EMERGING IDEAS OF GENERALIZATION, PROOF AND PROVING AMONG GRADE 6 STUDENTS

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This paper reports a small part of a design experiment which aimed at enabling Grade 6 students to make the transition from arithmetic to algebra. The focus on the intervention was to develop students' understanding of symbols and rules of operating on the symbols in the context of algebra by building a deep understanding of arithmetic. Further, it aimed at developing an appreciation of the use of algebra in rich contexts. In this paper, we discuss two of these contexts which are useful for developing two important ideas of mathematics, that of generalization and proof. We discuss Grade 6 students' first attempts to work on these tasks and grapple with these key ideas of mathematics. Though we feel that these students had sufficiently robust understanding of symbols and syntactic aspects of algebra, it was not sufficient to fully understand the demands of the two contexts.

INTRODUCTION

The recent years have witnessed a significant rise in interest in students'capacities to generalize, reason and argue mathematically, in other words to think mathematically. This has led to exploratory studies about students' abilities to generalise, reason, argue and prove as well as develop such capacities among students through teaching interventions. The emphasis on generalization and reasoning, recognized as among the most important processes in learning mathematics, also distinguishes the recent surge in process-oriented classrooms from the traditional product-oriented classrooms. Though both generalization and reasoning have assumed an important role in reform-oriented classrooms across the world, and when given appropriate opportunities students across elementary and secondary grades also are capable of engaging in these processes, the transition to formal proof and proving – the hallmark of mathematical knowledge, largely expected in the secondary school, is still to be understood. Moreover, the addition of these dimensions in the classroom has changed the nature and culture of classrooms, focusing on student participation and production and validation/ justification of mathematical ideas in interactions between participants of the classroom.

Algebra is a domain where both generalization and justifying/ proving play an important role. These provide the contexts which lend meaning to the symbols that get introduced in algebra and also have been found useful for promoting algebraic thinking. Algebra provides the language to succinctly describe the generalization in a pattern and also to justify specific number patterns (for e.g. among three consecutive numbers). Given the vast number of studies which document students' inability to understand symbols in algebra and to meaningfully use and manipulate them (see Banerjee, 2011 for a review), developing an understanding of algebra in rich contexts (like the ones mentioned above) seems to be a reasonable option (e.g. Rojano & Sutherland, 1991; van Reeuwijk & Wijers, 1997). Research in the past shows that these situations could be challenging (e.g. Stacey, 1989; Reid & Knipping, 2010) but creating appropriate classroom cultures which engage children in communication using natural language, signs, gestures, symbols could help them generalize and reason with algebraic symbols (e.g. Radford, 2003; Warren & Cooper, 2008; Rivera, 2010). In this paper, our attempt would be to illustrate Grade 6 students' emerging understanding of generalization and proof/ proving that arose in some particular contexts as they participated in a year-long intervention on beginning algebra. We would demonstrate how Grade 6 students tried to arrive at generalized rules for a pattern and highlight the contributions of various students towards the outcome together with the skills and knowledge that they brought to the situation. We would also highlight the difficulties which students face as they tried to make a transition from reasoning to proving with algebra.

METHODOLOGY AND FRAMEWORK OF THE STUDY

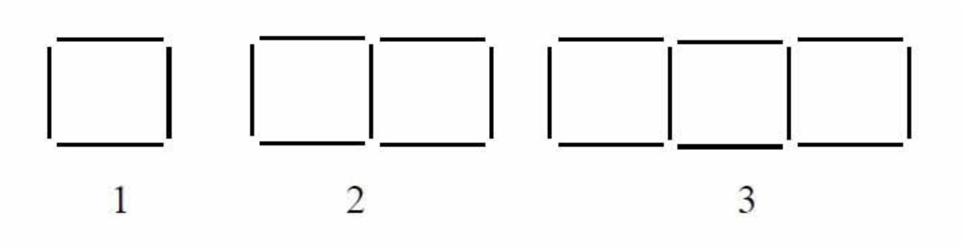
The aspect of the study that is being presented in this paper is part of a larger design experiment which consisted of five trials with Grade 6 students over two years, the first two being pilot trials. The later three trials formed the main study. The students came from two schools (one English medium and one vernacular medium), belonging to low and medium socio-economic backgrounds. Students were randomly selected from a list of volunteers after their Grade 5 final examination for the first trial in summer but they were subsequently invited to attend the latter two trials. The trials were held during the vacation periods of the school, during summer and Diwali. They were taught in the same medium of instruction as of the school by the research team members. None of the activities/ tasks that would be discussed in this paper were familiar to them.

The teaching intervention in all the trials included two kinds of activities - '*reasoning about expressions*' (which included discussions on possibilities and constraints on operations in the contexts of evaluating/ simplifying expressions, the meaning of symbols and comparing and judging equality/ equivalence of expressions) and '*reasoning with expressions*' (dealing with activities like representing relationships, pattern generalization, justification and proof). It aimed to bridge the gap between arithmetic and algebra by exploiting the structure of arithmetic and give meaning to beginning symbolic algebra. It was ensured that reasoning formed the core of the classroom culture, whether the tasks were computational tasks or non-computational. It was also hoped that once students understood algebraic symbols and transformations to signify some meaning and having some purpose, students would be able to use them in contexts and display meaningful actions on the symbolic representations they would create for the situations. Such situations were created in all the trials of the main study.

In this paper, we would focus on two tasks which were used in the third and final trial. We introduced the pattern generalization task (growing patterns of lines or dots or squares) and a game called the 'think-of-a-number' game (e.g. Think of a number. Multiply it by 2. Add 4. Subtract the original number. Subtract 2. Subtract the original number. What is the number you get? Do all of you get the same answer? Can you explain why this is happening?). These required students to represent the situation, generalize it and also provide some justification or proof. Even though these tasks were situated in the numerical world, they are characterized by the scope of generalization to a range of similar situations. We would try to understand the emergence of the crucial ideas of generalization and proof as they worked on these problems. We would also attempt to explore what advantages these students had while working on these tasks given their understanding of expressions revealed in the context of '*reasoning about expressions*' (Banerjee & Subramaniam, 2012) and what more is required in order to gain sufficient understanding of these situations.

Such problems/ tasks were introduced by the teacher as a whole group discussion/ activity followed by further working on the same or other similar tasks in pairs or individually. The pattern generalization task involved creating a generalized rule for continuing the pattern and identifying the equivalence of various rules proposed in the class. The 'think-of-a-number' game began as a puzzle proposed by the teacher inviting students to solve the mystery but subsequently required the students to create their own such puzzles. The purpose of getting students to work in pairs was not so much to analyse the usefulness of collaborative learning but to create a space where they could share their initial thoughts with their neighbours, given the complexity of the tasks. We spent a total of six 45 minute sessions on these two tasks. Data was collected for the trials by video-recording the classes, detailed teaching notes and reflections or each day, record of every day's work done by children, pre and post-tests and interviews after some duration of the second and the third trial. In this paper, we would use parts of this data to show their facility and difficulties with the processes of generalization and proving, revealed through the two tasks.

IDEAS ABOUT GENERALIZATION



How many matchsticks will be required to make the 4^{th} figure?

How many matchsticks will be required to make the 34^{th} figure?

How many matchsticks will be required to make the n^{th} figure?

Figure 1: Matchstick pattern of growing squares

After a short warm-up pattern generalization task involving polygons and the number of diagonals from one point, the first task in the classroom was to generalize a matchstick pattern of growing number of squares (Figure 1).

Due to their extensive exposure to arithmetic expressions, the first attempt was to write arithmetic expressions depicting recursive relations between consecutive figures in order to predict the number of matchsticks required to make some number of squares. There was no apparent difficulty in generating functional relationships which could help make the predictions. The following is a transcript from the first episode of this task from one of the classrooms (a similar discussion happened in the other one too).

Teacher: Think of an effective way of counting the matchsticks.

Joel: In first one it is 4, in second 3 is more

Teacher: 3 is more, what should I write?

Students: +3

Teacher: 4+3. So how are you saying that? What you said is there is 4 and then 3 has been added. Therefore. What about the next one?

Students: 4+3+3, 4+6

Teacher: you can also directly write 4+6. Next?

Students: 4+3+3+3

Teacher: What will be the 5th?

Students: 4+3+3+3+3, 5 times 3

Teacher: What about the next?

Students: 4+3+3+3+3+3 or 4+6+9

Teacher: Can you tell me something easier to write?

Students: 4+6+9, 4+5×3

Teacher: Okay. Now if I ask you about 12th figure?

Students: 4+3+3+

Some others: 4+11×3.

Teacher: Why did you choose this 11? Yes Mahesh.

Student: Teacher because we are increasing 6.

Teacher: We are increasing 6. I have not followed. Please explain.

Mahesh: 5+6=11

Teacher: 5+6, where did the 6 come from?

Aashish: If any number is there, we have to minus 1 and multiply by 3.

Teacher: Okay. This is a reasonably good explanation. If there is any number, you reduce it by 1 and multiply by 3. Here you decrease 12 by 1 to get 11 and multiply by 3.

Saurabh: 6 is 6 more than 12. That is why you add 6 to 5.

The first verbal expression of '…in second 3 is more' was an important articulation which was converted by the group as '+3', a symbol which they were quite familiar by now for representing change or relationship. The group preferred the expression with recursive addition of 3 rather than expressions of the kind '4+6+9', appreciating the possibility of generalization of the former. The recursive addition of 3 was converted quickly to a functional relationship connecting the number of squares and number of 3's in an expression. The last two interventions by Ashish and Saurabh were the first attempts to verbalize the pattern that they were seeing. Saurabh's rule of 'adding 6 to 5' (that is, add 6 more 3's to the existing 5 3's) was a more local rule, being generated on the basis of another expression they had written for making six squares: 4+5×3. This helped avoid the 'linearity error' many students make, which lead them to write the expression 4+10×3 (2 times 5 =10) for 12 squares. Ashish's intervention was more general, he was in fact stating the general rule of finding the number of matchsticks for any number of squares. This is the one the class used to find the number of matchsticks for 62 squares. In some time, to describe the rule for the n^{th} position or n squares in this pattern, a student gave the rule $4+n-1\times3$. The explanation given was 'n-1 is 61 [pointing to the 62^{nd} position] and like that n-1'. Another said 'whichever number minus 1 and in n it is minus 1'. However, this verbalization, although clear in their minds with regard to the sequence of operations, is ambiguously presented in the written expression, which they did not realize.

Subsequently, students across the two classrooms (English and the vernacular medium) generated a variety of expressions as rules for the same pattern. We could see two kinds of attempts with respect to the creation of the rule: expressions representing counting mechanism highlighting the structure of the pattern (e.g. $4+(m-1)\times3$, $3\times m+1$, $2\times m+(m+1)$) and complex expressions exemplifying pattern in the numbers (e.g. $4\times(m+1)-(m+3)$, $5\times m-(m\times2-1)$ and $3\times(m+1)-2$). They faced no difficulty with the process of generalization, the verbalization of a rule and converting it into an algebraic expression. They seemed to have grasped some important ideas of generalization. They did not approve of expressions which could not display the pattern suitably for all situations. They carefully identified the constancies in the expressions and pattern in the changes across the expressions leading to the rule. It can be argued that the expression is meaningful only when it bears a close resemblance to the structure of the pattern and is connected to the process of counting. However, we feel that both kinds of generalizations are important and valuable. In this case, students made sense of these expressions as representations for the process of counting or as exhibiting relationship between the numbers. This kind of 'playfulness and inventiveness' with expressions was possible due to the nature of understanding of expressions they had developed in the earlier part of the intervention. This process was particularly helped by allowing students to verbally state the general pattern that they saw in the series of arithmetic expressions. The students attempted to understand the symbols and identify appropriate syntax for the algebraic expressions which matched the verbal rule they stated. This cannot be considered meaningless simply because the resulting expression does not match the visual pattern. This process of verbalizing and generalizing in ways so that it has predictive value are generally found difficult in the various exploratory studies.

The think-of-a-number game provided another opportunity for us to evidence students' ability to generalize. When the instruction used in the task was simple like 'Think of a number. Add 6. Subtract 2. Subtract the original number. Subtract 3', students reasoned: '*50-50=0, 6-5=1*'. Though the student used a particular number to communicate her thinking, it displayed the structure of the argument clearly. This paved the way for using the letter to denote the starting number. The representation of the arithmetic or algebraic expression in this context closely matched the sequence of instructions thereby making it possible for students to start symbolizing the puzzle. A very simple situation created by one student in the class and its subsequent solution by another student illustrates the way they were trying to make sense of this task and their ability to relate the arithmetic and the algebraic solutions. In response to the question '*Think of a number. Add 2. Subtract 2. Add original number. Subtract original number*', a student remarked '*She told to add 2, then subtract 2. So we will get 0. And again add original number and subtract original number, so it becomes 0. x+2-2+x-x = x (+2-2=0, +x-x=0)*'. The translation of the verbal explanation to the symbolized form is another aspect of generalization. Students had developed a strong structure sense for both arithmetic and algebraic expressions and used consistent rules for transforming arithmetic or algebraic expressions (see Banerjee & Subramanium, 2012 for detailed discussion on this issue). This is likely to have helped them in this task as well, where they skillfully analyzed the operations and transformations on the initial number, often treating this number differently from the other numbers used in the context.

IDEAS ABOUT PROOF AND PROVING

Though students displayed some genuine understanding of generalization, they did not show much appreciation of the ideas of proof or proving that naturally arose in these two contexts. When asked whether all the rules generated for the square matchstick pattern were equivalent, some were not sure as they compared the expressions by focusing on the surface features, like comparing $3n+1$ with $n+n+n+1$ and arguing that in the first expression we have $3n$ whereas in the second we have n and $+1$ is same in both (derived from a strategy of comparing terms that was used quite successfully while comparing arithmetic and algebraic expressions without computation). Others argued that they must be the same as they are generated from the same pattern. A few students thought of substituting numbers in the expressions and check if all the expressions led to a common value during the classroom discussion. A large majority of them resorted to simplifying the expressions to the simplest form. Even after simplifying all the algebraic expressions and arriving at the same result for all, there was at least one student who expressed that '*unless the value of the letter is known, it would not be possible to judge the equivalence of the expressions*'. This leads to a doubt whether all students were able to see the purpose of using the letter or the idea of 'proof', a fairly well documented phenomena in the studies on proof and proving. In the earlier sessions of the trial, they had demonstrated sufficient understanding of equality and equivalence of expressions, which could be arrived at by carrying out various kinds of valid transformations on them. However, in this context, they did not see that expressions once shown to be equivalent do not need further corroboration with numbers and it is general enough to hold for all values of the letter. They also made syntactic errors while transforming them.

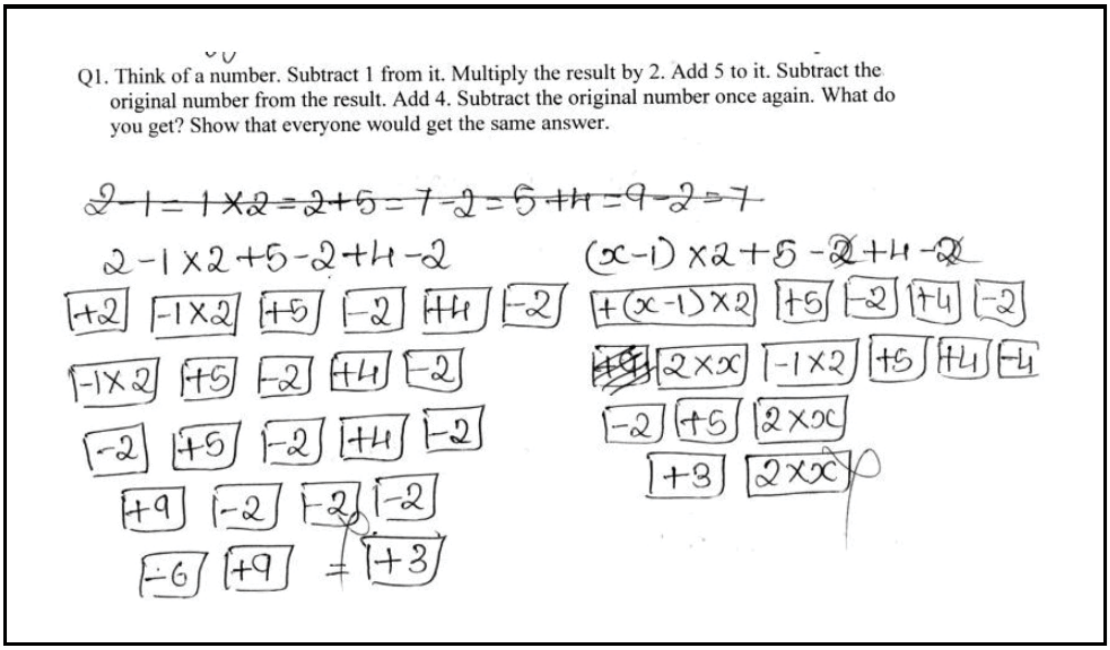


Figure 2: A student's attempt to prove using numbers and letter

In the 'think-of-a-number' game, some students often decided to try various numbers as the starting point and arrived at a result and then induced the pattern of relationship between the two. This process though useful in figuring out how the two (starting and the ending number) were related, did not throw much light on why this was the case. On various occasions of individual solving of such tasks given by the teacher or generating such problems on their own in the class, we came across ideas on proof and proving. Some believed that since the result holds well for multiple instances, it was sufficient to 'prove' the correctness of their conclusion (like, the sequence of instruction would lead to the number they started with or a constant). Some others, as in the verbal explanation stated above, treated numbers as 'quasi-variables' and manipulated expressions in particular ways to show the generality of the conclusion. See Figure 2 for such an attempt, which is incorrect but gives a glimpse of how students tried to use arithmetic expressions and algebraic ones to prove the result. The error is precisely because of the interference of the 'starting number' (2 in this case) with the use of letter. There were a few who could systematically symbolize the instruction algebraically and through manipulation explain why the conclusion holds true for all numbers. As the problems became more complex (posed by the teacher or generated by students), the mental tracking of transformations on the original number was harder and errors in representing and manipulating were observed.

Again, like in the case of showing equivalence of algebraic expressions in the pattern generalization task, they were not sure if the symbolic expression added any insight into the situation. This was further revealed as many of the typical errors students make while manipulating symbolic expressions re-emerged, sometimes even leading to meaningless manipulation. For instance, when they found that a particular set of instructions on the starting number led them back to the starting number and manipulation on a wrong representation led them to the value 0, they immediately concluded that x must be 0 as they did not know the value of x. They were of course not appreciating the goal or the purpose of the whole task. It was very hard for them to resolve the conflict as they did not anticipate what the manipulation should lead them to, thereby indicating a lack of its purpose.

The interviews conducted with seventeen students few months after the end of this trial further substantiated these ideas on proof and proving. The interviews explored students' solution to a pattern-generalization task and 'think-of-a-number' game, similar to the one used in the post-test of this trial. The richness of the classroom discussions were not reflected in the individual performance in the post-test or the interviews, with few students being able to complete the tasks successfully. The pattern-generalization task required students to generalize a pattern and show two rules to be equivalent. Students had variable ability to arrive at the generalized rule and only a few could show the equivalence of the different algebraic rules through simplification without any support. Besides asking for the utility of algebra in the 'think-of-a-number' game, the students were asked to represent a set of instructions, identify the correctness of an expression, judge equivalence of expressions and make a problem for an expression. Even though quite a few students correctly identified a valid representation, could make a question for an algebraic expression, could anticipate the result of simplification of the expression, they were unsure of the use of algebra for the situation. A typical response articulated by one of the students was '*a number represents a general number and if the same operations are carried out on the number, it can be shown that everyone would get the same answer*'.

DISCUSSION AND CONCLUSION

Students displayed sufficient capacity for generalizing in these contexts. This generalization was facilitated by their ability to articulate their reasoning in natural language and their prior experience of reasoning with such symbols. They had sufficient exposure to the arithmetic and algebraic symbols and had been reasoning about validity of syntactic transformations and equality and equivalence of expressions. This of course influenced the ways they saw the use of expressions in the contexts we discussed in this paper. They were readily seen to use expressions (arithmetic or algebraic) to represent the situations meaningfully. However, there were often syntactic errors in the representation which were hard to resolve. Similarly, though they were quite comfortable generating a representation in both the contexts, it was not obvious to them the purpose of using the letter. In the context of pattern generalization, the letter was surely a number and therefore it was more acceptable. However, when the discussion moved to showing the equivalence of two or more rules generated for the same pattern, it was not clear whether they all appreciated that the expressions must necessarily be equivalent or that simplification of expressions is general enough and does not need to be followed by substitution of a number. The 'think-of-a-number' game provided opportunities to the students to engage in representing a numerical situation algebraically. They could make sense of the expression but since they could explain the situation using narratives or using arithmetic expressions by treating numbers in a 'quasi-variable' way, they did not see algebra as adding much value.

Their earlier experience created a predisposition for symbolic representations and thinking and reasoning with an expression. However, fewer students could convert this understanding to one which could enable them to successfully complete the tasks of reasoning *with* expressions or appreciate the 'purpose of algebra'. The issue is not simply one of transferring the abilities from the syntactic world to the contexts where algebra is to be used as a tool or of creating a situation so that the letter gets embedded in the context and thus creating meaning for the letter or algebra. Two elements that play an important role in these tasks are (i) the culture of generalizing, proving and verifying, with which the students in traditional curricula have very little experience and which needs to be developed and (ii) students' belief about the effectiveness of using algebra in these tasks. One can probably explain the re-emergence of many syntactic errors and meaningless manipulation on the letter and the expressions as they failed to appreciate the purpose of algebra and the goal that they were trying to achieve.

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