good-enough metaphor also has a lot in common with the constructivist replacement of truth with viability, recently taken up by Proulx (2013) to analyse students' strategies or mathematical solutions as functional (instead of optimal) within a context. In a way, this is like Papert's little program in the video: it works well as far as to show "how a program can be made to find an answer", even if it is not optimal in terms of doing so (since it does not even provide the correct figure). I am not sure, however, of how much we actually value this kind of sort-of-right mathematics. It feels more like a necessary evil (when not simply taken as something we have to get rid of), not something to celebrate.

One key idea relating to sort-of-right mathematics that seems to emerge in mathematics education literature is that of *ambiguity*. Borasi (1993), for example, worked with some students on the impossibility of writing absolutely general and unequivocal definitions (*e.g.,* of a circle); Zaslavsky, Sela *et al.* (2002) explored the various interpretations of school-made mathematical concepts ("slope" in their case), and others have focused on the ambiguity of mathematical symbols (Mamolo, 2010) or representations (Schwarz & Hershkowitz, 2001). Some might even argue that everything related to mathematical modeling, statistics and probability is in fact all about coming up with sort-of-right answers. I would be tempted to say, however, that these are great occasions to offer sort-of-right mathematical experiences to students only if we emphasize the productive, creative, or simply interesting aspect of their sort-of-rightness, which is probably rarely the case.

Chain reaction?

The idea of sort-of-right mathematics surely has a disruptive power. In the beginning of the twentieth century, the mathematical community started to struggle in what we now call the foundational crisis. But as a result, new branches of mathematics were created, new heuristics were brought forth, and richer conceptualizations developed. What if we really tried exploring the potential of sort-of-right mathematics for teaching and learning? What if we deliberately set it out, and asked students for approximate answers, good-enough solutions, roughly viable procedures, and so on? What if we tried identifying new mathematical topics to explore with students, precisely to bring forth sort-of-right mathematics? I have already mentioned a few topics, and others could certainly be found in approximation theory (*e.g.,* asymptotic comparison, or situations in which precise answers are impossible to obtain, like with the quantic), or dwell upon various "kind of proofs" and their purpose (*e.g.*, zero-knowledge proofs aiming at establishing the existence of a proof; in FLM, see Hanna, 1995), and so on.

I cannot conclude without going back to Papert's comment and react to the idea that there is something "so wrong" with mathematics as it is taught in schools. If it is okay for mathematics to be sort-of-right, maybe it should also be the case for "mathematics as it is taught in schools". What I mean is that what we generally try to do in school with mathematics is incredibly difficult, if not simply impossible. I do not understand my job as a researcher in mathematics education as finding and fixing "what is wrong" in schools. My job is to provide ideas for what could be done in, out, around, for, with, or even against mathematics educational systems. Nothing more…but nothing less.

Notes

[1] See www.youtube.com/watch?v=4iIqLc0sjjs

[2] The word "right" comes from the Proto-Indo-European root *reg-* meaning "moving in a straight line".

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Where did/do mathematical concepts come from?

ROSSI D'SOUZA

I share three anecdotes from my teaching experiences and pose some open questions addressing mathematics education research. These three experiences, in hindsight, helped me address, if not entirely answer, the said questions. I emphasize the need to recognize and present mathematics as processes carried out by people (including students), rather than final products to be learned and applied.

Exploring odd and even numbers

I teach in a school for blind children. During a mathematics class, since the topic of odd and even numbers came up, I asked, "what are odd and even numbers?" The students stated that numbers which can be evenly divided by 2 are *even*, while those that could not are *odd*. When asked about *zero*, all seemed to arrive at a consensus that it is both odd and even. They reasoned that since zero leaves no remainder when divided by 2, it is even; however, we cannot divide zero by two since we have nothing to divide, making it odd. During further discussions, one student, Faizan, expressed discomfort with the idea of including zero as odd. He argued that numbers have the property that "odd +/– odd = even", or "even +/– even = even" and "odd +/– even = odd" and "even +/– odd = odd". However, if zero were to be included as an odd number, this pattern would not work. The pattern could be maintained only if zero was categorized as even, not odd.

On continuing the discussion, the number –4 turned up. The discussions included the following dialogue:

Me: So what about –4? Is it odd or even?

Faizan: Before deciding that, we need to know where did numbers like –1, –2 come from? I mean there has to be a reason. For example, when we found numbers that could be divided by two, we called them even and those that could not, as odd. So where did these numbers like –1, –2 come from?

Me: Maybe we can look at some examples of where these negative numbers can be found and then–

Faizan: Sir, when we visited the mall, the lift had numbers –1 and –2 to indicate the upper and lower basement.

The subsequent discussion emphasized the need to conceptualize negative numbers. Faizan suddenly interrupted stating that negative numbers are very old, while malls with basements are comparatively new. He later on hypothesized that maybe during the Harappan civilization, building structures which had some equivalent of basements could have given rise to the concept of negative numbers. The discussions continued with other hypotheses and examples that led the class to concur that it makes most sense to categorize –2, –4, … as even. These numbers fit into a continuous pattern of alternating even and odd numbers, whether read backwards or forwards.

This discussion got me thinking. If a student considers 0 as an odd number, is it a misconception? If a student invents a crazy concept like "*abc*-numbers" and "*xyz*-numbers" and decides to categorize, 3.5 in the set of *abc*-numbers, does 3.5 become an *abc*-number? If another student calls 3.5 a *xyz*-number, would it be a misconception? Similarly, is calling 1 a prime number, a misconception?

A lecture on divisibility

I was asked to conduct a lecture with students selected for a mathematics competition. I decided to talk about divisibility. On beginning with asking the definition of a prime number, the students answered in unison, "...a number that can only be divided by 1 and itself." I then asked about the number 1 since it fits the definition. However, the students refused to accept 1 as a prime number (possibly because their books "say so"?). Discussions followed until we decided to redefine the concept of primes as: "numbers greater than 1 that can only be divided by 1 and itself." This discussion included the Fundamental Theorem of Arithmetic.

In *Elements Book VII,* Euclid (1908, p. 278) introduces the idea of a prime number by defining it as "[that] which is measured by a *unit* alone" thus including 1 as a prime number. Reddick *et al.* (2012) present a list of sources in which 1 is included when referring to the set of prime numbers, even though mathematicians generally do not include it. Even until 1941, the number 1 was considered a prime number by some mathematicians, but non-prime by some others. Each had a justifiable reason for their claim. For example, Lehmer (1941) introduces the nomenclature for prime numbers by stating that "the letter p designates a prime which may be ≥ 1, > 1, or > 2 according to the context".

So does it make sense to ask if 1 *is* a prime number? Would it have made sense to ask the same, say, during the 1930s in some exam, with the assumption that there is only one correct answer? Is the concept of *prime numbers* now "finalized"? Can we take any specific mathematical concept and confidently say that now its truth cannot or should not be questioned? Did people in the 1930s realize that the definition of prime numbers would change later? Is it possible that our present definition will change later? When would a change be acceptable? If made by a mathematician? A student? Who is a mathematician? Someone with a university or state-designated position?

Exploring a sequence

A fellow mathematics educator expressed a special affection for the numbers 4, 2, 6 and 8, in that order. On asking her why, she refused to disclose her reason. Thinking that it might involve some special day in history, I looked up possible dates that could have been represented by 4, 2, 6, 8. Nothing! I thought it could mean: $4 + 2 = 6$ and $4 \times 2 = 8$. But that did not seem so fascinating. Another friend mentioned the Fibonacci sequence since $4 + 2 = 6$; $2 + 6 = 8$. I continued: $6 + 8 = 14$, (focus on the units places), $14 + 8 = 22$ and found a pattern. In "modulo 10", the numbers 4, 2, 6 and 8 follow a "fibonacci-ish" sequence: 4, 2, 6, 8, 4, 2, 6, 8, ... I was happy and then I rested (or so I thought). But another thought came to mind: could (4, 2, 6, 8) be the shortest sequence in mod 10? I immediately realized that it cannot, since (0, 0, 0, …) is a shorter sequence (of "length" 1) whereas (4, 2, 6, 8) has "length" 4. I dismissed this "trivial sequence" (0) because of it not being "beautiful". But then I realized that the sequence (5, 5, 0, 5, 5, 0, ...) which is "equivalent" to the "reduced sequence" (5, 5, 0) is smaller than the sequence (4, 2, 6, 8). But even (5, 5, 0), albeit not as trivial as (0), was a rather simple sequence. So, putting aside (5, 5, 0) and (0), could (4, 2, 6, 8) be the smallest "mod-10 fibonacci-ish sequence"?

I thought this conjecture to be too difficult to prove, considering the many possible sequences. But then realized that I would need just two, single-digit numbers to initiate a sequence. And these two numbers would be unique to only that sequence. For example, a sequence generated by the numbers 2 and 6 (I'll represent this by [2, 6]) would be 2, 6, 8, 4, 2, 6, 8, … which is "equivalent" to (4, 2, 6, 8). I then realized that there are now fewer possible sequences, since the total number would be strictly less than the total number of 2 digits. Also, [4, 2] is equivalent to [2, 6] which is equivalent to [6, 8]. *I.e.* [4, 2] = [2, 6] = [6, 8] = [8, 4]. Similarly, [0, 5] = [5, 5] = [5, 0]. This "equivalence" reduced the number of possible sequences to much less than 100 – 4 – 3 – 1 = 100 - 7 = 93. I tried another sequence, [0, 1]. This sequence generated: 0, 1, 1, 2, 3, 5, 8, 3, 1, 4, … which just went on! Could this be an infinite sequence? Of course not! (I felt silly!) The sequence would definitely have to repeat before at least 93 numbers were written. So I continued … 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0, 1, 1. Aha! I reached {1, 1} indicating the end of the sequence (of length 60).

Now {2, 2} was not present in the above sequence. [2, 2] generated (2, 2, 4, 6, 0, 6, 6, 2, 8, 0, 8, 8, 6, 4, 0, 4, 4, 8, 2, 0, 2, 2, …) with length 20. I tried [4, 4] and realized that it was equivalent to [2, 2] since {4, 4} was a sub-sequence of [2, 2].

I had now accounted for 1 + 3 + 4 + 20 + 60 = 88 out of the hundred "two-digits". Had I exhausted all possible sequences, the sum of the lengths of each sequence would have to add up to one hundred. I looked at a 2-number generator as a two-digit number. I saw that "13" was absent. So I tried [1, 3]. I got (1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2, 1, 3, …). thereby getting a sequence of length 12 = 100 - 88. This exhausted all possibilities. So I concluded that the only possible fibonacci-ish sequences are:

(0)

(5, 5, 0)

(4, 2, 6, 8)

(1, 3, 4, 7, 1, 8, 9, 7, 6, 3, 9, 2)

(2, 2, 4, 6, 0, 6, 6, 2, 8, 0, 8, 8, 6, 4, 0, 4, 4, 8, 2, 0)

(0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1)

with lengths 1, 3, 4, 12, 20, 60 respectively. This proved that (4, 2, 6, 8) was the smallest non-simple fibonacci-ish sequence in modulo 10. I think I have closure. However I had also felt a sense of closure on discovering that 4268 was a mod-10 fibonacci-ish sequence. Maybe something else will come up and I'll work on that too.

I then thought about my past excursions in mathematics. Then too I would invent concepts to communicate with peers or even myself. I jokingly thought of how unfair it would be to get students to prove or even understand claims and concepts that I invented for myself. I asked myself some silly questions like, "Is it possible for me have misconceptions about, fibonacci-ish sequences, length of a sequence, trivial sequence, reduced sequence, equivalent sequences, *etc.*?" Would it be fair to make children learn and be assessed on the concepts that I made for my own fun? Is it unfair only because these concepts have no use in the "real world"? What if I ask these questions about the mathematical concepts of "real mathematicians"? Are my ideas like length of a fibonacci-ish sequence, equivalent fibonacci-ish sequences, *etc.* mathematical concepts? If yes, then since when were they mathematical concepts? Since I invented them? Did I invent, or discover them? If I redefine fibonacci-ish sequences, will the old definition still be a part of mathematics? So what makes a fibonacci sequence (or odd and even numbers or prime numbers) a part of mathematics? Would some properties of fibonacci numbers also work for mod-10 fibonacci-ish numbers? Is this a legitimate mathematics question? Why? What if I asked "which properties, and why"? And if this is a valid mathematics problem, does that make me, the author, a mathematician? Are students mathematicians? Why not? What if students invent concepts themselves? Oh wait, they do!

Final words

We need to recognize (and teach) mathematics as an historically dynamic and ever changing process that people do. When learners engage with their life experiences and interests and even actively decide how to define concepts, they experience the joy of creating mathematics and developing a meaningful and a more holistic understanding of it.

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From the archives

Editor's note: *The following remarks are extracted (and slightly edited) from an article by Victor J. Katz (1986), published in FLM6(3).*

How many words can be formed from the letters of the Hebrew alphabet? When and where does the sun rise in Alexandria on August 16? What is the length of the circumference of a circle of radius 1? Why does $\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$ equal 4? Where should a 150lb man sit on a seesaw to balance his 50lb son? How should the stakes in a game of chance be split if the game is interrupted? It is the consideration of questions like these which historically has provided the impetus for the development of mathematics. The consideration of such questions with our students is one way that we can motivate and excite them.

I have found this historical approach to the topics of the mathematics curriculum to be a profitable one. I will present here some concrete examples of the use of historical materials in developing certain topics from precalculus and calculus. Some of these are ideas which can be introduced easily in the course of a standard treatment of the material. Others would require some reformulation of the curriculum. In those cases, the curriculum would benefit from the reformulation. […]

We begin with algorithms. This is the new "buzzword" in mathematics education today, with conferences, institutes,