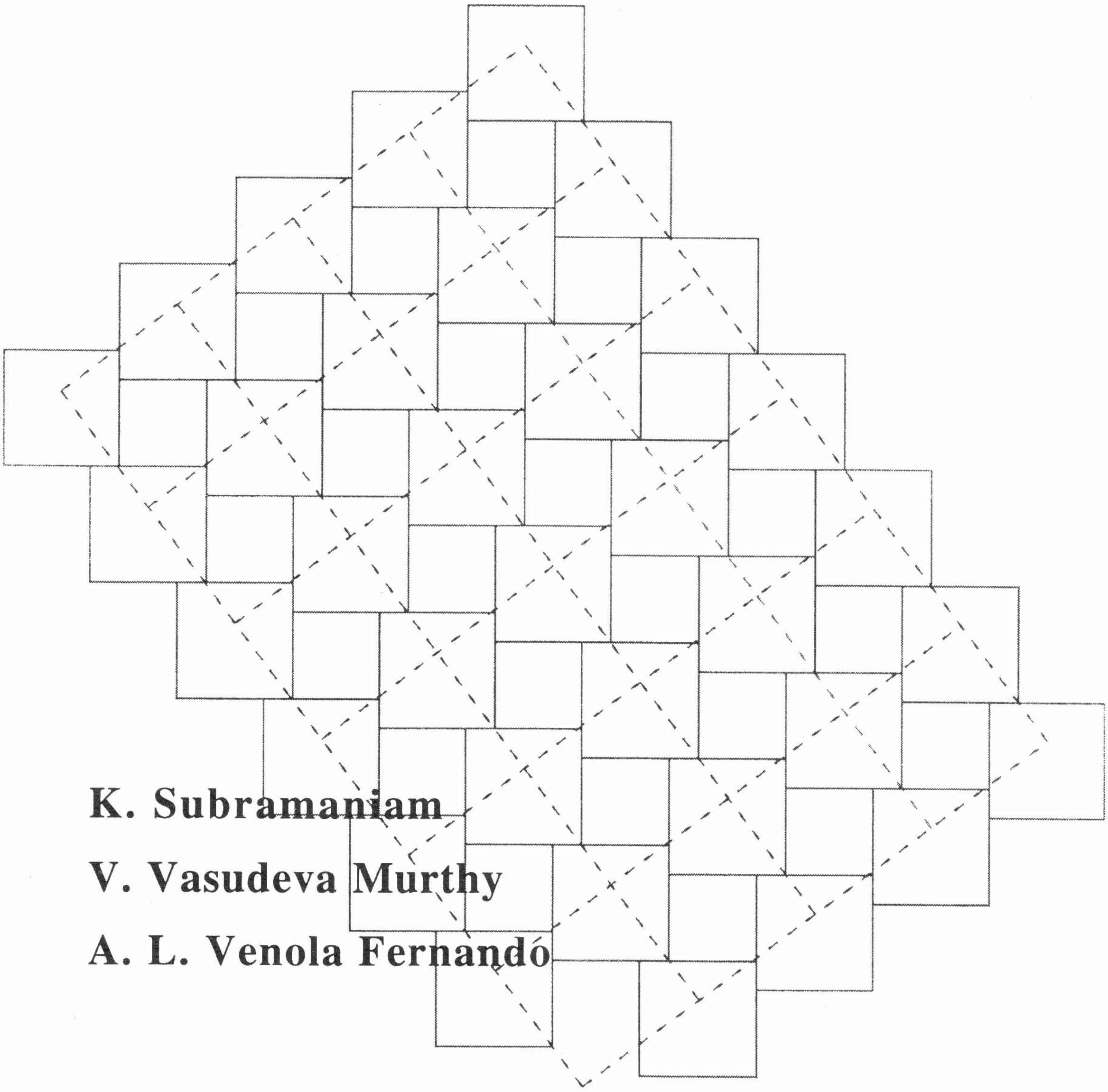
Technical Report No. 44

Exploring Mathematics

Dissections and Graphs



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Investigation 1 Pythagoras theorem by dissection

In this investigation our aim is to discover ways of proving the Pythagoras theorem. All the proofs that we will find involve showing that two areas are equal by dissection. That is, the areas concerned are cut into pieces so that one area can be transformed into the other by re-arranging the pieces. Dissection proofs therefore are a bit like jigsaw puzzles. It is a good idea to put the following things together before you begin the investigation: thick card paper to cut out triangles, squares, etc., some tracing paper or overhead transparency sheets, sketch pens or transparency pens, a pair of scissors. It is possible to do the investigation without these things, but having actual geometric shapes cut out from thick paper lets you explore different possibilities more easily.

Every student in high school has heard of the Pythagoras theorem and most students can state it. The theorem is probably one of the oldest theorems in mathematics and was known to the Babylonians in the second millennium B.C. One of the most impressive mathematical records from ancient history is the Plimpton 322 clay tablet (No. 322 in the Plimpton collection at Columbia University) from about 1800 B.C. This tablet gives a list of Pythagorean triplets, that is, sets of three whole numbers which can form the sides of a right angle triangle like $\{3, 4, 5\}$ or $\{8, 15, 17\}$. In another tablet, the Babylonians have written down the length of the diagonal of a unit square correct to about one millionth. These are good indications that they knew and used the Pythagoras theorem. Pythagoras, the Greek philosopher and mathematician, lived in the 6th Century B.C. The Pythagoras theorem clearly was known to people from different civilizations of the world from before that time. Why do we then call the theorem after Pythagoras? Some historians believed that he was the first to prove the theorem. However, many modern historians think that even this is doubtful. The name remains unchanged largely because of usage, because most people know the theorem by this name.

Let us recall the Pythagoras theorem. A common statement of the theorem runs something like this.

Theorem 1.1 The square on the hypotenuse of a right angle triangle is equal to the sum of the squares on the other two sides.

Without knowing the Pythagoras theorem if you just looked at the right triangle for long enough it is unlikely that you will ever guess the relation between the sides of the triangle. In this sense the Pythagoras theorem is non-obvious and non-trivial. It comes as something of a surprise and arouses our curiosity. We wonder what makes the theorem true search for a proof. Being the oldest theorem, it does not come as a surprise to learn that the Pythagoras theorem is also the most proved theorem. Elisha Scott Loomis, who wrote a classic book in 1927 on proofs of the Pythagoras theorem, claimed that there were over 370 different proofs, each one calling for a different figure!

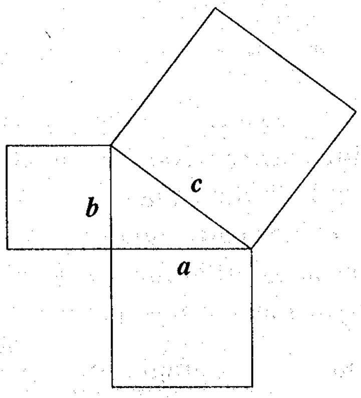


Figure 1.1: The Pythagoras theorem

Although the Pythagoras theorem is non-obvious, it is not difficult to prove it. One of the simplest proofs, which you have probably seen, has a square inscribed inside another square. Try to discover the proof if you cannot remember it. (If you really give up then you can see the figure shown on the next page.) You will notice that the proof involves some algebra, some manipulation of formulas. Most proofs of the Pythagoras theorem that you have seen probably involve some algebra. In this investigation we are going to try something different. We will try to find proofs which don't involve any algebraic manipulation at all.

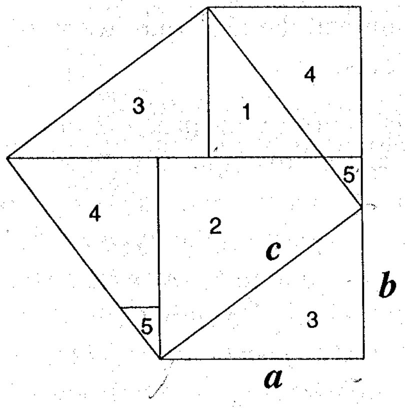
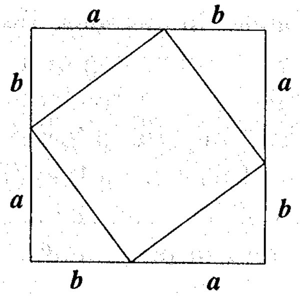


Figure 1.2: A dissection proof of the Pythagoras theorem

Figure 1.2 is an example of such a proof. In the right angle triangle, the two sides forming the right angle are a and b, and c is the hypotenuse. Identify the three squares a^2, b^2 and c^2. The figure shows how you can cut up the two smaller squares to form the big square.

Check whether the pieces in the big square are the same as (congruent to) the pieces in the smaller squares. The pieces numbered 1 and 2 are already



A simple proof of the Pythagoras theorem. You will need to work out the details of the proof. **Hint:** Use the formula for $(a+b)^2$.

in the big square. You have to check if the pieces numbered 3, 4 and 5 are congruent in both the squares. Compare Figure 1.2 with the proof that is described in the previous paragraph for which you turned the page. Do you see any connection? Instances of both these proofs are found in some of the ancient Indian texts in mathematics.

There are other examples of dissections which are found in ancient Indian texts. For example, in the *Aapastamba Sulvasutra* we find a procedure to compute the area of an isosceles trapezium by dissecting it and forming a rectangle. These examples are not proofs in the same rigorous sense as found in modern mathematics or even in Euclid. Nevertheless they are visually appealing and interesting. When we know that two sets of figures have equal areas, it is intuitively satisfying to find a simple dissection which can change one set into another. Moreover even something as simple and as well known as the Pythagoras theorem can give rise to problems that are both fascinating and sometimes very difficult. In this investigation and the next one, we will explore some of these Pythagorean dissection problems.

Let us return to our problem of dissection proofs of the Pythagoras theorem. We have seen an example of a dissection proof. Can you find other ways of cutting up the small squares on the sides forming the right angle and assembling the pieces to form the square on the hypotenuse? Take a right angle triangle with sides 3 units, 4 units and 5 units. How would you cut the two smaller squares to form the bigger square? Figure 1.3 shows one way of doing this. The 3×3 square has been cut up into two 3×1 rectangles and three 1×1 squares.

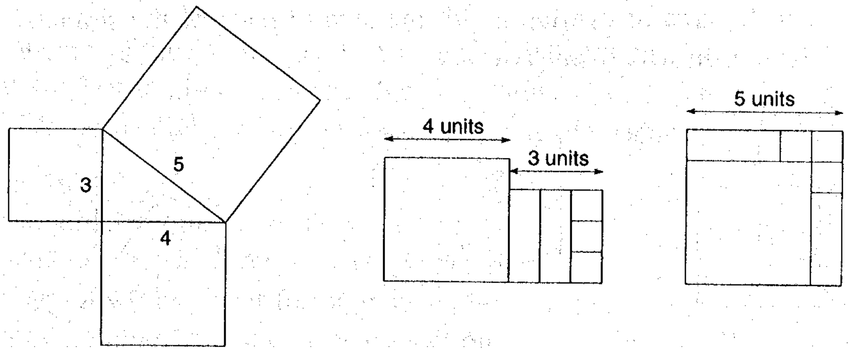


Figure 1.3: Cutting two small squares to make a bigger square

This method works for the particular triangle that we happened to choose. Will it work for other right angle triangles as well? What happens when the triangle has sides whose lengths are not whole numbers? What happens when the lengths of the sides are irrational, say, when the lengths are 1, $\sqrt{3}$ and 2?

It is now probably evident to you that cutting up the square into smaller rectangles and squares has limitations. We must look for other ways of cutting the small squares to obtain the large square which will work for any right angle triangle. Try this out and see if it looks like a hard problem.

By the time you go through this investigation, you should be able to come up with a number of ways of cutting up the small squares and assembling the large square. That is, you should be able to find different dissection proofs of the Pythagoras theorem. In fact, there is a method which gives several ways of proving the theorem by dissection including the one we saw in Figure 1.2.

Do the activities described below one after another. The connection between them may not be apparent at first, but they will all help in solving our main problem — of finding dissection proofs of the Pythagoras theorem.

ACTIVITY: OVERLAPPING OF SQUARES, CENTRE TO EDGE

Cut out two identical squares, say 6 cm × 6 cm from card paper, or better still from tracing paper or overhead transparency sheets. Put one square on top of the other so that a corner of the top square lies on the centre of the square below as in Figure 1.4(a). The edges of the two squares intersect at their mid-points. It is obvious that the area of overlap is $\frac{1}{4}$th the area of each of the squares. Can you spell out the reason why this is so?

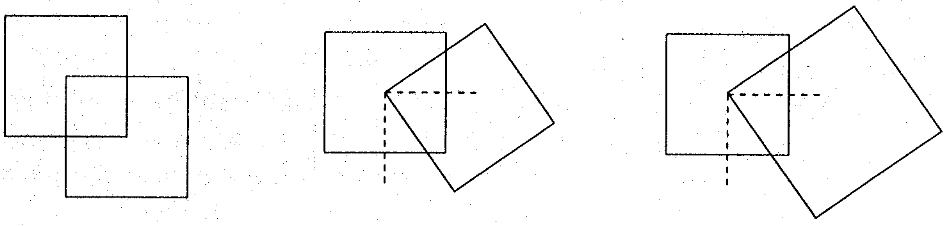

(a) (b) (c)

Figure 1.4: Overlapping squares

Rotate the square on top a little still keeping its corner over the centre of the square below as in Figure 1.4(b). How much of the area of the square at the bottom overlaps with the area of the square on top? Compare this with Figure 1.4(a). As the square rotates, some area of overlap is lost on one side and some area is gained on the other side. Compare the areas which are lost and gained. What can you now say about the area of overlap?

Now take a square which is larger than these squares, say 8 cm × 8 cm. Lay it on top of one of the 6 cm × 6 cm square so that the corner of the larger square lies on the centre of the smaller square as in the Figure 1.4(c). How much of the area of the smaller square overlaps with the area of the larger square?

If you have found that the area of overlap is still $\frac{1}{4}$th the area of the smaller square, find a way of cutting the smaller square into 4 pieces, each of them congruent to the overlapping portion. (Try extending the lines which form the edges of the bigger square.)

Write your results down. You will use them later. Now go on to the second activity.

ACTIVITY: THE ISOCELES RIGHT TRIANGLE

Consider the right angle triangle shown in Figure 1.5. It is a special right angle triangle because the two sides which form the right angle are equal. What will the ratio of the lengths of the sides forming the right angle to the hypotenuse be? From the Pythagoras theorem it follows that the area of the large square is double the area of each of the small squares. Can you show how the two small squares can be cut into pieces and put together to form the larger square? This is our dissection problem for the special case of the isoceles right angle triangle.

There are two simple but different ways of cutting the smaller squares to form the larger square. One of them is a four piece dissection, that is, it has only four pieces forming the larger square, and the other is a five piece dissection. If you have found only one of them, try finding the other one. If you are in a hurry you could look at the figure on the next page.

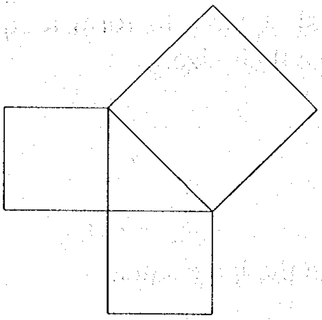


Figure 1.5: An isoceles right angle triangle

It is quite easy to find a dissection proof for the isoceles right angle triangle. It pays however to take a closer look at the dissections. We are going to suggest that you look at it in an entirely new way. This new way involves tiling patterns. Tiling patterns which do not have any gaps and can be extended to cover an infinitely large area are called **tessellations**. The study of tessellations and their properties is an interesting branch of mathematics. It is also an area where a lot of new discoveries are still waiting to be made. Tessellations have a relevance for the problem that we are pursuing. In fact, we are going to make use of tessellations to discover solutions to our dissection problem for right angle triangles in general. But first let us examine the dissection problem for the special case of the isoceles right angle triangle making use of tessellations.

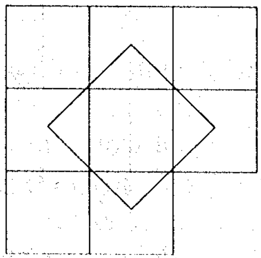
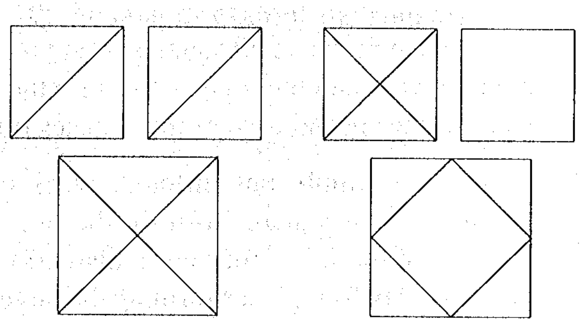


Figure 1.6: Square overlapping on a square tiling

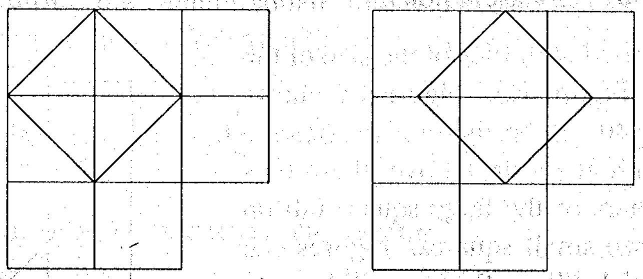
Imagine a floor tiled with tiles of the size of the small square in Figure 1.5. Figure 1.6 shows a portion of the tile. The figure also shows a tile of the larger square placed over these tiles so that two corners of the large square fall on the centers of two small squares. Figures can sometimes be misleading. So you will have to verify that the corners of the large square actually fall on the centres of the tiles. (Hint: find the horizontal and the vertical distance between the centres of the tiles). Now look carefully at



Two different ways of cutting equal squares to form a square twice their size.

the areas which overlap. Does this tell you how to cut two smaller squares and form the large square?

Move the larger square over the tiles without changing its orientation and try to find other ways of cutting the small squares to form the large square. You could, for example, let the corner of the large square fall on the corners of the smaller squares, or on the midpoint of their sides, and so on. Figure 1.7 shows these possibilities. Figure 1.7(a) yields a dissection that you have already seen. Identify which one it is. Figure 1.7(b) shows a new dissection. Go back to Figure 1.5 and show how the two small squares can be dissected to form the bigger square as suggested by Figure 1.7(b). How many pieces form the dissection?



(a) *(b)*

Figure 1.7: Different positions of the overlapping square showing different dissection possibilities

ACTIVITY: THE PYTHAGORAS THEOREM BY DISSECTION

Figure 1.8 shows a portion of a tiling of squares. Only four square tiles can be seen. Another square lies on top of the tiles so that the corners of the top square fall on the centers of the tiles below. Note that the square on top is the same size as the square tiles below.

Imagine now that the square on top is made slightly bigger (shown by the dotted line in the figure). The size of the tiles below is unchanged. How would you have to rearrange the tiles so that the corners of the square on top still fall on the centers of the tiles below? You are not allowed to move the tiles so that all contact between them is lost. Try and ensure that adjacent square tiles have as much contact along their edges as possible. It is useful to try this out by drawing various diagrams. It is better still to have cut outs of the square tiles with the larger square drawn on transparent sheets or on tracing paper.

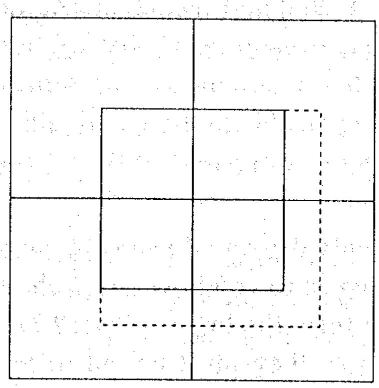


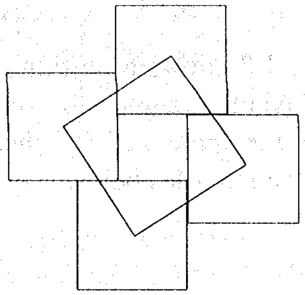
Figure 1.8: Square overlapping on a tiling

This is one of the important steps in finding Pythagorean dissection proofs. So take some time thinking about this problem. Only if you give up after a lot of thinking, look at the figure on the next page. (However, you are allowed to look at the previous figures and you can get a clue from looking at the tiling we drew for the previous activity, which is shown in Figure 1.6.)

After you have found a way of arranging the tiles so that the enlarged square on top falls corner-to-centre on the tiles (or if you have given up and turned to the next page), study the arrangement. How has the tiling changed? How many different squares can you see in the tiling? Look carefully at the portions of the square tiling covered by the square on top. If you use the results of **activity 1**, you should be able to find a relation between the areas of the smaller squares and the bigger one.

Take the sides of the two smaller squares, call them a and b, and form a right angle triangle out of them. What will be the area of the square on the hypotenuse? How

The figure shows how the square tiles should be arranged so that the corners of the larger square fall on their centers. Notice that the space formed in the centre has the shape of a square.



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is it related to the area of the bigger square in the tiling arrangement?

Also study the way the larger square is overlapping with the smaller squares. Does this suggest to you how one each of the smaller squares could be cut and reassembled to form the larger square? Draw a diagram of the two squares a^2 and b^2 and show how they must be cut to form the large square as suggested by the tiling figure.

It may be clear to you by now that we have indeed discovered a dissection proof of the Pythagoras theorem. The three different squares are the squares drawn on the sides of the right angle triangle. It is convenient to give names to these squares. In the right angle triangle, let the sides which form the right angle be a and b and the hypotenuse c. Let us choose a to be bigger than b. We will call the square drawn on side a the α-square. The square drawn on side b will be called the β-square and the square on the hypotenuse will be the Γ-square. The β-square is the smallest square.

As in the case of the isoceles right angle triangle (activity 2), we can obtain different dissections by moving the Γ-square to different positions on the tiling. Explore these arrangements and see how many dissections are possible. To see more possibilities, you may have to extend the tiling by adding more of the α and β-squares. Below in Figure 1.9 you will find some possibilities. Match the dissections shown on the left with the tiling arrangements shown on the right. At the beginning of the investigation we encountered a dissection proof (Figure 1.2). Find the tiling arrangement which gives this proof.

Notice that in each of the arrangements the corners of the Γ-square fall on congruent points of the α-squares. In the first tiling arrangement shown, for example, points A, B C and D are congruent points on congruent squares. By this we mean that the position of A within the first α-square is the same as the position of B within the second α-square. Such points are called **congruent points**. It is necessary for the corners of the Γ-square to fall on congruent points for the dissection to be possible. Is this condition satisfied as you move the Γ-square over the tiling arrangements? Check if the corners of the Γ-square fall on congruent points in the other arrangements in Figure 1.9. (What happens if you rotate the Γ-square?)

The question of the generality of the dissection proofs shown remains. Can the dissection proof that you have discovered be used for any right angle triangle? In the tiling arrangement shown the square at the centre, the β-square is smaller than the α square. What happens if you go on increasing the length of side b? Does the dissection still work? What happens if side b equals side a? How will you make the dissection work if side b is bigger than a?

We have been successful in finding a general method to obtain various dissections of the α and β-squares into the Γ-square. In the next investigation, we will generalize our problem and then look for solutions to new problems, all of which have to do with the Pythagoras theorem.

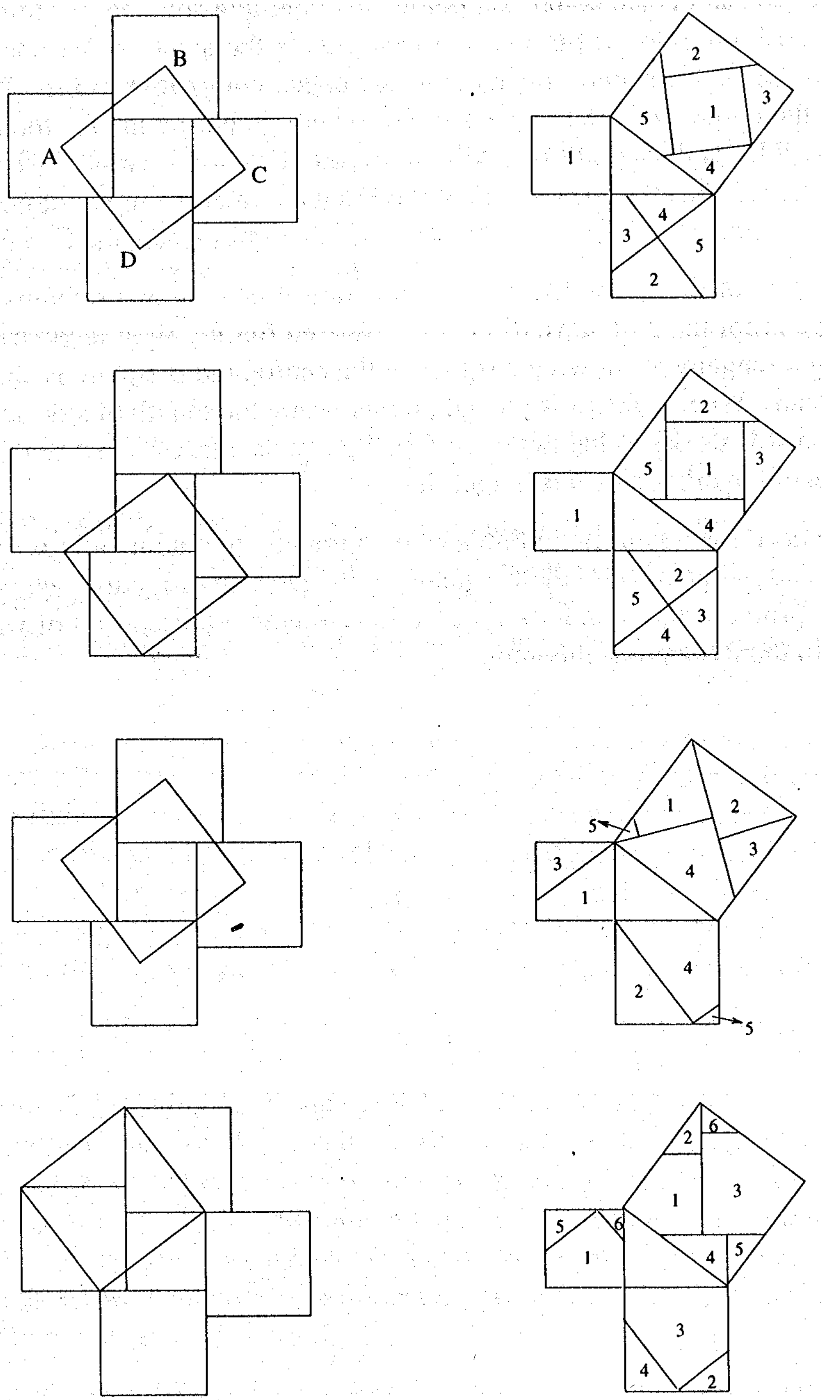
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Figure 1.9: The left hand side of the figure shows different positions of the overlapping square. The right hand side shows different dissection possibilities. Match the tiling arrangements on the left with the corresponding dissections on the right.

Investigation 2 Other Pythagorean dissections

We have explored different ways in which the Pythagoras theorem can be proved by dissection. In this investigation, we are going to explore a general version of this problem. Recall that the Pythagoras theorem in the standard version of the theorem speaks of squares drawn on the sides of the right angle triangle. Have you ever wondered why only squares have to be drawn? Can the figures drawn on the sides of the right angle triangle be different from squares? Let us modify the Pythagoras theorem by replacing all the squares in the theorem with equilateral triangles. Would the modified Pythagoras theorem still be true?

Theorem 2.1 In any right angle triangle the equilateral triangle on the hypotenuse is equal to the sum of the equilateral triangles on the sides forming the right angle.

Figure 2.1 shows equilateral triangles drawn on the three sides of a right angle triangle. Can you prove theorem 2.1 or show that it is in general false?

It is easy to find the formula for the area of an equilateral triangle. The height h of the triangle is given by the Pythagoras theorem in the standard version. (See Figure 2.2.)

$$\begin{aligned} h &= \sqrt{a^2 - \left(\frac{a}{2}\right)^2} \\ &= \sqrt{a^2 - \frac{a^2}{4}} \\ &= \sqrt{\frac{3a^2}{4}}, = \frac{\sqrt{3}a}{2} \end{aligned}$$

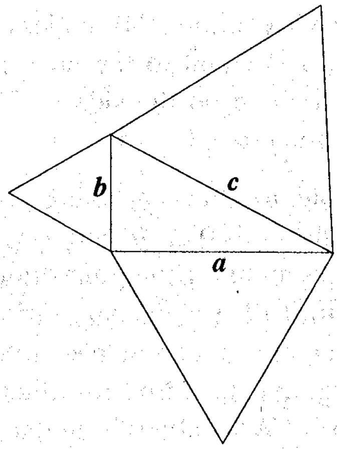


Figure 2.1: Right angle triangle with equilateral triangles drawn on the sides

$$\begin{aligned} \mathit{Area\ of\ an\ equilateral\ triangle} &= \frac{base \times height}{2} \\ &= \frac{\sqrt{3}}{4} \times a \times a \\ &= \frac{\sqrt{3}}{4} \times a^2 \end{aligned}$$

Hence,

$$\mathit{Area\ of\ an\ equilateral\ triangle} = \frac{\sqrt{3}}{4} \times \mathit{Area\ of\ a\ square}$$

We know from the Pythagoras theorem in the standard version (Theorem 1.1) that

$a^2 + b^2 = c^2$

Multiplying throughout by $\frac{\sqrt{3}}{4}$ we get,

$\frac{\sqrt{3}}{4}a^2 + \frac{\sqrt{3}}{4}b^2 = \frac{\sqrt{3}}{4}c^2$

from which it follows that the sum of the areas of the two equilateral triangles on the sides a and b is equal to the area of the equilateral triangle on the side c. Hence Theorem 2.1 is proved.

It is possible to think of other figures drawn on the sides a, b and c. Figure 2.3 shows regular pentagons drawn on the three sides of a right angle triangle. Is the Pythagoras theorem also true for these pentagons? Is it true for other regular polygons? What about irregular polygons?

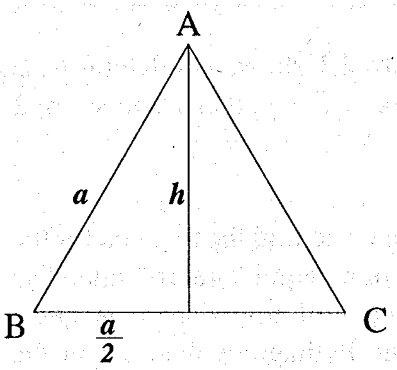


Figure 2.2: Equilateral triangle

Any polygon, regular or irregular, can be cut up into triangles. Figure 2.4 shows how this can be done by drawing all the diagonals of the polygon from one of the vertices. If two polygons are similar then it is not difficult to show that for each pair of corresponding triangles into which the polygon can be dissected, the triangles are similar.

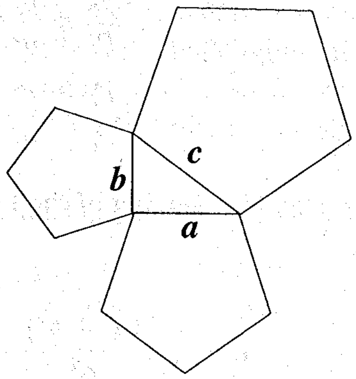


Figure 2.3: The Pythagoras theorem with pentagons on the sides of the right angle triangle

To prove this start with a triangle containing two sides of the polygon with the included angle as one of the internal angles of the polygon (△ABC and △A′B′C′ in the figure). These two triangles are similar to each other since the two pairs of sides {AB, A′B′} and {BC, B′C′} are in the same proportion and the included angle is the same. Now consider △ACD and △A′C′D′. Two pairs of sides, {AC, A′C′} and {CD, C′D′} are similar. The included angles ∠ACD and ∠A′C′D′ are equal, being the difference of the angle in the polygon and an angle of the first pair of triangles that we have already seen to be similar. Hence △ACD and △A′C′D′ are similar. In this way we can show that in each pair of corresponding triangles in the two polygons, the triangles are similar.

If two triangles are similar, the ratio of their heights is equal to the ratio of their sides. Hence the ratio of their areas is the square of the ratio of their sides. Since two similar polygons can be broken up into similar triangles, we can conclude that the ratio of the areas of similar polygons is the square of the ratio of their sides.

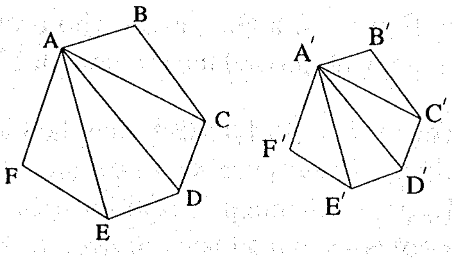


Figure 2.4: Similar polygons

Now imagine that similar polygons have been drawn on the three sides a, b and c of a right angle triangle, where c is the hypotenuse. Let P_a, P_b and P_c be the areas of these polygons respectively. We know that

$$\frac{P_b}{P_a} = \frac{b^2}{a^2} \quad \text{or} \quad P_b = P_a \times \frac{b^2}{a^2}$$

$$\frac{P_c}{P_a} = \frac{c^2}{a^2} \quad \text{or} \quad P_c = P_a \times \frac{c^2}{a^2}$$

From the standard version of the Pythagoras theorem $c^2 = a^2 + b^2$. Substituting, we have

$$P_c = P_a \frac{(a^2 + b^2)}{a^2} = P_a \times 1 + P_a \times \frac{b^2}{a^2} = P_a + P_b$$

Thus we have a general version of the Pythagoras theorem for similar polygons.

Theorem 2.2 In any right angle triangle, if similar polygons are drawn on the three sides, the area of the polygon drawn on the hypotenuse is equal to the sum of the areas of the polygons on the sides forming the right angle.

In fact, the Pythagoras theorem need not be restricted to polygons. The theorem would hold true for any set of similar figures composed of straight lines or curves, which are drawn on the three sides of the right angle triangle.

Since our main interest is in dissections, and since we will restrict our attention to polygons, we can now ask the question whether it is possible, in general, to cut the polygons on the sides forming the right angle and assemble the pieces to form the hypotenuse. A well known and powerful theorem, which was proved independently by two mathematicians is of relevance to us. The mathematicians were F. Bolyai, a Hungarian who proved the theorem in 1832 and P. Gerwien, a German, who proved the theorem in 1833. Let us state this theorem without proof.

Theorem 2.3 (Bolyai-Gerwien theorem) If there are two polygons of equal area, then it is always possible to cut one of the polygons into a finite number of pieces and reassemble the pieces to form the second polygon.

Is this theorem useful for the dissection of similar polygons drawn on the sides of a right angle triangle? We could always join the two smaller polygons drawn on the sides forming the right angle in any way we liked and get a joint or composite polygon. The Bolyai-Gerwien theorem is valid for polygons of any arbitrary shape. So we can cut this composite polygon into a certain number of pieces and assemble the pieces to form the polygon on the hypotenuse. However the theorem does not say anything about how many pieces we have to cut the first polygon into. In general, the number of pieces may be quite large. It is very challenging problem to find a dissection which involves the minimum number of pieces. We will call such dissections elegant. By an elegant dissection, we mean a dissection which is

1. valid for a right angle triangle of any size having polygons similar to the ones on the given right angle triangle,
2. which involves as few pieces as possible.

Condition 1 means that once we have found a way of dissecting the polygons for a given right angle triangle, we can change the lengths of the sides a and b of the triangle in any way we please. The dissection would still work and we don't have to search for a new dissection. Condition 2 is an incomplete condition since we do not know in advance what the minimum number of pieces required for the dissection is. However this is not an obstacle, rather provides a perennial challenge and great dissectionists have been spurred on to "go one better", that is, to reduce the number of pieces in a known dissection by finding a new one.

We will call an elegant dissection which transforms the two polygonal figures on the sides containing the right angle into the polygon drawn on the hypotenuse a Pythagorean dissection. Finding a Pythagorean dissection for any polygon, even for regular polygons is in general a very hard problem. We will explore two kinds of polygons below for which Pythagorean dissections can be found relatively more easily.

PYTHAGOREAN DISSECTION OF RECTANGLES

It turns out that we can use the tiling idea that we used for the Pythagorean dissection of squares also for rectangles. Let us draw rectangles on the three sides of a right angle triangle. Remember from theorem 2.2 that the three rectangles must be similar. That is, the ratio of the length and width must be the same for all the rectangles. Figure 2.5 shows similar rectangles drawn on the sides of a right angle triangle. We will call the sides common to the right angle triangle and the rectangles, the widths of the rectangles. So the width of the three rectangles are a, b and c respectively. For each rectangle let the height be k times the width. If k is more than 1 we have tall rectangles on the sides since the heights of the rectangles will be more than their widths. If k is less than 1, the heights will be less than the widths. The area of the three rectangles will be ka^2, kb^2 and kc^2 respectively. Following the convention that we have adopted we will call the rectangle with width a the α rectangle, the rectangle with width b the β-rectangle and the rectangle with width c the Γ-rectangle.

Let us now go back to the tiling arrangement that we used in Investigation 1 and try to make it work for rectangles. In the tiling arrangements the β-square is at the centre and the four α-squares are around it. In the same way let us place the β-rectangle in the centre and four of the α-rectangles around it as in Figure 2.6.

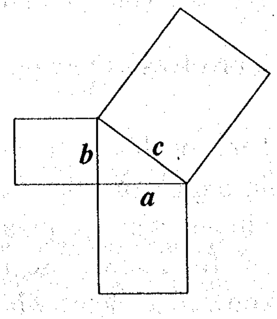


Figure 2.5: Rectangles drawn on the right angle triangle

In Investigation 1 the Γ-square was placed on the tiling arrangement so that its corners fell on the centres of the four α-squares. This is important because these four points are congruent points of the tiling arrangement. If we shift the Γ-square without rotation, its corners fall on congruent points of the tiling. We need to check if this happens with the tiling arrangement of Pythagorean rectangles.

Check if in Figure 2.6 the distance between the centres of two of the α-rectangles is equal to the length or to the width of the Γ-rectangle. That is, is the distance equal to c or to kc?

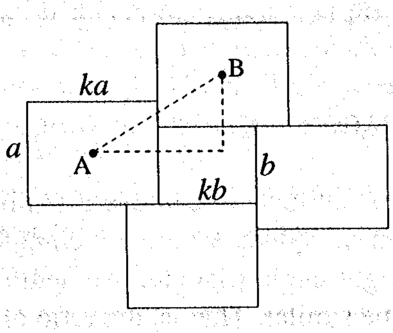


Figure 2.6: A tiling arrangement of rectangles

From the figure we find that the horizontal distance between the centres of the two α-rectangles is $\frac{ka}{2} + \frac{ka}{2} = ka$. If the small β-rectangle was not present in the centre of the tiling, the centres of the two α-rectangles would be on the same horizontal line. The β-rectangle pushes the centre up vertically by a distance b. So the vertical distance between the centres of the α-rectangles is b. From the Pythagoras theorem we can write down the distance between the two centres as $\sqrt{k^2a^2 + b^2}$. This is not equal to either kc or c.

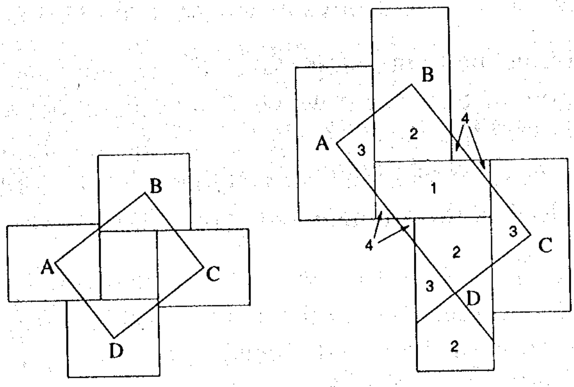
Look at the expression inside the square root in the previous paragraph. What should the terms be if we want the square root to reduce to kc? Clearly we need to have k^2b^2 instead of b^2. Can we change the arrangement of the rectangular tiles so that we have kb as the vertical distance? Don't be in a hurry to turn the page over and look at the arrangement. You will surely be able to find it.

If you have found the correct arrangement, study it. Do all the four corners of the Γ-rectangle fall on the centres of the four α-rectangles? Does this give you a dissection of the α and β-rectangles into the Γ-rectangle? In order to check this you will have to first find a way of cutting up the α-rectangle as suggested by the tiling arrangement. (Hint: Draw lines to extend the edges of the Γ-rectangle in the tiling arrangement.) Next you will have to show that the pieces in the Γ-rectangle are congruent to the pieces that you have cut the α-rectangle into.

Just as we did for the tiling arrangement for squares, move the Γ-rectangle to different points and see if each position yields a different dissection. Do the four corners fall on congruent points in the α-rectangles in these different positions?

Another question to explore is what happens when the relative sizes of a and b change? What happens as b becomes bigger, equal to a and then bigger than a? Does the tiling arrangement still give you valid Pythagorean dissections of rectangles? What happens if you change the value of k, the ratio of the length to the width of the rectangles? The figure on the following page also the tiling arrangement for some long rectangles.

We have seen that by means of a tiling arrangement, we obtained a successful Pythagorean dissection of rectangles just as we did for squares. Would tiling arrangements work for other polygons? One of the conditions for the tiling arrangement to yield a successful dissection is that the corners of the Γ-rectangle falls on congruent points of the α-rectangles. Unfortunately, this is not the case for other polygons. Hence the tiling arrangement does not help us in finding Pythagorean dissections of other polygons, even other regular polygons. This does not mean however that we cannot find dissections for other polygons as we shall see in the next section.



Tiling arrangements for the Pythagorean dissection of rectangles. Note the different arrangement for long rectangles.

PYTHAGOREAN DISSECTION OF EQUILATERAL TRIANGLES

Let us go back to Figure 2.1 which we drew at the beginning of this investigation. The figure shows equilateral triangles drawn on the three sides of the right angle triangle. We know that the sum of the areas of the equilateral triangles on the two sides forming the right angle is equal to the area of the equilateral triangle on the hypotenuse. Can we find a dissection of the two smaller equilateral triangles into the equilateral triangle on the hypotenuse?

We made a brief remark that the tiling patterns and arrangements do not yield a suitable dissection. We will leave it to you to check whether this is indeed true. A simple and elegant dissection of the equilateral triangles was published by Alfred Versady, a Hungarian, in 1989. Greg Frederickson, in his classic book on dissections, writes of this dissection: "it is humbling to wonder how this dissection was discovered". It is not always possible to follow a technique or a recipe or a thumb rule to make discoveries in mathematics. There are innumerable instances of discoveries which simply are brilliant and one fails to understand how the discoverer found them. And here is another interesting fact: Versady is not a mathematician. He is a technical draftsman and technical designer who lives in a small village called Metten in Hungary. He has made several brilliant discoveries concerning dissection problems.

Dissection problems are simple to understand and yet challenging and stimulating. That perhaps explains why, in this area of mathematics, amateurs still continue to compete with and often surpass professional mathematicians. Frederickson's book mentions a number of these amateurs who have contributed to geometric dissection problems. Engineers, draftsmen, designers, architects and artists jostle with mathematicians, physicists and computer scientists in the book. Dissection problems received repeated discussion in Martin Gardner's famous column 'Mathematical games' and became very popular. In the 1960s when Martin Gardner was writing about dissection problems, the world's leading expert on dissections was Harry Lindgren. Lindgren started his career as an electrical engineering draftsman and worked as a patent examiner of electrical specifications in Australia when he became the world's expert on dissections. Harry Lindgren wrote about the dissections that he discovered in another classic *Geometric Dissections* published in 1964.

Let us study the Pythagorean dissection of equilateral triangles discovered by Versady. Figure 2.7 shows the three equilateral triangles of sides a, b and c. It is convenient once again to refer to these triangles as the α, β and Γ-triangles respectively. The α and the β triangles are arranged edge to edge so that one of their vertices coincides at B. The vertex of the Γ-triangle coincides with another vertex of the β-triangle at D. The top edge of the Γ-triangle intersects an edge of the α-triangle at J which is the mid-point of the side AC. To show that the dissection is indeed correct we need to show that the pieces marked with the same numbers are congruent to each other.

The pieces numbered 1 and 2 are inside the Γ-triangle as well as inside the α or β-triangle. Compare $\triangle$DBG and $\triangle$DKH in Figure 2.7. These triangles are composed of the pieces numbered 3 and 4. $\angle$DBG and $\angle$DKH are both equal to 120°. $\angle$BDG is equal to $\angle$KDH since they are both equal to the difference of 60°and $\angle$NDK. The segments DB and DK are equal. Hence $\triangle$DBG $\cong$ $\triangle$DKH. By choosing the point M on DH such that DN = DM, we can ensure that the pairs of pieces formed by the numbers 3 and 4 are pairs of congruent triangles.

It is easy to show that the two remaining pairs of triangles (numbered 5 and 6) are pairs of similar triangles. (It is more difficult to show that they are congruent.) In Figure 2.7 consider the triangles numbered 5, $\triangle$JFL and $\triangle$HCL. $\angle$F and $\angle$C are each equal to 60°. The vertically opposite angles $\angle$HLC and $\angle$JLF are equal. So the two triangles are similar. By similar reasoning we can show that $\triangle$IEG is similar to $\triangle$IAJ. These are the triangles numbered 6.

We know that the area of the Γ-triangle is equal to the sum of the areas of the α and β-triangles. The pieces marked 1 and 2 are common to the Γ as well as the other triangles. We have seen that the pieces marked 3 are congruent and hence equal

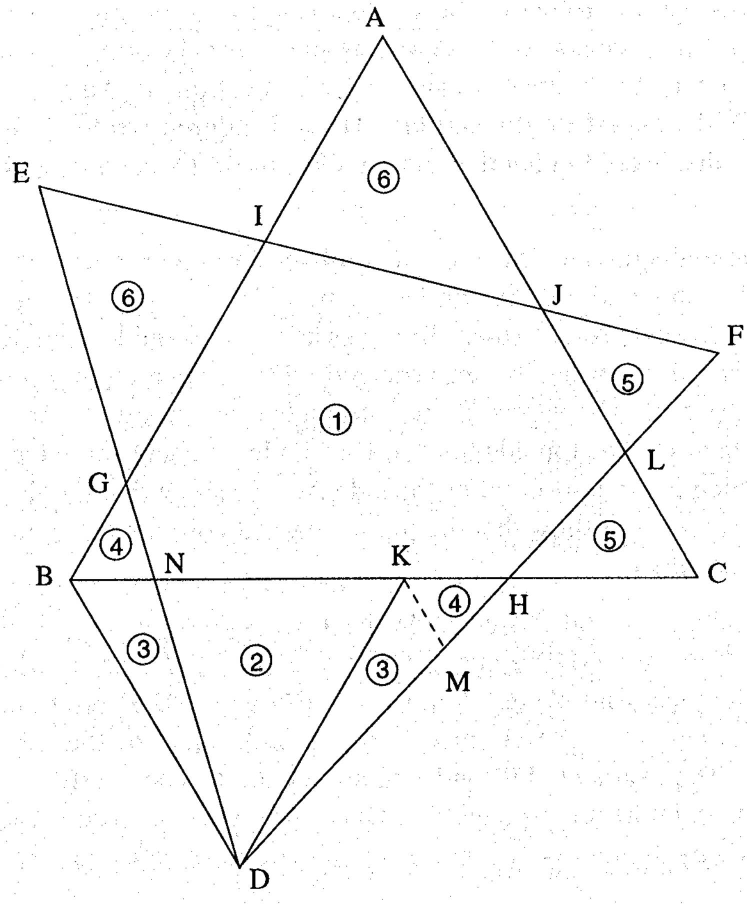

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Figure 2.7: Versady's Pythagorean dissection of equilateral triangles

in area. Similarly the pieces marked 4 are congruent and hence equal in area. It follows that the total area of the pieces marked 5 and 6 in the Γ-triangle is equal to the total area of the pieces marked 5 and 6 in the α and β-triangles. We will use this fact later on.

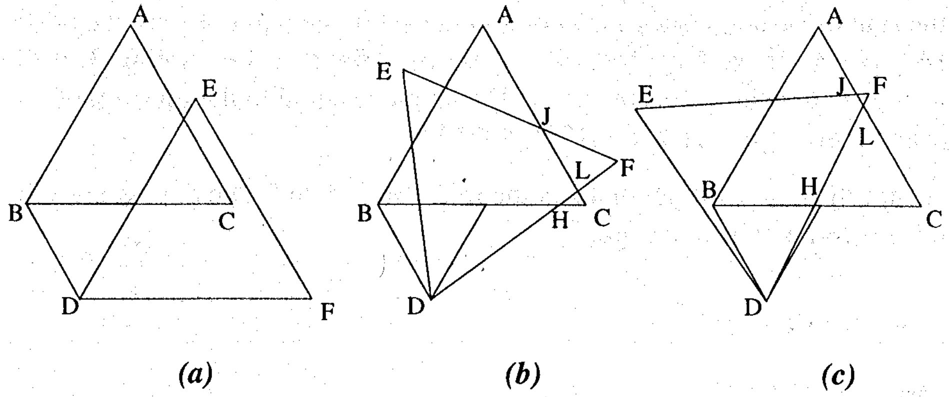

Figure 2.8: Rotating the Γ-triangle

How do we show that the pieces marked 5 are congruent to each other? We will show this indirectly. Let us keep the Γ-triangle with one of its vertices coinciding with D and one of its edges horizontal as in Figure 2.8(a). Now rotate the Γ-triangle anti-clockwise about the point D. The horizontal edge of the Γ triangle rises up to meet the base of the α-triangle. As the rotation is increased a small triangle, △JFL, begins to form over the edge AC as in Figure 2.8(b). Similarly a triangle, △HLC, forms over edge DF. In order for these triangles to form we need to show that DF is always greater than the distance DC. This in fact is true whatever be the relative proportions of a and b. We will leave the proof of this to you.

Compare the two triangles △JFL and △HCL. ∠F and ∠C are each equal to 60°. The vertically opposite angles ∠HLC and ∠JLF are equal. So the two triangles are similar. As we increase the rotation the areas of these two similar triangles change. Notice that in the beginning as the triangles are forming △JFL is bigger in area than △HCL. But after some rotation just before the point F crosses AC, △JFL is clearly smaller in area than △HCL. Since the rotation changes the areas continuously, we can assume that there is at least one point of the rotation where the areas of the triangles JFL and HCL are equal. Let us find out how many such points can be found.

Suppose that the area of $\triangle$HCL is equal to the area of $\triangle$JFL. We know that these two triangles are similar. Similar triangles can be equal in area only if they are congruent. Hence $\triangle$HCL $\cong$ $\triangle$JFL.

Now refer back to Figure 2.7. Each pair of corresponding pieces in the Γ-triangle and in the α and β-triangles are now congruent except the pair consisting of $\triangle$IEG and $\triangle$IAJ. However we have seen that these two triangles are similar. But their areas must be equal since the area of the Γ-triangle is equal to the the sum of areas of the α and β-triangles. Hence $\triangle$IEG $\cong$ $\triangle$IAJ.

Now it is not difficult to show that J is the mid point of AC. The details are left as an exercise for you. The steps are,

1. Show that EG = HF.
2. Show that HF = JC. Hence EG = JC.
3. Show that AJ = EG.
4. Hence AJ = JC.

We see therefore that if $\triangle$HCL is equal in area to $\triangle$JKL, EF cuts AC at its midpoint J. This produces the dissection that we require. Since there is only one midpoint on AC it follows that as the Γ-triangle DEF rotates about point D, there is only one position where the where the areas of $\triangle$HCL and $\triangle$JKL are equal.

However as the Γ-triangle is rotated about point D, we find that the side EF intersects twice with the midpoint of AC which is J. One of these positions is the one in Figure 2.7. The other position can be easily found and it is given by the reflection of EF about an axis passing through the point J. It is interesting to ask whether the other position also yields a dissection. The answer is 'yes' and we shall see this in the next section.

AN ALTERNATIVE PYTHAGOREAN DISSECTION OF THE EQUILATERAL TRIANGLE

Figure 2.9 shows the path traced by the point F as the Γ-triangle rotates about point D. The path is a circle with centre at D and radius equal to the side DF. The point E also moves on this circle. We can see that EF is a chord of the circle. E′F′ is

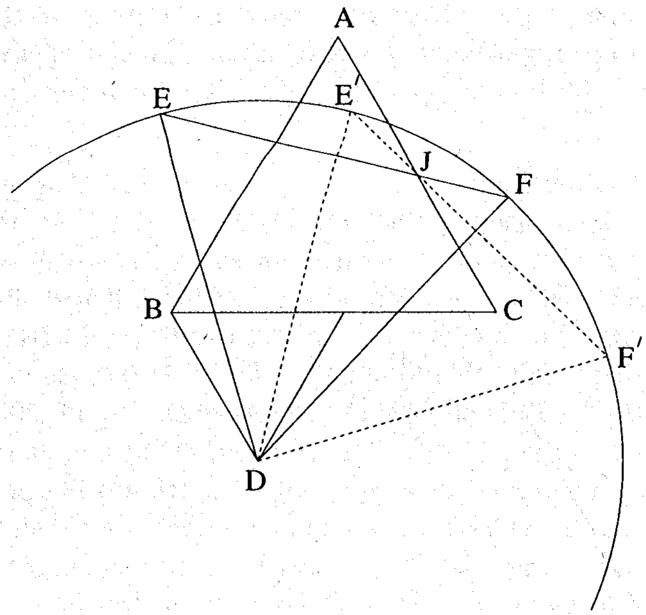

Figure 2.9: The path traced by points E and F

the reflection of this chord about a radius drawn through the point J and is hence equal to EF. The position of the Γ-triangle with its vertices at E′ and F′ is shown by dotted lines in Figure 2.9. This is clearly the other position in which the side EF of the Γ-triangle intersects AC at its midpoint J. Let us now examine whether this position yields a dissection of the α and β-equilateral triangles into the Γ-triangle.

Figure 2.10 shows how this dissection is possible. We need to make some constructions in order to obtain the right pieces for the dissection. The line segment BC is extended till it meets segment DF in I. The line segment DE is extended till it meets AC in H. The line segment EG is constructed such that ∠DEG = 60°. Mark point P on EF so that JP = JE. Mark point L so that JL = JH. Draw the segment PL. This dissection hence has 6 pieces like Versady's dissection. We now have to show that the pieces marked with the same numbers are congruent to each other.

The pieces numbered 1 and 2 are inside the Γ-triangle as well as inside the α or β-triangle. Next compare the quadrilaterals DBGE and DKIF which are made up of the pieces numbered 3 and 4. ∠DEG and ∠DFI both are equal to 60°. ∠DBG and ∠DKI are both equal to 120°. ∠BDE and ∠KDF are both equal to the difference of 60°and ∠NDK. It follows that the remaining angles in the two quadrilaterals

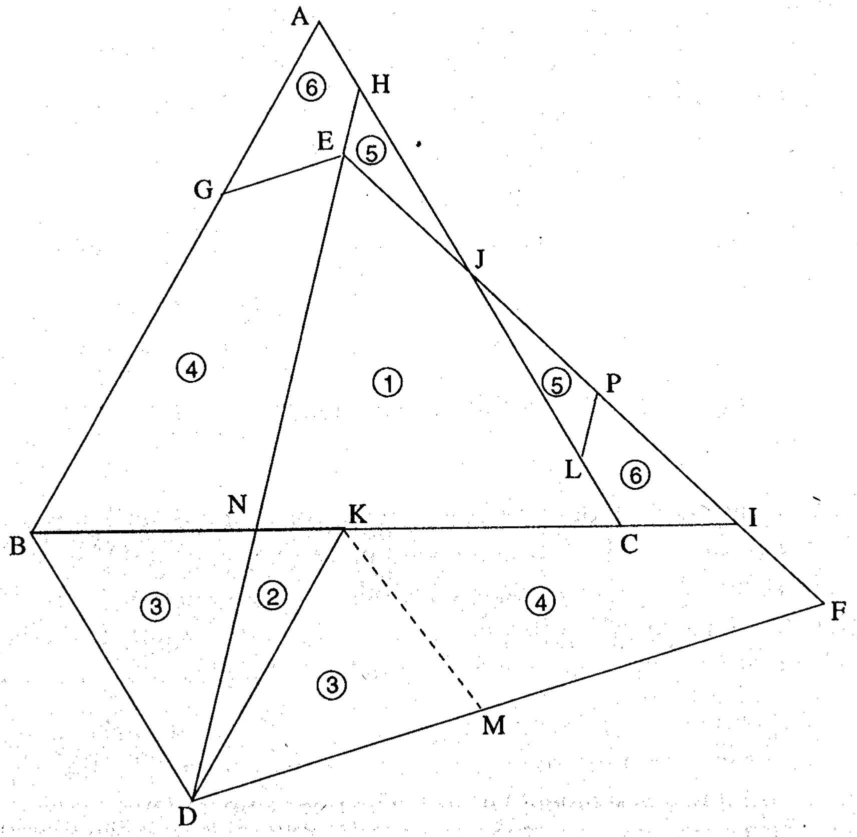
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Figure 2.10: An alternative Pythagorean dissection of equilateral triangles

∠BGE and ∠KIF are equal. Further the segment BD = DK. Therefore quadrilateral DBGE is congruent to quadrilateral DKIF. By choosing the point M so that DN = DM, we can ensure that the two pairs of pieces numbered 3 and 4 are pairs of congruent figures.

Next consider the pieces numbered 5 which are the triangles △JEH and △JPL. By construction JP = JE and JL = JH. The vertically opposite angles ∠PJL and ∠EJH are equal. Hence the two triangles are congruent. Now compare the quadrilaterals EGAH and CIPL which are the pieces numbered 6. ∠GEH and ∠ICL are both equal to 120°. ∠HEJ is equal to 120°since it is an external angle of △DEF. ∠LPJ = ∠HEJ = 120°since △JEH ≅ △JPL. ∠IPL forms a linear pair with ∠LPJ and hence ∠IPL = 60°. In the quadrilaterals EGAH amd CIPL, ∠GAH = ∠IPL = 60°. ∠AGE is congruent to ∠PIC as they form linear pairs respectively with congruent angles ∠EGB and ∠FIK which are interior angles of congruent quadrilaterals. We have seen that three of the angles in quadrilateral EGAH are equal respectively to three of the angles in quadrilateral CIPL. Hence the fourth pair of angles are also equal and the two quadrilaterals are similar. We know that the areas of the two quadrilaterals are equal since they are the last pair of pieces which are part of the α and the Γ-triangles. Similar quadrilaterals can have equal area only if they are congruent. Hence the two quadrilaterals EGAH and CIPL are congruent. So all the pieces marked with the same numbers are congruent to each other.

We have seen two possible Pythagorean dissections for equilateral triangles. Although the dissections themselves were simple, discovering them or proving that the dissections were correct, that is, that the pieces were congruent, was not so simple. In general, it is a hard problem to find Pythagorean dissections for even simple figures such as parallelograms or regular polygons with a small number of sides like the pentagon and the hexagon. We are not aware of Pythagorean dissections for these figures. We invite you to try and solve these problems and discover new dissections.

Investigation 3 Exploring Graphs

The word 'graph' is certainly familiar to you. It probably reminds you of functions like $y = x^2$ or $y = 3x + 2$, which you can plot on graph paper by drawing the coordinate axes. It might also remind you of bar graphs which we use to present data. The graphs that we are going to explore are very different from these graphs. Here are some examples of graphs of the kind we will explore in this investigation.

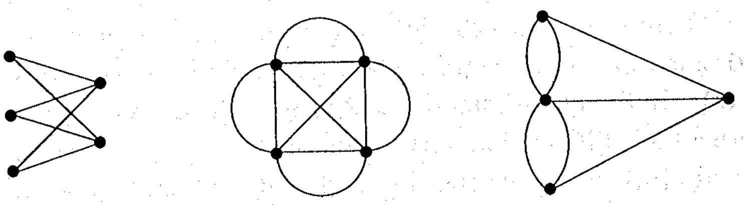


Figure 3.1: Some graphs

These kinds of graphs are sometimes called networks. Like the other kinds of graphs, these also are pictorial representations of some kinds of data. They are also mathematical structures with interesting properties. Graph theory is a branch of mathematics which studies the properties and applications of graphs or networks. As one can see from Figure 3.1, graphs essentially consist of points called **vertices** and lines that join two vertices called **edges**.

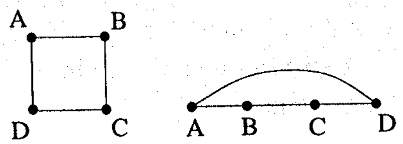


Figure 3.2: Isomorphic graphs

The exact shape of a graph is unimportant. What matters is which vertices are connected to which and by how many edges. For example in Figure 3.2 the two graphs shown are the essentially the same. The square graph has been stretched and squashed into a line. When we name the points appropriately we find that each point is connected to the same points in both graphs. A fancy way of saying that the graphs are the same is to say that they are **isomorphic** (iso = same, morphe = form). The graphs in Figure 3.3 are isomorphic with the graphs in Figure 3.1. Match the graphs which are isomorphic. One way to check if two graphs are isomorphic is to twist parts of the graph around in your imagination without breaking any of the connections. Another way is to find the correct way of naming the vertices, so that each vertex is connected to the same vertices in both graphs. Find the correct way of naming the vertices for each pair of isomorphic graphs in Figures 3.1 and 3.3.

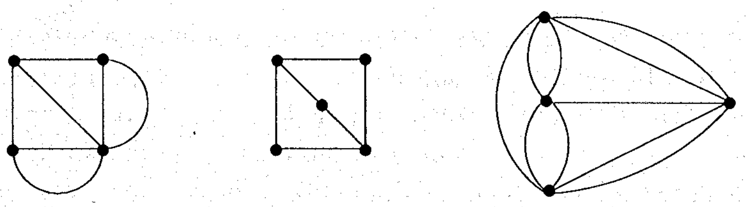


Figure 3.3: Can you recognize these graphs?

What kinds of data can graphs represent? Let us consider an example from cricket. There are six teams – a, b, c, d e and f playing in the world cup league. At a certain stage we have the following information about the matches that have been held.

a has played c, d and f.
b has played c, e and f.
c has played 2 matches.
d has played 3 matches.
e has played 3 matches.
f has played 4 matches.

It is convenient to draw a graph to represent this data. We draw the graph with 6 vertices shown in Figure 3.4. When two teams have played a match we join them by an edge. If we start by marking the 6 vertices on plain paper, you will find that it is easy to reconstruct all the information about which teams have played each other. Also from the graph it is very easy to answer other questions such as which matches have not yet been played.

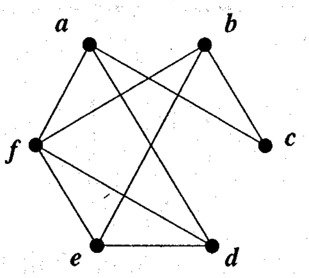


Figure 3.4: The world cup league

In this case we do not really need to draw a graph to answer questions about the tournament. As we proceed with exploring more problems and puzzles we will discover how useful it is to draw graphs to represent the data in the problems.

IMPOSSIBLE PATHS

Many school children are familiar with the diagram shown in Figure 3.5. The problem is simple — trace the diagram shown in one continuous line, that is, draw the diagram without lifting the pen and without tracing the same edge more than once. Try it out. The problem is frustrating and appears to be impossible. The solution lies in a trick. You fold a corner of the paper over one of the edges and move the pen over the folded paper. By doing this you are effectively retracing an edge but not actually drawing your pen over it. This of course is cheating.

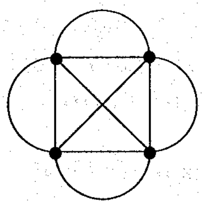


Figure 3.5: Tracing a graph

Now that you have seen some graphs in Figure 3.1, you would recognize that the diagram in the puzzle is a graph. Is it possible to trace the graph in one continuous line without cheating? If it is not possible, can one prove that it is impossible? Sometimes it is a useful technique in mathematics to change the conditions of a problem to see what effect it has on the solution. Let us change the graph a little and see if it can be traced in one continuous line. The graph has a square with diagonals and four 'ears' drawn round the sides. Rub out one of the ears of the graph and check whether it becomes possible to trace it. What happens if you rub out two ears? What happens if you rub out all the ears one by one? Check which of these graphs can be traced continuously. The graphs are shown in Figure 3.6. Can you find anything common among the graphs that can be traced?

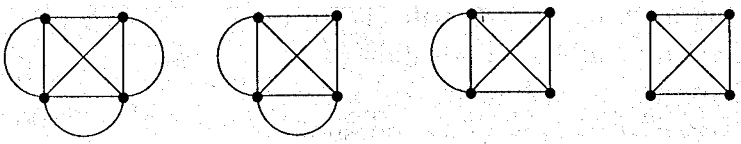


Figure 3.6: Which of these graphs can be traced?

If you have found out which can be traced and which cannot, you have probably made a hypothesis that symmetry has something to do with being able to trace a graph. But this is just an artifact of the graphs that we have chosen. It so happens that the graphs which look symmetric in Figures 3.5 and 3.6 cannot be traced and the graphs which can be traced are not symmetric. Just to resolve this question, try tracing the graphs in Figure 3.7. There is a symmetric graph which can be traced and an asymmetric one that cannot be. Try to find other symmetric graphs which can be traced and asymmetric ones which cannot be traced.

Think of the vertices of the graphs in the tracing problems as cities and the edges leading to them as roads. Suppose you are driving through the graph in a bus. You can enter a city more than once, but every time you must enter it and leave it through a different road. How many times can you then enter and leave the city? Every time you come to a city, you need a pair of roads – one road to enter it and one road to leave. What happens if the number of roads leading to a city is odd? You can enter it but you cannot leave! Is there any other possibility? Think about vertices with odd and even number of edges connected to them and you will figure out why some of the graphs in Figure 3.6 can be traced and why some of them cannot be.

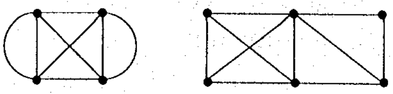


Figure 3.7: More graphs to trace

Let us look once again at some of these graphs. In Figure 3.8, the vertices have been named. Try to trace the graph by using different starting points. Start with A first and check if the graph can be traced. Then try starting with B, C and D in turn. Also make a note of the ending point each time. What can you say about the starting and the ending points? Examine the starting and ending points in the symmetric graph in Figure 3.7. How do you explain the coincidence of the starting and ending points?

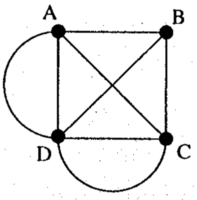
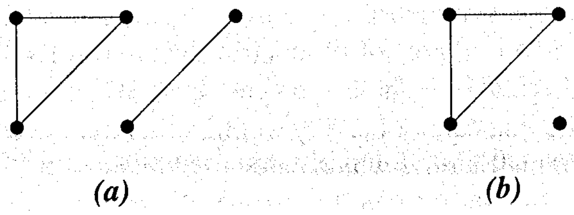


Figure 3.8: Different starting points

The number of edges which are connected to a vertex is called the **degree** of the vertex. When the degree of a vertex is zero, there are no edges connecting it with any other vertex, so the vertex is **isolated**.



(a) *(b)*

Figure 3.9: A graph which (a) is disconnected and (b) contains an isolated vertex

The solution to the tracing problem depends on **odd** and **even vertices**, that is, on whether the degree of a vertex is odd or even. You have probably discovered that a graph can have utmost two odd vertices to be continuously traceable. A graph with all vertices even can also be traced, like the one in Figure 3.7. What about a graph with only one odd vertex? First try and draw such a graph and then check if it is continuously traceable.

There is a very simple **relation between the degrees of the vertices and the number of edges**. The degree of a vertex is how many edges are connected to it. So the sum of the degrees of all vertices gives the sum of all the connections that edges make with vertices. Each edge is connected to two vertices, so has two connections. It follows that the sum of the degrees of all vertices is twice the number of all edges. In the light of this relation, can the sum of the degrees of all vertices be odd? What can you then say about the number of odd vertices in any graph?

A famous graph tracing problem lies at the historical origins of graph theory. The problem, well known as the Koenigsberg bridge problem, is only superficially different from the tracing problem but is essentially the same. Figure 3.10 is a simplified map of the bridge at Koenigsberg. There are two islands in the middle of the river Pregel. Seven bridges connect these islands to the opposite banks of the river and to each other. The amusement that the citizens of Koenigsberg had devised for tourists (and for themselves) was to walk

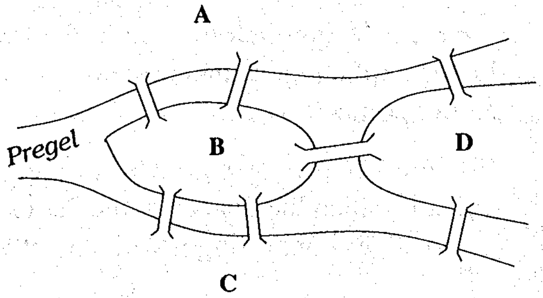


Figure 3.10: A schematic map of Koenigsberg

over all the bridges exactly once. As a tourist who knows about graphs you would of course have an advantage. Draw a graph which represents the Koenigsberg bridges. What will you choose to be the vertices and what the edges? Can you say whether the walk is possible?

The great Swiss mathematician Euler was the first, as far as we know, to solve this problem. He lived in the 18th century and spent most of his working life in St. Petersburg in Russia (now Leningrad). St. Petersburg was not far from Koenigsberg. (Both the cities are on the coast north of Poland and are about 800 km apart. Koenigsberg is also in Russia now and is called Kalaningrad.) Euler showed that it was impossible to walk through the seven bridges exactly once, by arguing in a manner similar to ours. In doing so, he sowed the seeds of graph theory.

For the rest of this investigation, we are going to place some restrictions on the kinds of graphs that we will discuss. We will stipulate that two vertices of a graph cannot be connected by more than one edge. We have relaxed this constraint for the graphs that we have seen so far. We will stipulate further that a vertex cannot be connected with itself. (All the graphs we have seen so far actually satisfy this constraint.) Graphs which satisfy these two constraints are sometimes called simple graphs. Henceforth the word 'graph' will just mean a 'simple graph'.

These constraints also force some restrictions on the degrees of a vertex. For a graph with n vertices, any vertex can have a maximum degree of $n - 1$ since it can be connected only once to the remaining vertices. The minimum possible degree of a vertex in any graph is of course zero. Suppose we have a graph with 4 vertices. What is the maximum number of edges possible? Each of the 4 vertices can be connected to 3 other vertices. So the sum of the degrees of all the vertices is $4 \times 3 = 12$. This is twice the number of edges. So the maximum number of edges a graph with 4 vertices can have is 6. (Draw the graph and verify this for yourself.) In general a graph with n vertices can have a maximum of $\frac{(n)(n-1)}{2}$ edges. A graph having the maximum number of edges possible is called a **complete graph**.

DEGREE SEQUENCES OF A GRAPH

The scout camp puzzle: Boy scouts and girl guides from all over the country attend a fortnight long camp. They have thoroughly enjoyed themselves and made many new and close friendships. After the camp, seven members of a new group of friends decide to write letters. Some of the friends are more enthusiastic at writing letters. Two of them write 5 letters each, two write 4 letters each, two write 3 letters each and the last friend writes only one letter. When the letters are delivered, to everyone's surprise, they find that they have received letters from exactly the same people they wrote to.

Is this scenario possible and can you figure out who wrote to whom? Note that each friend writes no more than one letter to any person. Also nobody writes a letter to himself or herself.

It is possible to represent this puzzle in the form of a graph. Each friend is shown by a dot and an exchange of letters by an edge connecting two dots. Note that in a graph when two vertices are connected by an edge, the relation between them is symmetric. In our case this condition is satisfied since in every case where a friend writes a letter to another friend, he also receives a letter from the same friend. If he wrote a letter to a friend but did not receive one, the condition of symmetry would not be satisfied. (Of course, in such a case, there would be no puzzle to be solved.) The graph in Figure 3.11 is an example different from the puzzle. It shows the case of 4 friends writing 2 letters each and receiving letters from the same persons they wrote to.

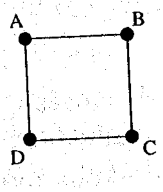
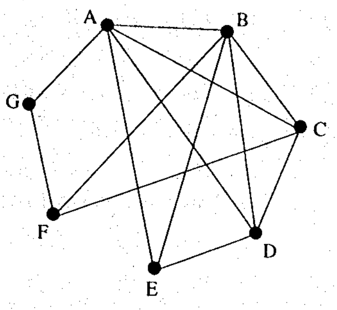


Figure 3.11: Solution to a simple puzzle: four friends write two letters each and receive letters from the same person they wrote to

The solution of the scout camp puzzle reduces then to drawing a graph that has 7 vertices. Each edge of the graph corresponds to each exchange of letters mentioned in the puzzle. Try and draw this graph. (There are several possibilities and one of them is drawn on the next page.)

In some cases it may be impossible to draw such a graph. Some of these cases are easy to identify quickly. It is impossible, for example, for any of the 7 friends to write 7 letters. At the most only six letters can be written since you are not allowed to write more than one letter to the same person or to write a letter to yourself. This corresponds to the condition that any vertex in the graph can have a maximum degree of 6.

Another case where the puzzle has no solution is the following. Let us suppose,



One of the possibilities for the scout camp problem. There are other possibilities too.

for example, that each of the seven friends writes one letter. It is impossible to have a graph with 7 vertices each of degree one. (Why? Think of how many odd vertices there can be.) This does not of course mean that the seven friends cannot write one letter each. It just means that the condition specified in the puzzle cannot be satisfied. It is not possible for each one of them to receive letters from exactly the same person to whom they wrote.

For similar reasons one can say that it is not possible for all the friends to write 3 letters each and receive letters exactly from the person they wrote to. What can one say about the following case: the friends write 5, 5, 3, 2, 2, 1, 1 letters respectively? Applying the same principle, that any graph must have an even number of odd vertices, shows that this graph too is impossible.

There are some cases which appear to have solutions but do not. Consider, for example, that the friends write 6, 5, 4, 4, 2, 2, 1 letters respectively. Here there are two odd vertices, and the maximum degree is 6. So it appears that the graph must be possible. However it turns out that this is not the case. Figure 3.12 shows an attempt to draw the graph for this case which has stopped half way. Of the 7 vertices, the first two A and B, have been shown correctly with degrees 6 and 5 respectively. The next vertex C must have a degree of 4. This is impossible if we re-

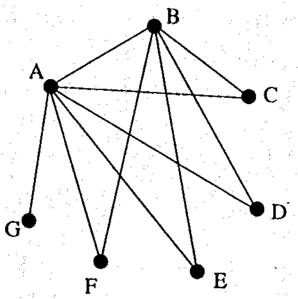


Figure 3.12: An incomplete graph showing an impossible condition for the puzzle

strict the degrees of the last three vertices to 2, 2 and 1.

How do we find out if it is possible to draw a graph for any case of the puzzle in general? If the number of vertices (friends) are fairly large, it would be very difficult to try and draw the graph by trial and error. Fortunately, a simple algorithm exists which allows us to find out if a certain distribution of letters is possible. This algorithm allows us to find out if a graph can be constructed for a given degree sequence, that is, whether a given degree sequence is graphic.

We first write down the degree sequence that we need to check making sure that it is in descending order. As an example let us consider the degree sequence 6, 6, 5, 5, 2, 2, 2, 2. Figure 3.13 shows how the algorithm can be applied to check if this degree sequence is graphic. The first step is to delete the first number in the sequence 6 and decrease the next six numbers in the sequence by one. In general, if the number we have deleted is k, we decrement the next k numbers by one. The same algorithm is then applied to the new degree sequence after ensuring that it is in descending order. We continue to apply the algorithm again and again till we arrive at a degree sequence which is simple enough for us to be able to say by inspection whether it is graphic. In Figure 3.13, after applying this algorithm three times, we arrive at the sequence shown at the end: 2, 0, 0, 0, 0. This sequence requires a graph with 5 vertices of which one vertex has degree 2, while all the other vertices are isolated. This is impossible. So the degree sequence 2, 0, 0, 0, 0 is not graphic. From this we conclude that the original degree sequence was also not graphic.

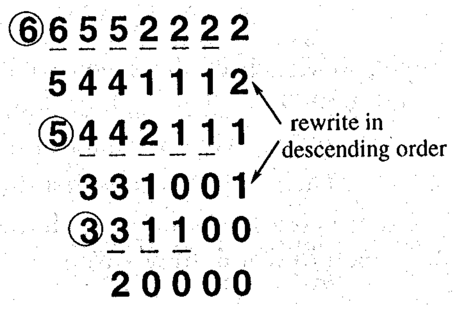


Figure 3.13: Applying the algorithm to check if a degree sequence is graphic

How does this algorithm work? When we delete the first number in the degree sequence, we are actually deleting a vertex. Since the degree sequence is in descending order, the vertex that we are deleting is the one with the highest degree in the graph. Suppose the degree of the deleted vertex is k. Along with the vertex, k number of edges which connect this vertex also are deleted. This we do by reducing the degree of the next k vertices by one. What remains is still a graph with one vertex and all the edges connected to this vertex removed. The algorithm works because for every graph it is possible to obtain a graph which is reduced by one vertex and the edges connected to it. The degree sequence of the reduced graph is obtained by the algorithm. If the degree sequence which results after applying the algorithm is not graphic, then it follows that the original sequence too was not graphic. Let us state this in the form of a theorem.

Theorem 3.1 If a given degree sequence (in descending order) is graphic then its reduced degree sequence is necessarily graphic. The reduced degree sequence is obtained by deleting the first number in the given (descending) sequence, say k, and decrementing the next k numbers in the sequence by one.

The theorem is of the form: if p is true then necessarily q is true. It follows therefore that if q is false, then p must be false. That is, if for any given sequence if the reduced sequence turns out not to be graphic, then the original sequence is also not graphic.

This form of the theorem is still insufficient to allow the use of the algorithm. If the reduced sequence is graphic, there is still no guarantee that the original sequence is also graphic. But this is easy to show. Add a vertex to the reduced graph and connect it to the first k vertices in the reduced degree sequence by edges. Now we have a graph which corresponds to the original degree sequence. Hence it is easily seen that if the reduced sequence is graphic, then the original sequence must necessarily be graphic. This is just the converse of Theorem 3.1. Although this converse of the theorem is necessary for the algorithm to work, we concentrate our attention on Theorem 3.1 whose justification we have not yet spelled out completely.

There is one assumption that we made in our explanation of the algorithm above. Did you notice this assumption? We assumed that the deleted vertex is actually connected to the next k vertices in the sequence. In such a case we could say that the graph is **sequenced**. A graph need not however be sequenced. Instead of being connected to the next k vertices, the deleted vertex could be connected to some other vertex, say, the mth vertex. So one of the next k vertices, say the ith vertex is not connected to v_1. Then the algorithm does not correspond to deleting all the edges connected to the vertex v_1. In fact, we will be deleting the wrong edge - one of the edges of the ith vertex while leaving all the edges of the mth vertex intact! What we are actually doing is worse. We are deleting one end of the right edge and one end of a wrong edge! We cannot leave half edges hanging in the graph. Does this mean that the algorithm can only be applied in cases where v_1 has degree k

and is connected only to the next k vertices?

Let us look at an example of an unsequenced graph and see what the effect of applying the algorithm is. Figure 3.14(a) shows the example graph that we have chosen. It has 6 vertices and its degree sequence is 4, 3, 2, 2, 2, 1. The first vertex of the sequence v_1 has a degree 4. It is not connected to the next vertex in the sequence which has degree 3 (v_2 in the figure), but is connected instead to the last one. This makes the graph unsequenced. Note that v_1 is also connected to v_3, v_4 and v_5. Will the algorithm work for this **unsequenced** graph?

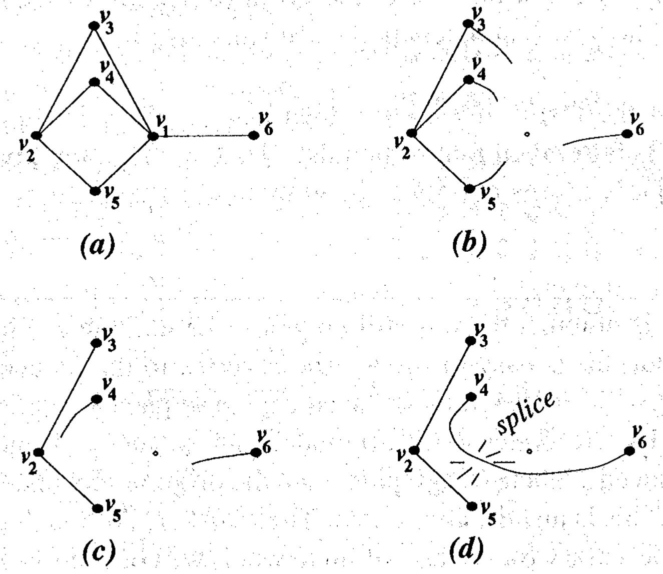


Figure 3.14: Repairing an unsequenced graph

Applying the algorithm to degree sequence we have chosen, we obtain the degree sequence 2, 1, 1, 1, 1. Let us see how this works on the graph itself. The first step starts with the deletion of vertex v_1. All the edges connected to this vertex are left dangling as in Figure 3.14(b). In step 2, one edge is deleted from each of v_2 through v_5 as in Figure 3.14(c). Now half an edge is left dangling from v_4 and half an edge from v_6. In Figure 3.14(d), the two edges which are dangling have been spliced together, thus leaving a graph which has the degree sequence 2, 1, 1, 1, 1. This is the degree sequence that we obtain by applying the algorithm.

We have been able, with the help of the splicing technique, to apply the algorithm for a particular unsequenced graph. The reduced sequence turned out to be graphic.

Will it be true for all unsequenced graphs that the reduced sequence obtained by applying the algorithm will be graphic? It is clear that whenever the deleted vertex is not connected to the next k vertices, we will have two half edges dangling. They can always be spliced together. The only time when this will fail is if the two vertices having broken edges are already connected by another edge. We are not allowed to have two edges connecting the same pair of vertices. So we cannot splice the broken edges together! Let us check whether this falsifies theorem 3.1.

When we wished to decrement the degree of v_2 in Figure 3.14(b) by deleting one of its edges, we chose to delete the edge connecting v_2 and v_4. v_4 was a rather handy vertex to have around since it was not already connected to v_6, and we had no problems in splicing the two half edges together. It is always possible to choose such an edge. This is because the degree of v_2 is higher than v_6 since it comes earlier in the degree sequence. So it is connected to some vertex that v_6 is not connected to. This vertex is v_4. So we chose to delete the edge joining v_2 to v_4.

Is it always possible to find a handy vertex like v_4 which is not connected to v_6? Let us rewrite this question in more general terms. Suppose that a graph is unsequenced. That is, to say v_1 of degree k, is not connected to v_i (one of the next k vertices), but is connected to v_m. Then deleting v_1 will leave a edge dangling at v_m and decrementing v_i will leave a edge dangling at some other vertex. Let this vertex be v_z. The question is – is it always possible to find a vertex v_z so that we can splice the two half edges together?

The answer to the question is 'yes'. This is so, simply because v_i appears earlier in the degree sequence than v_m. So its degree must be the same as v_m or higher. If it is the same, we just switch v_i and v_m and we obtain a sequenced graph. If its degree is higher then it must be connected to at least one vertex which is not connected to v_m. This is our vertex v_z.

We see therefore that theorem 3.1 holds true for an unsequenced graph. The only question that remains is whether it holds true if there are more than one unsequenced connections. But this does not pose any problem either. As we apply the algorithm and delete edges, we will always be able to splice the dangling edges one by one.

Graph theory is a relatively new branch of mathematics. Most of the work in graph theory has been done in the twentieth century. Theorem 3.1 was proved independently by two graph theorists, Havel in 1955, and Hakimi in 1962.

BIPARTITE GRAPHS

Bipartite graphs are a class of graphs where the vertices fall into two distinct groups. None of the vertices in a group is connected to another vertex in the same group. Vertices of one group always connect to vertices of the other group. Let us look at an example of such a graph.

The four fairy-tale musicians of Bremen, the donkey, the dog, the cat and the cock are having an ice cream treat in the thieves' den. The donkey likes strawberry, vanilla and tooti-frooti flavours. The dog likes chocolate and vanilla. The cat likes butterscotch, tooti-frooti, kesar pista and vanilla. The cock wants to taste all the flavours that the others like. Let us draw a graph of the ice cream treat.

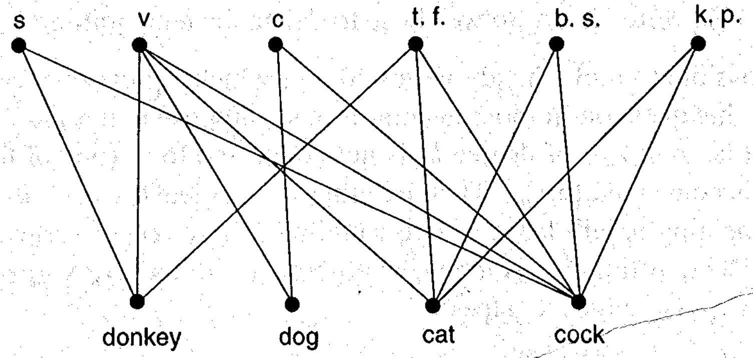


Figure 3.15: The ice cream treat

We see that the vertices fall into two distinct groups – those who eat and the things that are eaten. Obviously none of the musician friends are interested in eating each other!

There are many real life problems which can be characterized by bipartite graphs. (One of the graphs in Figure 3.1 and its isomorph in Figure 3.3, is a bipartite graph.) In the puzzle that follows, we will look at a class of bipartite graphs that have an especially interesting structure.

The black and white party puzzle: A friendly lady in the colony hosts an unusual fancy dress party for children. At the beginning of the party, she distributes a cap, a jacket and a pair of trousers to each child. All of these come

> in only two colours – black and white. She also makes sure that no two children are wearing the same combination of dresses.
>
> Each child gets three gold or three silver coins neatly wrapped up in colour paper. Before the party is over each child must exchange all of his or her coins. The rule is that the child must exchange a coin only with another child who has only one garment different from his or her own. At the end of the party the lady is puzzled. Although none of the children could see which coins they had, all of the silver coins have been switched with gold coins. She tries to remember how she had distributed the coins. Help her find out.

The first thing to do here is to get as much data as we can from the puzzle. There are three garments each in two different colours – black or white. How many different different dresses are possible? This tells us how many children there are.

If you have figured this out you can skip to the next paragraph. If you have not, think of how many dresses are possible with the white cap. You can have either a white jacket or a black one. With each kind of jacket you have a choice of white or black trousers. So that gives you $2 \times 2 = 4$ dresses with the white cap. Four more dresses with the black cap and that makes eight dresses. So eight children attend the party.

We also know that four of these children received gold coins and four received silver coins. At the end of the party, after all the exchanging was done, those who had gold coins ended up having all silver coins and vice versa. We now have to find out who had gold coins and who had silver. It is not immediately obvious how we must draw a graph for this puzzle. In fact, we will reserve that for later, for after we have solved the puzzle.

Let us work out a notation for showing what dress a child is wearing. If a child is wearing, say, a white cap, a white jacket and a black pair of trousers, we can write that down as wwb. What would the dress bwb be? To simplify the notation some more let us write 0 for black and 1 for white. Then we can write down all possible dresses in the following way;

000 001 010 011 100 101 110 111

Notice that the string contains all the binary numbers up to the third place starting from zero. The biggest three place binary number is **111** which equals 7. We can call these 'binary number names' since they stand for each combination of dress that the children are wearing. Let us assume to begin with that the child wearing the dress 000 (all black) had three gold coins. She must exchange each of her coins with somebody who has only one garment different from hers. Find who these three other children are.

These three other children must all have had silver coins, because at the end of the exchange child 000 had only silver coins left. Each of the other children exchanged one silver coin with child 000. With whom did they exchange their other silver coins?

When you have worked this out you will probably have found all the children who received gold coins and all those who received silver coins. Write them down as two lists under 'Gold' and 'Silver'.

Now we can draw a bipartite graph like the one in Figure 3.16(a). On the left, the two vertices represent all the children who received gold coins at the beginning and on the right are the children who received silver coins to begin with. Each edge represents an exchange of coins. Fill in the appropriate binary number names for each of the vertices. Convince yourself that the solution to the puzzle is correct

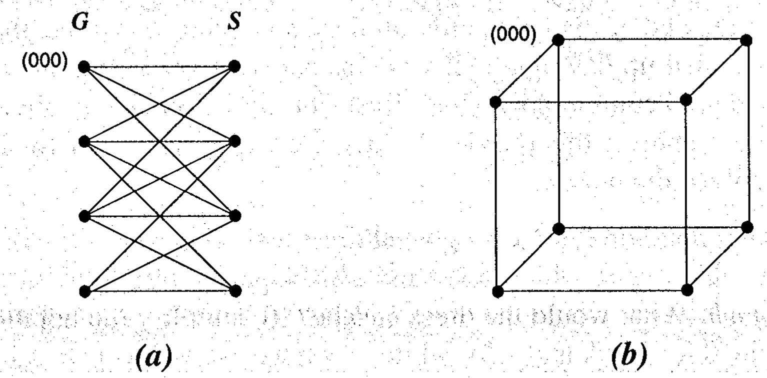


Figure 3.16: The graphs for the black and white party problem

The graph for this puzzle can be shown in a far more interesting way than Figure 3.16(a). Figure 3.16(b) shows a cube or rather a wire-frame model of a cube. Notice that one of the vertices of the cube is marked (000). Mark the other vertices taking care to see that connected vertices have the same digits in two of their places. For instance the vertices (000) and (001) are connected since they differ only in the 3rd place. The vertices (001) and (010) are not connected although their digits are the same because they differ in *both* the second and the third place.

It is surprising to discover that the wireframe cube is actually a bipartite graph! Its vertices fall into two neat groups. No two vertices in the same group are connected to each other. An interesting question to ask is whether this is true of the cube because of its special 3-dimensional structure. Let us check this: what is the analog of the cube in 2-dimensions? That's easy – it is the square. A square has only 4 vertices. Is it a bipartite graph? If we number the vertices using our binary notation we find that it is indeed a bipartite graph.

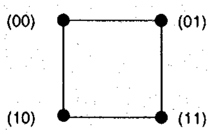
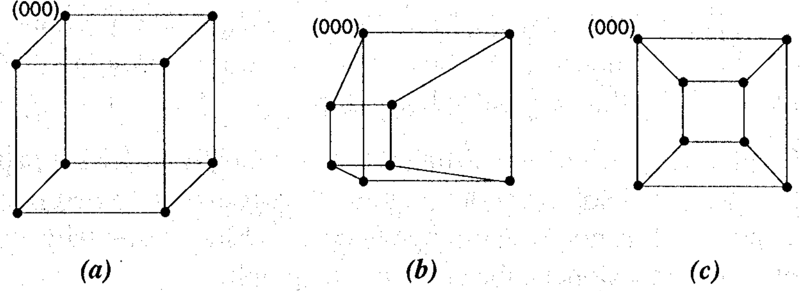


Figure 3.17: The square is a bipartite graph



(a) *(b)* *(c)*

Figure 3.18: The graphs for a 3-d cube

What about the analog of the cube in higher dimensions? In 2-d, the cube analog, the square, has $2^2 = 4$ vertices. In 3-d it has $2^3 = 8$ vertices. We can guess that the 4-d analog of the cube, called the hypercube, would have $2^4 = 16$ vertices. We can also guess that in order to give number names to the vertices in binary notation we need 4 places. The problem remains of drawing the graph corresponding to the 4-d cube on 2-d paper.

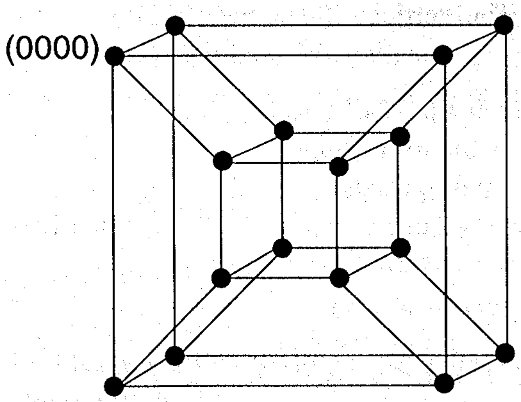


Figure 3.19: The graph for a 4-d cube

We can do this in one way – by squashing a 3-d cube. In Figure 3.18, we make one of the faces of the cube smaller and squash the cube flat to obtain the graph in Figure 3.18(c). Obviously this graph is isomorphic with the wire frame cube in Figure 3.18(a). Now we take another copy of the squashed cube and join the corresponding vertices of both the squashed cubes by parallel lines as in Figure 3.19. Name the points appropriately starting from the corner with (0000). Remember the rule that every vertex is connected to those vertices whose binary number differs in only one place. Verify that you indeed get a bipartite graph.

For cubes of higher dimensions it is difficult to draw the graph on 2-d paper. But you can write down a table of the vertex points and sort them into two groups like we did for the puzzle. It is not difficult then to spell out the reason why any graph of the form of an n-dimensional cube is a bipartite graph.

We have had a brief and cursory introduction to three topics in graph theory. As you have probably gathered, many of the problems in graph theory are both interesting and accessible without a long mathematical training. One can also have a lot of fun by solving puzzles using graphs or even designing some new puzzles that can be tackled using graphs. The black and white party puzzle, for instance, was one that we made up while writing this booklet. We invite you to create and solve more puzzles and problems.

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