation was sure that 5/1 is more than 5/5, but was hesitant about drawing a picture or describing precisely how much each portion was.

From the interviews, it appears that students readily draw on both the interpretations, especially on the sharing interpretation for comparison. The share meaning in some cases did not lead to a clear picture of how much the fraction exactly was. In the case of the fraction 14/9, it was difficult to draw a picture showing the sharing completely and students either hesitated to do so or withdrew after trying. We interpret this as suggesting that both the interpretations are useful and mutually reinforce each other. The results taken as a whole indicate that instruction emphasizing the measure and share meaning can positively contribute to students' understanding of fractions and can supplement part-whole understanding, which by itself is inadequate.

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