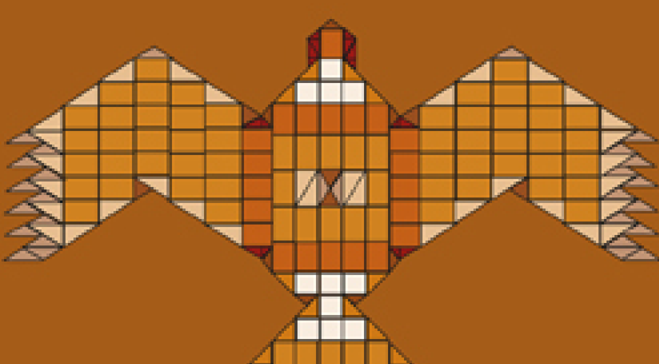
Mathematics education in India

Status and Outlook



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The diagram on the cover is a Vedic chiti or sacrificial altar made of geometrically shaped bricks (See Chapter 2).

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Preface

This collection of articles on the status of and outlook for mathematics education in India provides a detailed background for the Indian National Presentation at the 12th International Congress on Mathematics Education (Seoul, July 2012). It provides a viewpoint on the vast and varied landscape of the subject, and offers an insight into not only the problems and potential of mathematics education in India but also how they are approached by scholars active in this arena.

Mathematics is embedded deeply into the life and culture of people in the Indian subcontinent, attested by a long history of engagement with mathematics in art, craft, work and abstract disciplines of thought. This has also meant a tradition of socially embedded modes of education and learning in aspects of mathematics as well. Such a historical perspective on mathematics and its education in India is illustrated briefly in this collection.

Most of the articles centre on school mathematics, reflecting the current centrality of concern in Indian education. With the Right to Education legislated by the Indian Parliament in recent years, universalization of school education is becoming an imminent reality. On the other hand, the need for strengthening mathematics education at school level is acknowledged by all policy planners. While the overall expansion and development of higher education remains an important issue, problems of curriculum and pedagogy, assessment and teacher professional development have their roots in mathematics at school, and these are discussed at some length. The paucity of research in mathematics education and its influence on policy is pointed out.

India is characterized by diversity and cultural riches, as well as endemic poverty and social division, and this is reflected in mathematics education as well. Despite the tremendous challenges, also visible are a number of innovative initiatives, some small and some on a large scale. While a short collection like this cannot hope to evaluate the effectiveness of such initiatives, it does point to them with a sense of hope towards the future.

Any short set of articles and authors on mathematics education in India must necessarily be selective and this collection very likely excludes many important and interesting issues. Yet we hope that it adds value to discussions on this theme, not only among Indian educators, but among the international community of mathematics educators as well.

This collection is an outcome of the National Initiative in Mathematics Education (NIME) launched under the auspices of the Indian National Science Academy. The NIME

initiative and the publication of this book were supported generously by the National Board of Higher Mathematics. Similarly, the Homi Bhabha Centre for Science Education extended support to the NIME initiative by publishing this collection and in other ways. We thank the organizers and participants of the NIME regional and national conferences and the National Seminar on the History and Cultural Aspects of Mathematics Education organized by the Indira Gandhi National Open University. The material in this collection draw on the proceedings of these conferences. We thank the members of the NIME Steering Committee for guidance and support.

The authors of the collection have made great efforts in putting together information and ideas, often when there was insufficient or no prior work to draw upon. They have managed extremely tight deadlines and have co-operated in every possible way. We are grateful to them for making the collection possible. We thank K. Ramasubramanian for permission to use the diagram on the cover. The members of the mathematics education group at the Homi Bhabha Centre pitched in to ease the labour of production. Manoj Nair, as always, was the person who saw the production through till the end.

June 2012

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1. Mathematics education in India – An overview

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Introduction

The spirit of modernity and development in nations is reflected in their investment in children's education in general. If science education is often termed as societal investment in the envisioned future, education in the "high roads of mathematics" perhaps constitutes their hope for the as-yet unvisioned future. Presidents and Prime Ministers remind their people that science and mathematics education need to equip the nation's youth to meet the challenges of the 'new economy'. Modern nations see value in building a mathematically literate society and hope for a strong mathematical elite that can shape the knowledge economy of the 21st century. At the same time, mathematical proficiency is universally considered hard to achieve.

India, with its strong mathematical traditions, may be expected by the world to produce excellence in mathematics. But this may be an unreasonable expectation, since India is grappling with problems of endemic poverty, and even universalising education is a challenge. Yet, despite adversity, India has managed to produce mathematicians like Ramanujan and Harish-Chandra. All this adds up to an intriguing picture.

In contrast to the expectations of the global elite, one should consider the hopes and aspirations of the Indian people themselves. In a population that is largely poor (by any standards), education is seen as the key instrument to break out of poverty. As many adult education programmes in India demonstrated, the non-literate or neo-literate poor see the ability to 'calculate', to 'estimate' and to 'predict' as essential life-skills that education must (and hopefully does) impart, skills whose natural home in the school curriculum is mathematics. Once again, what one perceives is a sense of disappointment that school education does not impart such skills. In a *public hearing* in 2006 when a curriculum group met members of the general public, a grocer bitterly complained that he could never find educated young recruits who could *calculate* when stocks would need to be replenished and by how much.

What then characterizes mathematics education in India? We suggest that it is this mix, of severe systemic challenges, but yet a growing young population approaching them with a sense of hope, in a land of many innovations and initiatives, a system operating rather chaotically. In this article, we attempt to give a bird's eye-view of the vast landscape of mathematics education in India.

Systemic Challenges

The landscape of mathematics education in India calls for a very broad vision to encompass and comprehend. It is not only a matter of scale and magnitude in numbers of children and teachers that constitute the system, but also messy but democratic modes of functioning in which there are pulls from many social and political aspirants of society. We want every child to learn mathematics and enjoy it; the reality of achieving this with millions of children and teachers by democratic means provides a major systemic challenge. Before we look at how this affects mathematics education specifically, we need an understanding of the vast system it operates in.

The law called Right of Children to Free and Compulsory Education Act (abbreviated as Right to Education or RTE Act) came into force in India as recently as April 1, 2010. It guarantees 8 years of elementary education to every child in the age group 6-14 in an age appropriate classroom in the vicinity of his/her neighbourhood. This implies the right of every Indian child to quality mathematics education as well.

The subcontinent

Education in India is provided and controlled by three levels: the central government in Delhi, the state governments and local sources (largely private). It is regulated by both the centre and the states; this has crucial implications for designing and implementing curricula and pedagogic practices, policies for hiring and training teachers, monitoring schools, for setting standards and ensuring them, procedures for certification and ensuring overall systemic health. The states are responsible for these functions, the centre being largely regulatory but helping with funding. This at once enables many decentralized efforts as well as challenges attempts at national or centralized education reform.

The linguistic and cultural diversity of the Indian subcontinent accommodates a range of voices and approaches, and offers multiple ways of approaching mathematical experience. Many states in India are themselves geographically as large as some European nations and often larger in population. Education within these states is administered in further divisions of educational districts, but there is little decentralization within the state. Curricular and pedagogic processes are not locally shaped, and the state educational authority is as remote as the central government from the viewpoint of a school. While this enables curricular homogeneity, it tends to stifle local pedagogic ingenuity.

Structure

India's education system is structured by developmental stages from pre-primary to post-graduate level as shown in Figure 1. Elementary education (primary and upper primary) is managed separately from secondary (including higher secondary) education. Undergraduate education is typically for three years, and 4-5 years for professional degrees. Universities are regulated centrally but managed within the state, with a system of affiliated colleges providing undergraduate education.

The Ministry of Human Resource Development governs the overall Indian education system, with each State government having its own Education Ministry, and a Central Advisory Board on Education providing the platform for exchanges between the centre and states (as well as between states). In all 43 Boards of School Education operate in the country and they are the ones that formulate syllabi, train teachers and offer certification.

For school education, the National Council of Educational Research and Training

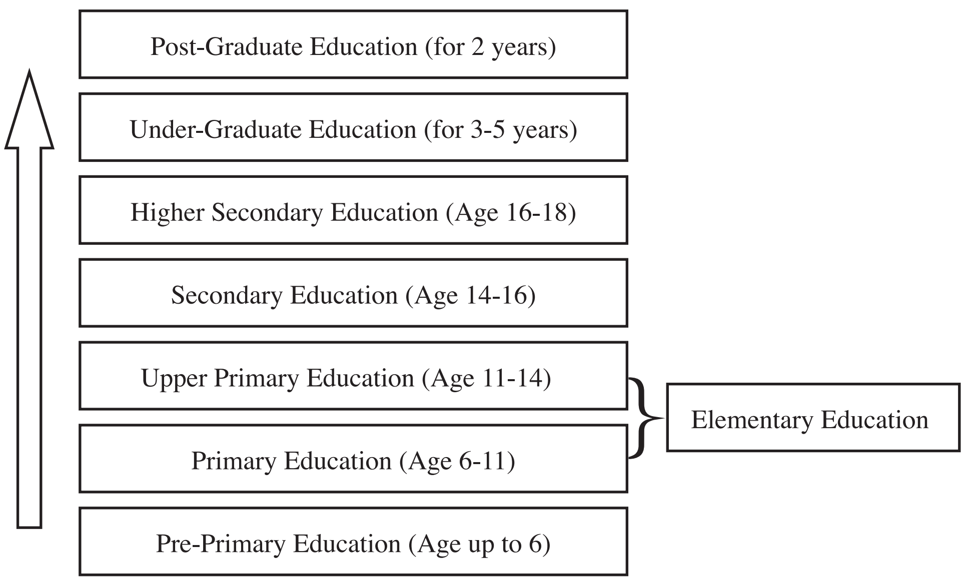

Figure 1: Levels of Education in the Indian System

(NCERT) is the apex body for curriculum related matters, but except for the Central Board of Secondary Education for which it designs curricula, its role is largely advisory vis a vis the other Boards of education. At the University level, every University formulates its own curricula but the University Grants Commission regulates their functioning.

There is a vibrant Open University system as well as the National Institute of Open Schooling that seek to provide access to education cutting across potential barriers formed by these structures.

Large numbers

Even a cursory look at the numbers shows how daunting implementation can be, and we take up only data from primary education for discussion.[1]

	Total	Number in rural areas
Number of children (ages 6-11)	134 (boys - 69, girls - 65)	108
Number of schools	1.28	0.8
Number of teachers*	5.8	4.5

*May include teachers also teaching in the upper primary grades

Table 1: Numbers related to primary education in India (in millions as of 2009)

The numbers already reveal a picture of a large education system, largely rural and millions of missing girls. That the government is the main provider of education for the population becomes clear when we note that, of the 1.28 million schools, 1.03 million are government-run. Of these, 0.8 million have classes only for the primary stage and 0.23 million have classes up to upper primary sections. The average number of teachers in a primary-only school is 2.98 and 6.96 for a school that has classes 1 to 8. The average pupil to teacher ratio is 36 for primary-only schools and 33 for schools with primary and upper primary levels.

The Quality Dimension

Why should it matter for teaching arithmetic and basic algebra whether the number of schools is one thousand or one million? After some threshold value of n (possibly 3 or 4), perhaps teaching mathematics to 10^m children is the same as 10^n children for any $m > n$. To some extent, this is the viewpoint embedded in much of educational administration in India, and mathematics educators are deeply aware of the injury such attitudes can bring to children's education.

According to some Indian scholars, the central challenge of Indian education is dealing with

[1] All data in this section are from the Final Report 2008-09 of Sarva Shiksha Abhyan, the Education for All project of the Government of India. They are intended to be indicative of the current scenario.

the metaphorical triangle of *quantity, quality* and *equality*. The state sector in education is beset by major shortage and uneven spread of resources, as witnessed by the large percentage of single classroom schools, as high as 38 percent in a rather large populous state like Andhra Pradesh. Such extreme shortage of resources presents a tremendous quality constraint. Much worse, and especially relevant to mathematics education, is lack of qualified and committed teachers. No system can rise above the quality of its teachers, and content knowledge of mathematics is crucial for mathematics education. Set against this is the data that nearly 43 percent of teachers in India in elementary education do not possess a college degree of any kind, let alone in mathematics.

Indian society is division-riven and this provides a great challenge for quality and equality in education. Mathematics being a compulsory subject of study, access to quality mathematics education is every child's right. On the other hand, there is considerable research (though not specific to mathematics classrooms) to suggest that teacher preconceptions, bias and behaviour, causes discrimination against children from the groups with low socio-economic status, the so-called "Scheduled Castes" (SC) and "Scheduled Tribes" (ST).

We have spoken of the missing millions among girls. The girls who do come to school are subject to social discrimination as well. In rural areas preconceptions such as mathematics being "unnecessary" for girls can be observed even among teachers. Despite the better performance of girls in Board examinations than boys in recent years, the stereotype that boys are better at mathematics than girls is seen to persist.

The social context of Indian education is reflected in the sharp disparities between different social and economic groups, which are seen in school enrolment and completion rates. Thus, girls belonging to SC and ST communities among the rural and urban poor and the disadvantaged sections of religious and other ethnic minorities are educationally most vulnerable, and data confirm this.

Set against such a bleak picture is also hope, arising from several wellsprings of activity:

1. Against all odds and amidst extreme diversity, we find children who take to mathematics and teachers committed to mathematics education. Statistically small, they still make up a large number given the size of the Indian population.

2. While social barriers are a great challenge, the confidence and energy released by overcoming them is very positive. Mathematics, being the discipline of thought without great need for texts, laboratories and other paraphernalia, and being the discipline that greatly inspires confidence and self-esteem, becomes then an instrument to break out of adversity for children from these disadvantaged sections, especially girls.

3. Southern India has seen how the growth of computing and Information Technol-

ogy industry offers a sense of hope to people, and perhaps due to the popular perceptions of computing, to a surge in interest in mathematics education. Among this is a noticeable increase in the participation, in mathematics learning, of girls and children from underprivileged sections.

4. The educational reform process initiated in the last decade has seen a churning across the country within school mathematics, in terms of attitudes and approaches to it. While it is too early to tell whether these efforts will lead to radical shifts, the trend is positive.
5. Lastly, the use of technology, only recently coming in as a factor, may help India solve some of the systemic problems discussed above.

Reforms

While mathematics was seen to be an essential part of any curriculum from early on, perspectives differed. The Zakir Husain committee in 1937 saw it in relation to work. The National Policy on Education in 1986 saw it as a "vehicle to train a child to think, reason, analyze and to articulate logically." However, the shape of mathematics education has remained largely the same over the last 50 years. In response to global curricular processes in India too there has been considerable curricular acceleration in school Mathematics. For instance, calculus which was only taught in college three decades ago is taught now at the higher secondary level. On the other hand projective geometry has almost entirely disappeared from the school. At the undergraduate level, the core curriculum remains much the same, though the influence of computer science and other modern disciplines can be seen in the course mix on offer.

In all this, one strain that has been persistent is the experience of anxiety and failure associated with Mathematics. Excessive use of procedure and the pressure of Board examinations and entrance examinations for access to prestigious institutions have created a culture of highly competitive preparation among the urban elite, and this has taken a toll on meaningful mathematics. On the other hand, in almost all Boards if there are specific disciplines that record failures, mathematics is principal among them. It is often referred to as the 'killer' subject and studies showed that a large number of children were failing or dropping out before completing elementary school because they could not cope with the demands of the curriculum.

Over the end of the last century, a perception that mathematics education was increasingly becoming burdensome and ineffective had gathered momentum. The Report 'Learning Without Burden' (Ministry of Human Resource and development, 1993) had pointed out that children were in fact not 'dropping out' but were being 'pushed out', owing to the 'burden of non-comprehension', as a result of an irrelevant curriculum, distanced from the lives of the majority, and often rendered 'boring and uninteresting' by outdated teaching strategies. This shift from conventional 'deficit theories', which attribute children's inability to learn to some 'deficit' in their mental abilities or their home background, led to a critical review of the curriculum and the traditional teaching learning process based on rote memorisation of facts.

The National Curriculum Framework (henceforth "NCF 2005") responded to this and guided the development of new curricula and textbooks based on how children actively construct knowledge, rooted in social and cultural practices (National Council for Educational Research and Training [NCERT], 2005). The NCF 2005 position paper on the teaching of mathematics (NCERT, 2006a) begins by stating that the primary goal of mathematics education is the "mathematization of the child's thought processes" and the development of the "inner resources of the growing child." It goes on to argue for a "shift from content to process", recommending a multiplicity of approaches, to liberate school mathematics from the "tyranny of the one right answer obtained by applying the one algorithm that has been taught". It emphasized the need for processes such as "formal problem solving, use of heuristics, estimation and approximation, optimization, use of patterns, visualization, representation, reasoning and proof, making connections, and mathematical communication".

Subsequent to this, many Boards of education in the states undertook a curricular review exercise and the last few years have witnessed a churning. While the lofty goals articulated above may be hard to achieve, there have been some significant shifts visible in textbooks and pedagogic processes, especially in elementary education. However, secondary education, weighed down by the shadow of Board examinations, remains hard to reform.

The end-of-school Board examinations remain landmark events in the lives of children, and as passports to economic mobility, they critically inform attitudes to education. These exams cast long shadows and inordinately influence classroom assessment. In fact, the traditional pattern of examinations in mathematics have been a matter of serious concern and have not only intimidated children but have often dissuaded more creative teachers too, since their classroom efforts to encourage sense making tend to get obliterated by the focus on procedural questions devoid of meaning and contextual relevance.

In this context, the pressures of a democratic society on Board examination results have to be acknowledged as well. When single subject failures tended to be high in mathematics, the pressure to set exams that fail fewer pupils became strong. This has led to a situation where pass rates have increased among those who appear for Board exams, but many who give up, drop out much earlier. This also means that high achievement in many of these exams may not attest to high competence or mastery of the subject either.

One solution to this has been attempted in many parts of the world, that of streaming students into Basic Mathematics and Advanced Mathematics, with the former constituting mathematical literacy that the state considers essential for its citizens, and the latter dictated by disciplinary objectives. But this is problematic in India, since they can become yet another form of social discrimination, with the latter course simply not being offered in many schools which children from poorer sections attend. Indeed, this was the experience in many Indian states in which such streaming existed till the 1960's. In a society that is already deeply riven by many social schisms, the possibility that the rights of disadvantaged children to quality education in mathematics might be subverted presents a major problem.

The reforms we have spoken of have come about because outside the formal system the country has had a range of educational initiatives, largely experimental and small scale but nevertheless carried out by passionately committed educationists. The valuable lessons got from such work contributed significantly to the national reform process. Such work is still visible in India, across geographic regions, from primary schools to university education.

Higher stages

We have spoken at length about elementary education. The situation is similar in secondary and tertiary education, but the fact that India has the third largest higher education system in the world (after China and the USA) suggests that there is a great deal of mathematics around as well.

According to India 2009 Reference Annual (Ministry of Information and Broadcasting, 2009), India has 20 universities run by the Central Government and 215 run by States. In addition there are 100 autonomous institutions deemed-to-be universities that do not get their funding directly from Governments. Nearly 16000 colleges are affiliated to these universities, among them 1800 exclusively for women.

India is also home to some institutions where world class research in mathematics is carried out. A strong group of Indian mathematicians have been contributing to the development of many areas of mathematics. The legendary genius Srinivasa Ramanujan has inspired generations of young Indians towards taking up mathematics as a calling.

India boasts of institutions of technology and medicine that have been globally acclaimed for their standard of undergraduate education. These, and the boom in Information Technology industry (and its generation of jobs) in the last two decades, have led to a greater emphasis on mathematical training, and the nation seeks to expand a pool of scientifically equipped manpower.

This creates a situation in India where higher education in mathematics forms a very sharp pyramid. A few elite institutions offer excellent opportunities for mathematics research, and a small number for mathematics education as a part of technology or engineering education, or in some instances, management studies. However, among the large number of universities and a vast number of affiliated colleges, which provide the bulk of tertiary mathematics education, there is an overall rigidity in curriculum, pedagogy and modes of assessment that make mathematics education often ineffective, and this affects the prospects of building a strong pool of mathematics teachers for the future. Small innovative initiatives towards constructing a meaningful interactive pedagogy at the undergraduate level give hope for solving this problem on a larger scale in the future.

The major challenge

If one were asked to isolate and point to one single challenge as the most important among the plethora of problems that we have mentioned, it would have to be that of *creating a pool of good mathematics teachers in the required numbers*. At the elementary stage, the numbers exist, but not with the required understanding of mathematics or attitudes towards mathematics or comprehension of how children learn (or fail to learn) mathematics. The social inequalities in India and the resource-poor rural schools call for greater competence on the part of teachers than richer, more democratic societies. This calls for new modes of teacher professional development that are yet to be formulated.

At higher stages, the numbers are daunting. The existing pool of teachers is woefully inadequate for meeting the requirements, especially with universalisation of school education becoming a conceivable reality within a generation. With the numbers, the problem of rigour and depth in mathematical knowledge and practice becomes more acute. Devising systemic measures to achieve quality in teacher preparation is perhaps the most urgent need in the Indian mathematics scenario today.

Research

An important agenda for mathematics education in India is research in mathematics education. University departments, while undertaking research in education, by their typical structure, tend to attract largely people who are neither mathematically trained nor thus inclined. Further, the idea of research providing solutions to curricular conundrums or pedagogic trauma remains outside the framework of decision making in education. This is not to belittle the tremendous contributions made by governmental as well as non-governmental initiatives towards reform that have been characterised by innovation and commitment. However, these do not rest on a scaffolding of research and rigorous critique as yet. The system needs to build a way of actively pursuing research on several fronts towards well formulated questions and use the answers to influence policy. It should be noted here that India provides a large enough arena, with tremendous diversity, to even allow a self-contained universe for analysis and research, and international influences can only add to this richness.

The agenda for such research includes not only internalist critique from the discipline of mathematics and its pedagogy and practices. Indian society and its cultural and work-based practices also offer avenues for mathematical explorations that a pedagogue could incorporate into a toolkit. However, a body of research needs to be built to make realistic use of such possibilities.

Last words

This heady mix can be summarized, perhaps a bit crudely, as follows:

1. The challenge of providing quality mathematics education for all at school level is immense, and the country has some way to traverse to achieve this.
2. The need for a large body of teachers with expertise in mathematics and training in pedagogy is acute.
3. The Government is the central player in Indian education, but it is not monolithic either.
4. On the other hand, India's diversity has given rise to a range of initiatives, some small, some large, including some from the Government.

We have spoken of problems endemic to the Indian mathematics education system, but many of them are not unlike problems encountered in mathematics education in other societies and nations. The immense size and diversity of the Indian subcontinent, low levels of resources and an almost ungovernable polity complicate, but the sense of hope that prevails suggests that India may yet solve these problems, that force us to take a hard look at mathematics not only in terms of curricula (in diversity), pedagogy (in widely varied milieu) but in social context as well.

One thing is for sure: when India manages to provide quality mathematics education for all, mathematics education as a discipline would have new insights and new formulations to work with.

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2. Glimpses of the History of Mathematics in India

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Introduction

Starting from the representation of numbers, through the way of arriving at the solutions of indeterminate equations, to the development of sophisticated techniques in handling the infinite and the infititesimals, there has been a wide variation in the choice of working style amongst mathematicians of different cultures. By working style we mean the approach taken by mathematicians in formulating the problem, in internally visualizing the solution, in externally representing their consolidated understanding, and so on.

Most of the mathematics in India starting from 5th century CE has been handed down in the form of highly compressed and cryptic verses, not to mention the aphoristic sytle adopted prior to this period. As the transmission of knowledge was primarily oral, these verses/aphorisms used to be memorized and passed on orally form generation to generation—traces of which can be seen even today in several parts of India. The Indian mathematicians were so adept in metrical composition that even infinite series expansion of trigonometrical functions have been presented in the form of beautiful verses, which sometimes have a *double entendre*. While on this topic, it may also be mentioned that the purpose for which the power series were arrived at in India (around 14th century) as well as the route taken by mathematicians to arrive at them, are quite different from the trajectory adopted by mathematicians in Europe a couple of centuries later.

The purpose of this article in not to exalt a particular tradition or a working style over the other, but to provide a bird's eye view of the origin and development of mathematics in India by citing several passages form the original sources so as to enable the reader to have a flavor of the working style of Indian mathematicians and the kind of practical or application-oriented mathematics they developed. This apart from giving a broad picture of development of mathematics in India, would also help them acquire a cross-cultural perspective that would enable them to have a better appreciation of the evolution of mathematics across different cultures.

Mathematics in India has a very long and hallowed history. *Śulbasūtras*, the oldest extant texts (prior to 800 BCE) explicitly state and make use of the so-called Pythagorean theorem apart from giving various interesting approximations to surds, in connection with the construction of altars and fire-places of different sizes and shapes. By the time of Āryabhaṭa (c. 499 CE), the Indian mathematicians were fully conversant with most of the mathematics that we currently teach at the elementary level in our schools, which includes the methods for extracting square root, cube root, and so on. Among other things, Āryabhaṭa also presented the differential equation of sine function in its finite-difference form and a method for solving linear indeterminate equation. The *bhāvanā* law of Brahmagupta (c.628) and the *cakravāla* algorithm described by Jayadeva and Bhāskarācārya (12th cent.) for solving quadratic indeterminate equation are some of the important landmarks in the evolution of algebra in India.

The Kerala School of Astronomy and Mathematics pioneered by Mādhava of Saṅgamagrāma (c. 1350)—stumbling upon the problem of finding the exact relationship between the arc and the corresponding chord of a circle, and problems associated with that—came very close to inventing what goes by the name of infinitesimal calculus today. Particularly, Mādhava seems to have blazed a trail by enunciating the infinite series for $\frac{\pi}{4}$ (the so-called Gregory-Leibniz series) and other trigonometric functions.[1] As mentioned earlier, it is quite interesting to note that almost all these discoveries are succinctly coded in the form of metrical compositions in Sanskrit. To the present day reader, having got so much accustomed to the use of symbols, it may even be difficult to imagine a recursion relation, or an infinite series, or for that matter the derivative of a trigonometric function to be couched in the form of chaste prose or charming poetry (sometimes with an intended pun). But amazingly, that is how it has been presented to us at least from the time of *Śulbasūtras*, most of which were supposed to have been composed by 5th century BCE, till late 19th century CE. In what follows we attempt to provide a flavor of this mathematics with plenty of quotations from the original source works.

For the sake of convenience, we divide the paper into three sections (leaving out the first one on introduction). Section 2 deals with Mathematics in Ancient India (prior to 5th century CE), which would be followed by the section on Mathematics in the Classical period (500–1350 CE). In Section 4 we will discuss Mathematics in the Medieval period (14th – 16th cent.) which is described as the Golden Age of Mathematics in India. Before we embark upon the details, it wouldn't be out of place to quote the beautiful verses of Mahāvīrācārya (c. 9th cent.) conveying the ubiquity of mathematics.

[1] Interesting proofs of these results are presented in the famous Malayalam text *Gaṇita-Yuktibhāṣā* (c. 1530) of Jyeṣṭhadeva (Sarma, 2009) as well as in the works of Saṅkara Vāriyar, who was a contemporary of Jyeṣṭhadeva.

Mahāvīrācārya, in order to impress upon the importance of the study of mathematics, right at the very beginning of his classical treatise *Gaṇita-sāra-saṅgraha*, eloquently puts forth the diverse disciplines in which mathematics finds its application:

> लौकिके वैदिके चापि तथा सामयिकेपि यः। व्यापारस्तत्र सर्वत्र सङ्ख्यानमुपयुज्यते॥
> कामतन्त्रेऽर्थशास्त्रे च गान्धर्वे नाटकेऽपि वा। सूपशास्त्रे तथा वैद्ये वास्तुविद्यादि वस्तुषु॥
> छन्दोलङ्कारकाव्येषु तर्कव्याकरणादिषु। कलागुणेषु सर्वेषु प्रस्तुतं गणितं परम्॥...
>
> बहुभिर्विप्रलापैः किं त्रैलोक्ये सचराचरे। यत्किञ्चिद्वस्तु तत्सर्वं गणितेन विना न हि॥
>
> Whether the dealings have to do with worldly affairs or spiritual matters or religious practices, enumeration is very much involved. In affairs related to love, in economics, in music, in drama, in cooking, in practicing medicine, in the fields like architecture, in using metrics, in [employing] figures of speech, in [composing] literature, in logic, in grammar, in arts, etc, the mathematics [that is going to be discussed] is extremely important. ...
>
> Why keep talking much? In all the three worlds consisting of living and non-living entities, whatever be the transaction, it cannot be executed without mathematics!

Mathematics in the Ancient period

Śulbasūtras

The Vedic priests had developed a class of manuals that would assist them in the construction of altars (called *Vedis*) used for performing sacrifices. These texts called *Śulbasūtras* primarily dealing with the geometry related to the design of the *Vedis*, are considered to be a part of a larger class of texts known as *Kalpasūtras*, which in turn are considered to be one of the six *Vedāṅgas*.[2] The word *śulba* stems from the root *śulb* which means 'to measure'. Since all the measurements were done using ropes or chords in the very early times—traces of which can be found in practice even today—it seems the word in due course was synonymously employed to refer to the chords themselves.

[2]The term *Vedāṅga* is used to refer to six branches of knowledge namely *śikṣā, vyākaraṇaṃ, kalpaḥ, niruktaṃ, jyotiṣaṃ* and *chandaḥ*. In olden times, all these branches used to be studied by every Vedic priest either after completing his studies of the *Veda*, or simultaneously along with it.

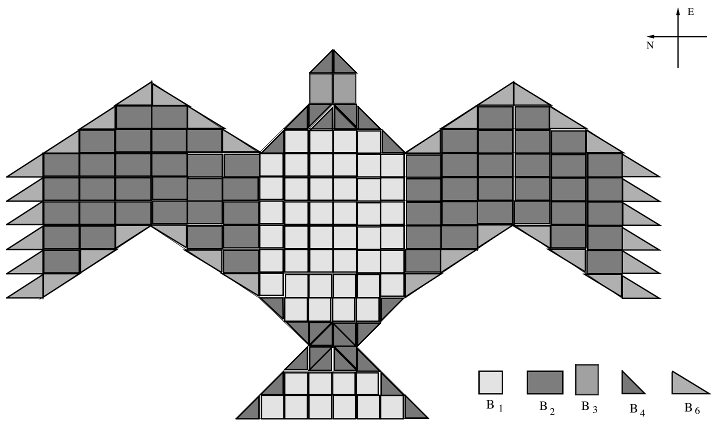

Figure 1: The first layer of the altar *Śyenaciti*.

Some of the geometrical constructions such as the *śyenaciti* (see Figures 1 and 2) prescribed by the Śulbakāras (the authors of the *Śulbasūtras*) are quite involved[3] and cannot be simply constructed without having a mastery over certain techniques that include the procedures for determining the east-west direction at a given location, for drawing straight lines that are at right angles to each other, for constructing a square whose side is surd times an integer, for finding the area of certain geometrical objects, and so on. We now proceed to discuss some of these basic tools as explained in *Śulbasūtras*.

Finding the cardinal directions

Having chosen the location at which the sacrificial altar is to be constructed, the first thing that needs to be done is the determination of the east-west direction at that location.[4] The procedure determining it is described as follows:

> समे शङ्कुं निखाय शङ्कुसम्मितया रज्वा मण्डलं परिलिख्य यत्र लेखयोः शङ्क्वग्रच्छाया निपतति तत्र शङ्कू निहन्ति सा प्राची।
>
> Fixing the gnomon (*śaṅku*) on level ground and drawing a circle with a cord measured by the gnomon, he fixes pins at points on the line (of the circum-

[3] In fact, there are a number of constraints that need to be fulfilled in the construction of *śyenaciti* such as, the number of bricks in each layer should be constant (200), the area of all the bricks put together must be equal to a specified number, and so on.

[4] In fact, this knowledge was a prerequisite for any kind of ritual prescribed in the Vedic literature and not necessarily the construction of altar or fireplace described in the *Śulbasūtras*.

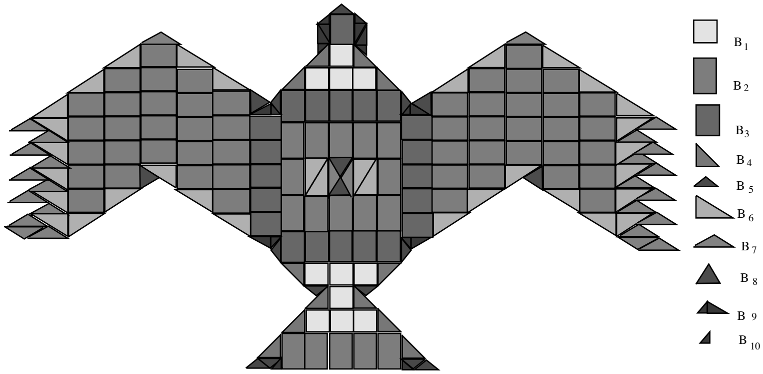

Figure 2: The second layer of the altar *Śyenaciti*.

ference) where the shadow of the tip of the gnomon falls. That is the east direction (*prācī*).

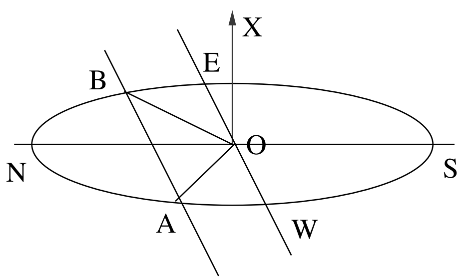

Figure 3: Determining the east-west line using *śaṅku*.

Asking the question as to why perform this experiment with *śaṅku*[5] in order to determine the direction, and not simply look at the sunrise and sunset and be with it, the commentator Mahīdhara observes:

[5]The term *śaṅku* refers to a very simple contrivance in the form of a rounded stick of a suitable length and height with a sharp tip at one of its edges. This has been extensively employed by Indian astronomers for conducting a variety of experiments to determine the cardinal directions at a given place, the latitude of the place and so on.

…तस्य उदयस्थानानां बहुत्वात् प्रतिदिनं भिन्नत्वात् अनियमेन प्राची ज्ञातुं न शक्या। तस्मात् शङ्कुस्थापनेन प्राचीसाधनमुक्तम्। दक्षिणायने चित्रापर्यन्तमर्कोभ्युदेति। मेषतुलासङ्क्रान्त्यहे प्राच्यां शुद्धायामुदेति। ततोऽर्कात् प्राचीज्ञानं दुर्घटम्।

Since the rising points are many, varying from day to day, the [cardinal] east point cannot be known [from the sunrise point]. Therefore it has been prescribed that the east be determined by fixing a *śaṅku*. …

The theorem on the square of a diagonal

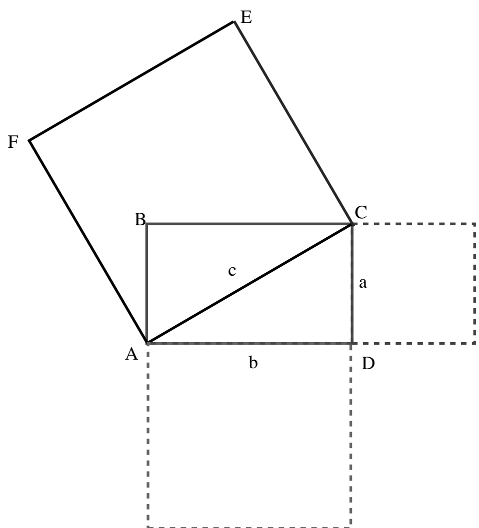

Figure 4: *Śulva* theorem: the theorem on the square of a diagonal.

The so called Pythagorean theorem (*bhujā-koṭi-karṇa-nyāya*) is given by Bodhāyana in his *Śulbasūtra* is as follows:

दीर्घचतुरश्रस्याक्ष्णया रज्जुः पार्श्वमानी तिर्यङ्मानी च यत्पृथग्भूते कुरुतः तदुभयं करोति।

The diagonal of a rectangle produces [an area] that is produced by the length and the breadth separately.

It may be noted (see Figure 4) that the actual enunciation of the theorem in *Śulbasūtras* is not with respect to the right-angled triangle but with respect to the sides and diagonals of squares and rectangles. It is well known that the famous mathematician-astronomer *Bhāskara* (12th century) gave an elegant proof of the theorem.[6] Yet another interesting proof (using dissectional method) is provided by Jyeṣṭhadeva in his *Yuktibhāṣā* (Sarma, 2009, pp.179-180).

To draw a square whose area is equal to n times a given square

Kātyāyana gives an interesting method for obtaining a square whose area is equal to the sum of the areas of a large number (say n) of squares.

> यावत्प्रमाणानि समचतुरश्राण्येकीकर्तुं चिकीर्षेत् एकोनानि तानि भवन्ति तिर्यक् द्विगुणान्येकत एकाधिकानि। त्र्यस्रिर्भवति तस्येषुस्तत्करोति।
>
> As many squares (of all side) as you wish to combine into one, the transverse line will be [equal to] one less than that; twice a side will be [equal to] one more than that. It will be a triangle (*tryasri*).[7] Its arrow (i.e., altitude) will do that.

Consider that there are n squares each of area a. It is desired that we obtain a square whose area is equal to the sum of all the n squares. The procedure given by Kātyāyana is to construct an isosceles triangle say ABC whose base is of length $(n-1)a$ and sides of length $\frac{(n+1)a}{2}$. It is said that the altitude of the triangle (AD) will give the side of a square ($\sqrt{n}a$) whose area will be na^2.
In Figure 5, $BD = \frac{1}{2}BC = (\frac{n-1}{2})a$. Considering the triangle ABD,

$$\begin{aligned} AD^2 &= AB^2 - BD^2 \\ &= \left[\frac{(n+1)a}{2}\right]^2 - \left[\frac{(n-1)a}{2}\right]^2 \\ &= \frac{a^2}{4}\left[(n+1)^2 - (n-1)^2\right] \\ &= na^2. \end{aligned}$$

[6]Making a note of this, Burger and Starbird in their recently published book (2010, p.233) observe:

> The proof presented here that was discovered by the Indian mathematician Bhaskara in the 12th century exemplifies aesthetics and beauty in mathematical arguments.

[7]Literally the word *tryasri* means a three-sided figure.

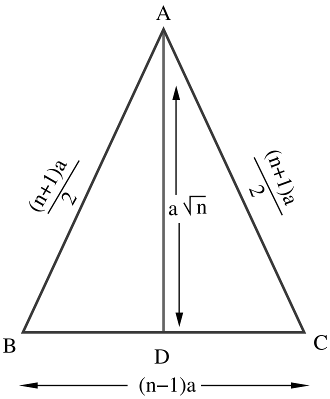

Figure 5: Scheme for drawing a square whose area to n times a given square proposed by Kātyāyana.

The prescription given above may look pretty straightforward and simple. But what is noteworthy is the amalgamation between geometry and algebra that it requires in order to come up with this prescription in its most 'general' form.[8]

Transforming a square into a circle

As mentioned earlier, the Vedic priests constructed altars of different sizes and shapes. In doing so, they had also imposed the constraint that altars of different shapes be of the same area. This naturally gives rise to the problem of transforming a square into a circle—over which mathematicians of all civilizations have struggled for ages. The prescription given by Bodhāyana for this problem is:

> चतुरश्रं मण्डलं चिकीर्षन् अक्ष्णयार्धं मध्यात् प्राचीम् अभ्यपातयेत्। यद्यदतिशिष्यते तस्य सह तृतीयेन मण्डलं परिलिखेत्।
>
> Desirous of transforming a square into a circle, may the [length of the] semi-diagonal (*akṣṇayārdhaṃ*) be marked along the east direction starting from the centre. Whatever portion extends [beyond the side of the square], by adding one-third of that [to the semi-side of the square] may the circle be drawn.

[8]By 'general' form we mean the usage of *yāvat-tāvat* (as much-so much), which has been denoted by n in our explanation of the *sūtra*.

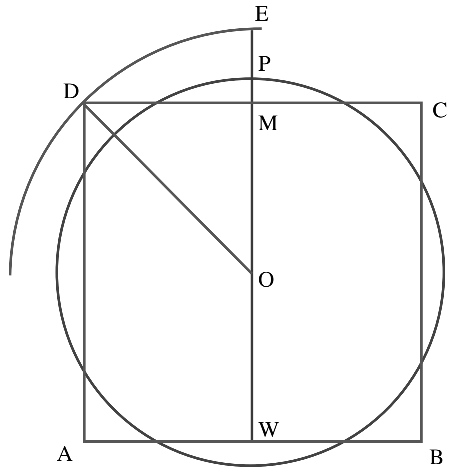

Figure 6: Constructing a circle whose area is same as that of a square.

In Figure 6, $AB = 2a,\ OP = r$ and $OD = a\sqrt{2}$. Hence $ME = a(\sqrt{2} - 1)$. Now the radius of the desired circle ($OP = r$) is given by

$$\begin{aligned} r &= a + \frac{a}{3}(\sqrt{2} - 1) \\ &= \frac{a}{3}(2 + \sqrt{2}). \end{aligned}$$

The above expression for radius involves finding the value of $\sqrt{2}$.

Value of $\sqrt{2}$ given in *Śulbasūtras*

The following *sūtra* given by the *Śulbakāras* (Bodhāyana, Āpastamba and Kātyāyana) presents an interesting rational approximation to $\sqrt{2}$:

प्रमाणं तृतीयेन वर्धयेत्, तच्चतुर्थेन, आत्मचतुस्त्रिंशोनेन, सविशेषः।

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}. \tag{1}$$

It must be noted that we have introduced an 'approximate' symbol in the above equation and not that of 'equality'. This is to signify the fact that the *sūtra* quoted above clearly states that the value specified is only approximate (*saviśeṣaḥ*, literally 'with a distinction/speciality').

Before we move on to the next topic, it may simply be mentioned that several scholars have attempted to offer explanation as to how the *Śulbakāras* might have arrived at the above expression,[9] which gives the value 1.4142157... that is remarkably close to the actual value 1.4142136....

The Bakhshālī Manuscript

The Bakhshālī Manuscript, a compendium of rules and illustrative examples related to arithmetic and algebra, was incidentally discovered by a farmer in the year 1881 in the course of excavation at a village called Bakhshālī in the north-west frontier of India.[10] This manuscript, which contained about 70 folios in birchbark (not all in readable condition), is found to be written in *Śāradā* script. Though the author and period of composition are not known, scholars are of the view that this should have been composed anywhere between 4th–7th century CE, if not earlier.[11]

Among other things, the manuscript presents certain interesting problems involving indeterminate equations along with their solutions. We present an example below.

Example: A jewel is sold among five merchants together. The price of the jewel is equal to half the money possessed by the first together with the moneys possessed by the others, or $\frac{1}{3}$rd the money possessed by the second together with the moneys possessed by the others, or $\frac{1}{4}$th the money possessed by the third together with the moneys possessed by the others, or $\frac{1}{5}$th the money possessed by the fourth together with the moneys possessed by the others, or $\frac{1}{6}$th the money possessed by the fifth together with the moneys possessed by the others. Find the cost of the jewel, and the money possessed by each merchant.[12]

Solution: If m_1, m_2, m_3, m_4, m_5 be the money possessed by the five merchants, and p be the price of the jewel, then the given problem may be represented as

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[9]See for instance, the insightful article by Henderson (2000), and the one by Dani (2010), with incisive remarks.

[10]This place Bakhshālī is about 50 miles from Peshawar (currently in Pakistan).

[11]For erudite discussions on this issue see the introduction in (Sarasvati, Svami Satya Prakash & Jyotishmati, 1979), and an exclusive chapter devoted to this in (Hayashi, 2005).

[12]The statement of the problem commences as follows (Sarasvati, Svami Satya Prakash, & Jyotishmati, 1979, p.30):

> पञ्चानां वणिजां मध्ये मणिर्विक्रीयते किल। तत्रोक्ता मणिविक्रीत्रा मणिमूल्यं कियद्भवेत्॥ ...अर्धं त्रिभाग पादांश पञ्चभाग षडंश च।

Here it may be mentioned that though the solution to the problem is available in greater detail, the statement as such is not fully decipherable from the manuscript (see for instance, (Hayashi, 2005, pp. 174-175), and hence what has been presented above is a partially—yet faithfully—re-constructed version of it (see (Srinivasiengar, 1967, pp.39-40)) by gathering the information available in bits and pieces.

$$\begin{aligned}
\frac{1}{2}m_1 + m_2 + m_3 + m_4 + m_5 &= m_1 + \frac{1}{3}m_2 + m_3 + m_4 + m_5 \\
&= m_1 + m_2 + \frac{1}{4}m_3 + m_4 + m_5 \\
&= m_1 + m_2 + m_3 + \frac{1}{5}m_4 + m_5 \\
&= m_1 + m_2 + m_3 + m_4 + \frac{1}{6}m_5 \\
&= p.
\end{aligned}$$

Hence we have

$$\frac{1}{2}m_1 = \frac{2}{3}m_2 = \frac{3}{4}m_3 = \frac{4}{5}m_4 = \frac{5}{6}m_5 = q \text{ (say)}.$$

Substituting this in any of the previous equations we get $\frac{377}{60}q = p$. For integral solutions we have to take $p = 377\,r$ and $q = 60\,r$, where r is any integer. In fact, the answer provided in Bakhshālī manuscript is $p = 377$ and $m_1, m_2, m_3, m_4, m_5 = 120, 90, 80, 75, 72$ respectively.

Mathematics in the Classical Age

Starting with Āryabhaṭa in the 5th century, and extending upto Nārayaṇa Paṇḍita of the 14th century, the Indian mathematicians have blazed a trial in the study of several branches of mathematics that include obtaining recurrence relation for the construction of sine table, finding solutions to indeterminate equations of the first and the second degree, obtaining rules for finding the sum of arithmetic and geometric progression, finding the sum of sums, construction of magic squares, and so on. Some of the prominent mathematicians (most of them astronomers as well) who belonged to this period include Bhāskara I (c. 600), Brahmagupta (c. 628), Mahāvīra (c. 850), Pṛthūdaka (c. 860), Muñjāla (c. 932), Śrīpati (c. 1039), Bhāskara II (c. 1150), and Nārāyaṇa Paṇḍita (c. 1350). Due to the constraint on the length of the article, here we confine our discussion only to a select few topics listed above.

Solutions to indeterminate equations

The problem of finding integral solutions to indeterminate equations of the first and the second order has been of considerable interest to Indian mathematicians and astronomers of this age. An explicit algorithm for finding the general integral solution of the first order indeterminate equation of the form

$$ax + by = c, \tag{2}$$

called *kuṭṭaka* in Indian mathematics—popularly known as Diophantine equation—is found in *Āryabhaṭīya*. Āryabhaṭa as usual has been very cryptic and has presented the algorithm in just two verses (32 and 33 of *Gaṇitapāda*). However, it is interesting to note that Bhāskara I provides almost 30 examples in his commentary to illustrate the application of *kuṭṭaka* method prescribed by Āryabhaṭa. This clearly mirrors the need that was felt for solving such problems that occur in real life as well as in the context of solving certain problems in astronomy.

While Āryabhaṭa confined himself to dealing with first order indeterminate equation, Brahmagupta, a brilliant mathematician, who live about a century and a quarter later has attempted to solve a much harder problem of solving quadratic indeterminate equation of the form

$$Dx^2 + 1 = y^2, \tag{3}$$

where D is a positive integer that is not a perfect square. The principle invoked by Brahmagupta in solving equations of the above form has been referred to as *bhāvanā*. A detailed exposition of the *bhāvanā* principle, and the significant role it plays in modern algebra and number theory has been nicely brought out by Datta in one of his recent articles (2010). The solution developed by Brahmagupta has been improved upon by later Indian algebraists of whom special mention may be made of Jayadeva (early 11th century?) and Bhāskara II. The improved algorithm known as *cakravāla* has been illustrated by Bhāskara II by taking difficult[13] numerical cases like $D = 61$ and $D = 67$. A lucid explanation of Bhāskara's *Cakravāla* algorithm given in his *Bījagaṇita* (12th century), and its efficiency over Brounker-Wallis-Euler and Lagrange algorithm (17th century and 18th century) along with a numerical example, has been provided by Sriram in one of his recent articles (2005).

Sum of series & Sum of sums

Āryabhaṭa (c. 499 CE), in the *Gaṇitapāda* of *Āryabhaṭīya*, deals with a general arithmetic progression in verses 19–20. Following this, e gives the sum of the squares and cubes of natural numbers in verse 22:

[13] Difficult because, the smallest positive integer solution for the case $D = 61$ happens to be $x = 226153980$ and $y = 1766319049$.

सैकसगच्छपदानां क्रमात् त्रिसंवर्गितस्य षष्ठोंऽशः।
वर्गचितिघनः स भवेत् चितिवर्गो घनचितिघनश्च॥

> The product of the three quantities, the number of terms plus one, the same increased by the number of terms, and the number of terms, when divided by six, gives the sum of squares of natural numbers (*varga-citi-ghana*). The square of the sum of natural numbers gives the sum of the cubes of natural numbers (*ghana-citi-ghana*).

In other words,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{4}$$

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + n^3 &= [1 + 2 + 3 + \ldots + n]^2 \\ &= \left[\frac{n(n+1)}{2}\right]^2. \end{aligned} \tag{5}$$

Construction of sine-tables

The Indian astronomers took upon themselves the task of constructing sine-tables as accurately as possible, for their procedure for finding planetary positions—which in turn was crucial in the making the calendar, called *pañcāṅga*—was filled with sine and cosine functions. For constructing the sine-table, to be more precise the Rsine-table called *paṭhita-jyā*,[14] the circumference of a circle is divided into $21600'$ and usually the Rsines are tabulated for every multiple of $225'$, thus giving 24 tabulated Rsines in a quadrant. Using the value of $\pi \approx \frac{62832}{20000} = 3.1416$, given by Āryabhaṭa, the value of the radius turns out to be $3437'\ 44''\ 19'''$. This value, which is correct to seconds, was usually approximated to $3438'$.[15]

In the *Gītikā-pāda* of *Āryabhaṭīya* (verse 12), we find the following verse[16] that gives a table of Rsine-differences (the first differences of the values of trigonometric sines expressed in arcminutes):

मखि भखि फखि धखि णखि ञखि
ङखि हस्झ स्ककि किष्ग श्घकि किघ्व।

[14]In the Indian astronomical-mathematical treatises, the sine and cosine values were specified in minutes of arc and not in radians. The notation 'Rsine' is used to mark this distinction.

[15]Using a more accurate value of π, Mādhava (c. 1340–1420) gave the value of the radius correct to the thirds as $3437'\ 44''\ 48'''$ which in *Kaṭapayādi* notation is given by *devo-viśvasthalī-bhṛguḥ*.

[16]This verse is perhaps the most terse verse in the entire Sanskrit literature that the author of the paper has ever come across. Only after several trials would it be ever possible to read the verse properly, let also deciphering its content.

> घ्लकि किग्र हक्य धकि किच
> स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्याः॥
>
> 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, and 7—these are the Rsine-differences [at intervals of 225′ of arc] in terms of the minutes of arc.

Incidentally it may be noted that the values presented here are correct to minutes, and this was perhaps the first ‘sine-table’ ever constructed in the history of mathematics.[17] How did Āryabhaṭa arrive at the above table?

In *Gaṇitapāda* (verse 12) Āryabhaṭa gives an ingenious method of computing the Rsine-differences, making use of the important property that the second-order differences of Rsines are proportional to the Rsines themselves:

> प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम्।
> तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि॥
>
> The first Rsine divided by itself and then diminished by the quotient will give the second Rsine-difference. The same first Rsine, diminished by the quotients obtained by dividing each of the preceding Rsines by the first Rsine, gives the remaining Rsine-differences.

Let $B_1 = R\sin\,(225')$, $B_2 = R\sin\,(450'), ...,$ $B_{24} = R\sin\,(90°)$, be the twenty-four Rsines, and let $\Delta_1 = B_1$, $\Delta_2 = B_2 - B_1, ...,$ $\Delta_k = B_k - B_{k-1}, ...$ be the Rsine-differences. Then, the above rule may be expressed as[18]

$$\Delta_2 = B_1 - \frac{B_1}{B_1} \qquad (6)$$

$$\Delta_{k+1} = B_1 - \frac{(B_1 + B_2 + \ldots + B_k)}{B_1} \qquad (k = 1, 2, \ldots, 23). \qquad (7)$$

This second relation is also sometimes expressed in the equivalent form

$$\Delta_{k+1} = \Delta_k - \frac{(\Delta_1 + \Delta_2 + \ldots + \Delta_k)}{B_1} \qquad (k = 1, 2, \ldots, 23). \qquad (8)$$

From the above it follows that

$$\Delta_{k+1} - \Delta_k = \frac{-B_k}{B_1} \qquad (k = 1, 2, \ldots, 23). \qquad (9)$$

[17] First because, the tables of Hipparchus (now lost) and Menelaus, as well as those of Ptolemy are all tables of chords and not of half-chords, as in the case of the table given by Āryabhaṭa.

[18] Āryabhaṭa is using the approximation $\Delta_2 - \Delta_1 \approx 1'$.

Since Āryabhaṭa also takes $\Delta_1 = B_1 = R\sin(225') \approx 225'$, the above relations reduce to

$$\Delta_1 = 225' \tag{10}$$
$$\Delta_{k+1} - \Delta_k = \frac{-B_k}{225'} \qquad (k = 1, 2, \ldots, 23). \tag{11}$$

In his scholarly preface to a recently published volume, David Mumford (2010) describes the procedure given by Āryabhaṭa as "the discrete analog of the result that sine solves the harmonic equation $y'' + y = 0$".

Nārāyaṇa Paṇḍita's general formula for *Vārasaṅkalita*

In his *Gaṇita-kaumudī*, Nārāyaṇa Paṇḍita (c. 1356) gives the formula for the r^{th}-order repeated sum of the sequence of numbers 1, 2, 3, . . . , n (Dvivedi, 1936, p.123):

> एकाधिकवारमिताः पदादिरूपोत्तरा पृथक् तेंऽशाः ।
> एकाद्येकचयहरास्तद्घातो वारसङ्कलितम् ॥
>
> The *pada* (number of terms in the sequence) is the first term [of an arithmetic progression] and 1 is the common difference. Take as numerators [the terms in the AP] numbering one more than *vāra* (the number of times the repeated summation is to be made). The denominators are [terms of an AP of the same length] starting with one and with common difference one. The resultant product is *vāra-saṅkalita*.

Let
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = V_n^{(1)}. \tag{12}$$

Then, Nārāyaṇa's result is

$$V_n^{(r)} = V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)} \tag{13}$$
$$= \frac{[n(n+1)\ldots(n+r)]}{[1.2\ldots(r+1)]}. \tag{14}$$

The Cow problem

A notable feature of Nārāyaṇa—a feature that he shares with Bhāskarācārya, the author of *Līlāvatī*—is that he presents several interesting examples drawn from day to day life, that would not only be appealing to the reader, but also impress upon him the importance of the topic that is being discussed. After presenting the formula for finding the sum of sums, Nārāyaṇa illustrates the use of the above formula by choosing an interesting problem related to the estimate of population of cow.

> प्रतिवर्षं गौः सूते वर्षत्रितयेन तर्णकी तस्याः।
> विद्वन् विंशतिवर्षैः गौरेकस्याश्च सन्ततिं कथय॥
>
> A cow gives birth to a [she] calf every year [and] their calves themselves [begin giving birth], in 3 years time. O learned, tell the number of progeny produced by a cow in 20 years.

Recalling

$$
\begin{aligned}
V_n^{(0)} &= 1+1+\ldots+1 = n \\
V_n^{(1)} &= V_1^{(0)}+\ldots+V_n^{(0)} = 1+2+\ldots+ = \frac{n(n+1)}{2} \\
V_n^{(2)} &= V_1^{(1)}+V_2^{(1)}+\ldots+V_n^{(1)} = \frac{n(n+1)(n+2)}{1.2.3}
\end{aligned}
$$

For the sake of convenience we represent the solution of the problem in the form of a table (see Table 1).

Contribution of the Kerala School

The Kerala School, pioneered by Mādhava (c. 1340–1420) and followed by illustrious mathematicians and astronomers like Parameśvara, Dāmodara, Nīlakaṇṭha, Acyuta and others, by introducing several new ideas and techniques built an elaborate mathematical edifice that forms part of what is known as Calculus today. Our aim in this section, is to provide a glimpse[19] of some of the brilliant discoveries—such as the sum of infinite geometric series, the infinite series for $\frac{\pi}{4}$, its fast convergent approximations, and so on—that were made by the mathematicians of the Kerala School, anticipating some of the developments in Europe made almost two centuries later. Before we embark upon the details it may be mentioned that a systematic exposition of the work of the Kerala School, is to be found in the famous Malayalam work *Gaṇita-yukti-bhāṣā* (Rationales in Mathematical Astronomy) (Sarma, 2009) composed by Jyeṣṭhadeva (c. 1530)—a disciple of Dāmodara and junior to Nīlakaṇṭha.

[19] For a detailed exposition of the Development of Calculus in India, the readers are referred to the article by Ramasubramanian and Srinivas (2010).

Year	1^{st}gen.	2^{nd}gen.	3^{rd}gen.	4^{th}gen.	5^{th}gen.	6^{th}gen.	7^{th}gen.
1	1						
2	1						
3	1						
4	1	$V_1^{(0)}$					
5	1	$V_2^{(0)}$					
6	1	$V_3^{(0)}$					
7	1	$V_4^{(0)}$	$V_1^{(1)}$				
8	1	$V_5^{(0)}$	$V_2^{(1)}$				
9	1	$V_6^{(0)}$	$V_3^{(1)}$				
10	1	$V_7^{(0)}$	$V_4^{(1)}$	$V_1^{(2)}$			
11	1	$V_8^{(0)}$	$V_5^{(1)}$	$V_2^{(2)}$			
12	1	$V_9^{(0)}$	$V_6^{(1)}$	$V_3^{(2)}$			
13	1	$V_{10}^{(0)}$	$V_7^{(1)}$	$V_4^{(2)}$	$V_1^{(3)}$		
14	1	$V_{11}^{(0)}$	$V_8^{(1)}$	$V_5^{(2)}$	$V_2^{(3)}$		
15	1	$V_{12}^{(0)}$	$V_9^{(1)}$	$V_6^{(2)}$	$V_3^{(3)}$		
16	1	$V_{13}^{(0)}$	$V_{10}^{(1)}$	$V_7^{(2)}$	$V_4^{(3)}$	$V_1^{(4)}$	
17	1	$V_{14}^{(0)}$	$V_{11}^{(1)}$	$V_8^{(2)}$	$V_5^{(3)}$	$V_2^{(4)}$	
18	1	$V_{15}^{(0)}$	$V_{12}^{(1)}$	$V_9^{(2)}$	$V_6^{(3)}$	$V_3^{(4)}$	
19	1	$V_{16}^{(0)}$	$V_{13}^{(1)}$	$V_{10}^{(2)}$	$V_7^{(3)}$	$V_4^{(4)}$	$V_1^{(5)}$
20	1	$V_{17}^{(0)}$	$V_{14}^{(1)}$	$V_{11}^{(2)}$	$V_8^{(3)}$	$V_5^{(4)}$	$V_2^{(5)}$
Sum	20	153	560	1001	762	210	8

Table 1: The population of cow in 20 years.

Sum of an infinite geometric series

While deriving an interesting approximation for an arc of a circle in terms of the *jyā* (Rsine) and the *śara* (Rversine),[20] Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya* presents a de-

[20]Considering a circle of radius R, if $s = R\theta$ is the arc of a circle, subtending an angle θ (in radians) at the centre, then

$$\begin{aligned} jy\bar{a}(s) &= R\sin\theta \\ \acute{s}ara(s) &= \text{Rvers}\,\theta \;=\; R(1-\cos\theta). \end{aligned}$$

tailed explanation of how to sum an infinite geometric series. The specific series that arises in this context is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}\,. \tag{15}$$

At the outset, Nīlakaṇṭha poses a very important question (Śāstrī, 1930, p.106):

> कथं पुनः तावदेव वर्धते तावद्वर्धते च ?
>
> How do you know that [the sum of the series] increases only upto that [limiting value] and that it certainly increases upto that [limiting value]?

Proceeding to answer the above question, Nīlakaṇṭha first obtains the sequence of results

$$\begin{aligned}
\frac{1}{3} &= \frac{1}{4} + \frac{1}{(4.3)}\,, \\
\frac{1}{(4.3)} &= \frac{1}{(4.4)} + \frac{1}{(4.4.3)}\,, \\
\frac{1}{(4.4.3)} &= \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)}\,,
\end{aligned}$$

and so on, which leads to the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right). \tag{16}$$

Nīlakaṇṭha then goes on to present the crucial argument: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by RHS of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality expressed in (15).

Mādhava series for π

The infinite series for $\frac{\pi}{4}$ enunciated by Mādhava in the form of a verse,[21]

[21] This verse is quoted by Śaṅkara Vāriyar in his commentary *Kriyākramakarī* on *Līlāvatī* (Sarma, 1975, p.379).

> व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
> त्रिशरादिविषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात्॥
>
> The diameter multiplied by four and divided by unity [is found and saved]. Again the products of the diameter and four are divided by the odd numbers like three, five, etc., and the results are subtracted and added in order.

is the well known series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots \quad . \tag{17}$$

We shall now present the derivation of the above result as outlined in *Yuktibhāṣā* (Sarma, 2009 pp. 183–198). For this, let us consider the quadrant OP_0P_nS of the square circumscribing the given circle (see Figure 1) of radius r. Divide the side P_0P_n into n equal parts (n very large). The resulting segments P_0P_i's ($i = 1, 2, \dots, n$) are known as the *bhujās* and the line joining its tip and the centre OP_i's are known as *karṇas*. The points of intersection of these *karṇas* and the circle are denoted by A_i. The *bhujās* P_0P_i, the *karṇas* k_i and the east-west line OP_0 form right-angled triangles whose hypotenuses are given by

$$k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2 . \tag{18}$$

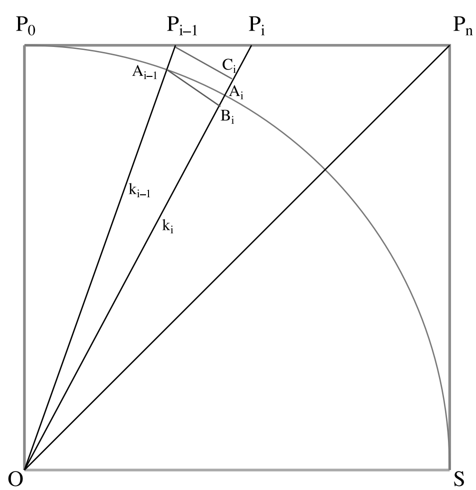

Figure 7: Geometrical construction used in the proof of the infinite series for π.

Considering two successive *karṇas*—ith and the previous one as shown in the figure—and the pairs of similar triangles, $OP_{i-1}C_i$ and $OA_{i-1}B_i$ and $P_{i-1}C_iP_i$ and P_0OP_i, it can be shown that

$$A_{i-1}B_i \;=\; \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \tag{19}$$

Now the text presents the crucial argument: When n is large, the Rsines $A_{i-1}B_i$ corresponding to different arc-bits $A_{i-1}A_i$ can be taken as the arc-bits themselves. Thus, $\frac{1}{8}$th of the circumference of the circle can be written as the sum of the contributions given by (19).

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0k_1}\right)+\left(\frac{r^2}{k_1k_2}\right)+\cdots+\left(\frac{r^2}{k_{n-1}k_n}\right)\right]. \tag{20}$$

It is further argued in the text that the denominators $k_{i-1}k_i$ may be replaced by the square of either of the *karṇas* i.e., by k_{i-1}^2 or k_i^2 since the difference is negligible. Thus (20) may be re-written in the form

$$\begin{aligned}\frac{C}{8} &= \sum_{i=1}^{n}\frac{r}{n}\left(\frac{r^2}{k_i^2}\right)\\ &= \sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+\left(\frac{ir}{n}\right)^2}\right)\\ &= \sum_{i=1}^{n}\left[\frac{r}{n}-\frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right)+\frac{r}{n}\left(\frac{\left(\frac{ir}{n}\right)^2}{r^2}\right)^2-\ldots\right]\end{aligned} \tag{21}$$

In the series expression for the circumference given above, factoring out the powers of $\frac{r}{n}$, the summations involved are that of even powers of the natural numbers. It was known to Indian mathematicians that

$$\sum_{i=1}^{n} i^k \;\approx\; \frac{n^{k+1}}{k+1}\,. \tag{22}$$

Now, using the estimate (22) for these sums when n is large, we arrive at the result[22]

$$\frac{C}{8} = r\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right), \tag{23}$$

[22] In modern terminology, the above derivation amounts to the evaluation of the following integral

$$\frac{C}{8} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+\left(\frac{ir}{n}\right)^2}\right) = r\int_0^1 \frac{dx}{1+x^2}.$$

which is same as (17), the well known series for $\frac{\pi}{4}$.

Discussion

Some of the novel insights and techniques of handling the infinitesimals and the infinite sum developed by the Kerala school of mathematicians, was first brought to the notice of the western scholarship by Charles Whish (1834) in early nineteenth century.[23] Somehow this seems to have gone unnoticed among the historians of mathematics for more than a century, that as late as 1940s the renowned historians like Carl Boyer (1949, pp.61-62) make infelicitous remarks:

> They (Hindus) delighted more in the tricks that could be played with numbers than in the thoughts the mind could produce ... The Pythagorean problem of the incommensurable, which was of intense interest to Greek geometers, was of little import to Hindu mathematicians, who treated rational and irrational quantities, curvilinear and rectilinear magnitudes indiscriminately.

However, subsequent studies have led to a somewhat different perception of the Indian contribution as may be gleaned from the following quotation from a recent work on the history of mathematics (Hodgekin, 2005, p.168):

> We have here a prime example of two traditions whose aims were completely different. The Euclidean ideology of proof which was so influential in the Islamic world had no apparent influence in India (as al-Biruni had complained long before), ... To suppose that some version of 'calculus' underlay the derivation of the series must be a matter of conjecture.
> The single exception to this generalization is a long work, much admired in Kerala, which was known as *Yuktibhāṣā*, by Jyeṣṭhadeva; this contains something more like proofs—but again, ...

In the recent past there has been an attempt to assess the Indian contribution to the development of calculus by several scholars. To the question whether the Kerala school invented calculus, while some have answered in the affirmative, the others have some

[23]Though this remarkable paper of Whish got published only in 1934, from the notings made by John Warren in his *Kālasaṅkalita* (1825), we understand that Whish had communicated his findings to some of the senior officers like George Hyne as early as 1825. However, the views maintained by Whish that the infinite series were found by 'Natives' themselves were not received favorably. George Hyne in one of his correspondence observes: "the Hindus never invented the series; it was communicated with many others, by Europeans, to some learned Natives in modern times... the pretensions of the Hindus to such a knowledge of geometry, is too ridiculous to deserve refutation". For more details of the episode see Sarma et.al (2010).

reservations in accepting this position (Katz, 1995; Raju, 2001; Bressoud, 2002; Divakaran, 2007)—as easily evident from Hodgekin's observation quoted above.[24] In this connection, we would like to present a few facts before the readers.

Indian work on calculus 'primarily' stems from solving problems in Astronomy. To be more precise, it got developed as a part of the continuous endeavor on the part of astronomers to improve the precision of their calculations that involves sine and cosine functions, and their derivatives. All the astronomer-mathematicians starting from Āryabhaṭa to Mādhava had a conception of planetary model wherein they had to deal with only circles, and sine and cosine functions (whose differentials repeat after two orders). This, along with the fact that the pursuit of mathematics in India was 'primarily' calculation or application-oriented, explains why arbitrary functions and curves were not considered by Indian mathematicians.

However, recalling the fact that there are essentially three founding pillars on which the edifice of calculus rests upon—one, splitting the curve into infinitesimal parts, two, locally linearizing them, and three, summing up their 'infinite' infinitesimal contributions—and the fact that all the three are found in their full blown form in the derivation of Mādhava series for $\frac{\pi}{4}$ as presented above, and also the fact that "the muse of mathematics can be wooed in many different ways",[25] we leave it to the readers to judge for themselves, 'how' and 'where' to place the contribution of Mādhava,[26] in the context of narrating the grand story of the discovery of calculus.

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[24]Interestingly, there are a few scholars who have come up with models for the transmission of calculus from India to Europe (Joseph & Almedia,2007; Raju, 2007).

[25]Towards the end of his review of the book *Mathematics in India* (Plofker, 2008) David Mumford, the renowned mathematician and Fields medalist observes:

> Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge. ...the muse of mathematics can be wooed in many different ways and her secrets teased out of her (Mumford, 2010).

[26]Though Mādhava has been the acknowledged fountainhead of the profound ideas that emerged from the Kerala School, unfortunately none of his works on mathematics per se, but for a couple of works in astronomy, are extant now. It is only from the quotations and citations made by the later astronomers and mathematicians of this School that we come to know of some of his brilliant contributions.

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3. Indigenous traditions and the colonial encounter: A historical perspective on mathematics education in India

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Introduction

In this article, we try to provide a broad overview of the historical processes associated with the evolution of modern mathematics education in India, with a focus on elementary mathematics education. We try to organize this narrative around two issues that are of contemporary relevance. One, contending ideals about objectives of mathematics education in relation to functionality as a desired goal (or otherwise) in school mathematics. Second, the disconnect between the learning processes in school and those outside of it. This does not amount to projecting present concerns on to the past. Attempts to reconstruct a history of mathematics education in India during the eighteenth and the nineteenth centuries tend to reveal the above two aspects as central strands to what is otherwise a complex story. The sources for this complexity, it should be mentioned at the outset, lie in the huge diversity that India has imbibed over centuries in the social, economic and cultural terrain, shared and sustained by its people.

Indigenous traditions of learning mathematics

Such diversity as sustained by the people in the spheres of production and culture in varied ecosystems and landscapes is evident in traditions of knowledge and processes of their transmission. Indigenous traditions of education is one sphere where we can recognize such a tremendous diversity in the material and cultural practices of the people. India had a very rich and widespread culture of institutional education during the precolonial era, both for elementary as well as higher branches of learning. While the elementary institutions of learning were known as *pathshalas*, the higher institutions were known as '*tols*', akin to colleges. There were also *madrasas*, seats of Arabic learning. Dharampal's work has shown us that they were widespread, dynamic institutions of learning, which pervaded the entire Indian rural and urban landscape in the precolonial era (Dharampal, 1983). The *pathshalas* were elementary schools of a locality. These were single teacher schools, for a village or a group of villages, which catered to the upper and the intermediate caste groups, and excluded the lower/manual labouring caste groups. The children were divided into classes not with respect to age, but in accordance with their capability to learn language and arithmetic. There was no standardized curriculum across regions and its orientation was thoroughly local. The idea was not to produce scholars, but to enable students to study further to become one, if they chose to. The *pathshalas* were integral to the material and social world of the people.

The unique feature of *pathshalas* was the strong element of functionality in the curriculum probably a result of their complete dependence on local patronage of the community with its caste hierarchy (Radhakrishnan, 1990). These *pathshalas* were open to male children across occupational groups. They were trained in reading, writing and arithmetic (Basu, 1982; Acharya, 1996). The locus for the curriculum came from the need of the community to enable children to become competent/skilled participants in the transactions of letters and numbers within the local society. At the same time, it was also a culture of learning that celebrated exposition of skills and competence in public. Acquisition of skills through learning in the *pathshalas* over a period of four to five years continuously involved the participation of the local public in the affairs of learning, which not only patronized the teacher through economic support but also evaluated his labour (and the teacher was typically male). This meant that learning had to address local concerns of relevance. While the local orientation of the curriculum rendered such diversity to the *pathshalas* in the various regions of the country, the character, competence and the learnedness of the teacher also added to this diversity within regions–sometimes within the same village or the town. This is particularly so in the case of learning languages in the *pathshalas* where the basic objective of learning to read and write correctly was often accompanied by encouraging familiarity with certain texts either popular in the region or due to the choice of the individual teacher. This feature of a certain autonomy in what counted as learning within the *pathshalas* had to contend with a strong evaluative participation of the local public that often tested what was perceived as functional or relevant learning. But its autonomy also rendered to these institutions certain possibilities to transcend the local.

Along with diversity in curriculum, the *pathshalas* seemed to share a culture of pedagogy grounded in a form of memory very different from the modern associations of memory with rote or mechanical mode. This could be characterized as *recollective memory* where memory practices constituted a distinct mode of learning and not merely aids to learning. Oral recitations were central to this form of learning while the role of writing was to assist recollective memory, making the distinction between the oral and the written ambiguous. Learning under the aegis of recollective memory in itself constituted understanding, especially in a culture that appreciated exposition and celebrated remembering. Learning arithmetic in this mode cultivated computational abilities that often attracted attention among visitors so much so that it became common to characterize Indians as having a natural proclivity towards computation: "the natives of India are remarkable for the facility with which they acquire the mathematics; and indeed they excel in anything in which figures or numbers are concerned. Their system of arithmetic is almost entirely committed to memory and the power which the little schoolboys display in mental arithmetic is quite astonishing to the European" (Rhenius, 1841, p. 269-70). Jean Baptiste Tavernier traveling in 1665 mentions the quick abilities of Indians to sum and perform mental calculations. Remarks such as "the Banias, by the strength of his brain only, will sum up his accounts with equal exactness and quicker dispatch than the readiest Arithmetician can with his pen" are not uncommon even during the initial days of the British presence in India. The East India Company offered rewards (twenty pounds!) to its soldiers if they would learn arithmetic from the natives (Sarma, 1997).

The probable basis for such perceptions could be reconstructed from the pedagogic practice of the *pathshalas*. The dominant presence of the various number tables in the curriculum of the *pathshalas* in the various languages and region of the country are yet to be studied properly, while there remain scattered hints and reticent guesses about their plausible role in the learning of arithmetic in the past. But their integral place in the curriculum of the widespread *pathshalas* could be discerned from various sources. The Bombay Gazetteer mentioned,

> "...the vania boy commits to memory a number of very elaborate tables. These tables of which there are no fewer than twenty contain among others two sets for whole numbers, one table of units up to ten multiplied to as high as forty times; the other for numbers eleven to twenty multiplied by eleven to twenty times. There are fraction tables giving the results of multiplying $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $1\frac{3}{4}$, $2\frac{1}{2}$ and $1\frac{1}{2}$ into units from one to one hundred; interest tables showing the monthly rate of one percent on sums from Re. 1 to Rs. 1000, the amount due for each quarter of a month; tables of squares of all numbers from one to 100 and a set of technical rules for finding the price of a part from the price of the whole" (Sarma, 1997).

D. D. Kosambi (1962), the historian-mathematician, mentioned the use of such tables in the Marathi speaking region after the name of Hemadri, who was Chancellor of exchequer under the last Yadavas of Devagiri in the thirteenth century and was an outstanding practitioner of computations. John Taylor of the Bombay Medical Establishment, who translated Lilavati from Sanskrit in the year 1816, mentions that in the Marathi schools the tables of multiplication consisted in multiplying ten numbers as far as 30 and in the Gujarati schools, in multiplying ten numbers as far as one hundred. In Bengal, *Subhankara* was an household name for the repository of mathematical or computational expertise as Lal Behari Dey writing about the life of the Bengali peasant writes, "the village school master was the first mathematician of the village. He had not only *Subhankara*, Indian cocker at his fingertips but was acquainted with the elements of *vijaganita* or Algebra". Even in the Hindi speaking areas of the country it was a common practice to learn multiplication tables and the scholar Sudhakara Dwivedi traces the Hindi word 'pahara', which denotes multiplication tables to the famous poet Tulsidas who has this profound metaphor of table nine – the sum of digits in each multiple of nine is always nine; just as nine is inherent in all its multiples, so is the Lord Rama ever present (Sarma, 1997).

Along with such scattered evidence that point to a certain basis for cultivation of recollective memory with respect to the learning of numbers, there are four distinct aspects that further point towards a dynamic culture of engagement with transmission of mathematical knowledge in the Indian past. They are i) math tables which could have been manuals used in the *pathshalas* in the different regional languages ii) compilations of mathematical techniques in regional languages iii) various forms of problem posing and problem solving in popular consciousness and iv) certain kinds of practices inherent to various occupational or artisanal groups. In the following, we shall attempt to reconstruct the *pathshala* learning based on these four different types of evidence available to us.

Mathematical tables

The early nineteenth century British surveys on Indian education to the extent that they paid attention to the curriculum and pedagogy of the indigenous institutions of learning have pointed out how integral the use of math tables of various kinds were in the teaching and learning of elementary arithmetic. The study of such tables in the Tamil speaking region of the country where the *pathshalas* were called the *tinnai* schools (veranda schools or the pyal schools as the British called them) shows that the structure and organization of these tables were mnemonic in nature, whose primary purpose was to cultivate recollective memory. There is a strong pedagogic basis to their very organization and they are not textbooks in the modern sense. But they are products of learning, from within the tinnai arithmetic practice. During the process of memorizing the arithmetic tables, students wrote their own manuals, almost as an end product of their training. Drawing upon our study (Senthil Babu, 2007), we briefly discuss some of the salient features of the Tamil system to point out certain possibilities for further studies in the other regional language traditions.

Every number in the primary number series would be memorized in a particular order, in the pattern of integrating the sound of the number name, visual recognition of the symbol, loud recital and writing, with concurrent testing at each level by the monitor or the teacher. This elementary number series consisted of both whole numbers and an extensive system of fractions in the Tamil system. Such an extensive system of fractions, when represented as addition-based iterations, became the organizing basis to learn numbers in the memory mode of learning. There were separate sessions in the *tinnai* routine, where the children would stand up and recite the entire series in unison, loudly in front of the teacher, one series after the other, repeatedly, day after day till the logic of addition as the basis of number organization is cognitively internalized along with the process of building memory registers for the numbers in a particular order. This would mark the the memory learning of the elementary number series in Tamil called the *Ponnilakkam*, (*pon* = gold; *ilakkam* = number place, in the literal sense), denoting a particular order of numbers, as quantities. Next in line is the *Nellilakkam* (*nel* = paddy; *ilakkam* – number place) which is a number series that takes the units of Tamil volumetric measures as numbers and proceeds along similar lines as that of the *Ponnilakkam*. Here, the standard numbers that occur in the series are the standard units for grain measures in Tamil. This series was also organized on the principle of iterations of addition, where the basic unit of grain measure, the *cevitu* becomes the number to be added repeatedly till the highest unit, the *kalam* is reached. The standard units of the grain measure, that occur on the way from *cevitu* to *kalam* are *cevitu, alakku, ulakku, uri, nali, kuruni, patakku, tuni and kalam*. In a similar pattern, all these units that occur below the unit *kalam* would be represented in combinations with each other, paired by addition. Although there is no information for the time taken by a student to become proficient in *Ponnilakkam* and *Nellilakkam*, it seems that this alone took about two years to complete. Each student, after completely committing to memory the entire series of the *Ponnilakkam* and *Nellilakkam*, would actually write his own book on palm leaves, out of his own memory, without assistance from the teacher or the monitor. This also marks a process by which natural memory ability was trained into a cultivated memory, where reading and writing were only incidental to the learning process, not ends in themselves (Carruthers, 1990, p.70).

Followed by this was the learning of the Tamil multiplication table book called the *Encuvati*. The *Encuvati* is a compilation of several kinds of multiplication tables. All the numbers learnt during the course of *Ponnilakkam* and *Nellilakkam* would be subjected to multiplication with each other, to yield an entire set of tables, that were to be committed to memory. The organizing basis of the *Encuvati* was multiplication, represented in a tabular format, which helped secure an order, helping memory. There are several layers of multiplication tables involved. Followed by this is the learning of squares, called the *Kulimattu*. Representations in tabular order further assist recollection, allowing the possibility of identifying a median (one easy number in the middle of a table, say five, fifty, five hundred) so that both sides from that point could be remembered and recollected. Even though logical constructions involving numbers of this order are universal, (in contrast to words, that require habit and repeated practice for recollective memory), in the pedagogic practice of the *Encuvati*, we find a situation where language plays a central role, integrating itself strongly to number learning, when not a single number name would appear strange to a child growing up in a community which thrived in production and exchange practices that involved multiple modes of measurement as integral to their material culture.

Cultivating a culture of problem solving

The *Encuvati* learning sessions involved regular afternoon sessions of problem posing and solving in the *tinnai* schools. Here, recollective memory would have to score well in an algorithmic context. The students carried back problems to their homes, where problem solving happened in a non-institutional context, entirely orally to aid the process of strengthening the cognitive apparatus of associative memory. The next morning, the results were collected and discussed. The school and its pedagogic strategies were immersed in the cultural context, imbued with shared learning, where creativity in a child was associated with a whole set of agents outside the institution. This mode of public involvement points to a culture of learning arithmetic where a common pool of problems with variations in techniques was shared by both the students and the local people. This culture of learning also constituted the basis on which the public evaluated and participated in affairs of learning (Senthil Babu, 2007, pp. 28-30).

Mathematics in the regions

There has yet been no systematic attempt in documenting the various sources which would provide us clues about the nature of the problems from this common pool, which survive as remnants of the arithmetic learning culture of the *pathshalas* in various parts of the country. One such significant repository lies in the various compilations of mathematical problems in the regional languages. It requires a great deal of disciplinary labour and conviction to initiate documentation of the regional mathematical traditions and to engage with them with the rigorous attention that they deserve, while at the same time understanding the historiographical and political processes associated with the lack of engagement with this tradition so far. Though it has been recently stated that the relationship between the regional and the so called pan-Indian Sanskritic tradition of mathematics was mutual and complementary (Sarma, 2011, p. 202), there is still a long way to go before attempting any such characterization about the relationship between the multiple-regional and the canonical 'Indian mathematical tradition'.

S. R. Sarma (2011) in a recent initiative has attempted to document certain evidence regarding the presence of various such 'regional literature' of mathematics. He points to the close association between the professional scribal communities like the *kayasthas* in Northern India and the *karanams* in the south and the merchant communities of western India. In the eastern region, Orissa seems to have a rich heritage of mathematical treatises, like the *Lilavatisutra* which was a very popular text in Orissa for all age groups to study mathematics through works of addition, subtraction, multiplication, division, mensuration, trigonometry and so on. The diversity and variations in the regional corpus of this tradition was also due to the fact that there were such enormously diverse systems of metrological practices in various regions, and often within the same region speaking a single language. Sarma states that even if some of such regional texts could be translations from the prevalent Sanskrit texts like in the case of *Ganitasarakaumidi* composed by Thakkur Pheru, a learned Jaina assay master at the court of Khaliji Sultans, which was almost a phonetic translation of Sridhara's *patiganita*, the examples in the regional literature were drawn from the different localized professions involving traders, carpenters, masons. Sections on solid geometry contained rules for calculating volumes of bridges, crop yields and magic squares. *Kayasthas* in the northern region who were professional record keepers had their variety of arithmetic called '*kaitheli Amka'* in verse form, which was published as '*Kautuk Aru Kaitheli Amka'*. Even in the case of texts of non-professional genre like the *Pavuluruganitamu* in Telugu which seems to have been a translation of Mahavira's *Ganitasarasangraha* by Mallana in the eleventh century, there are interesting variations. If *Ganitasara* had five methods of squaring and seven methods of cubing, Mallana had only one each and avoided the algebraic route. In case of examples, more interestingly, there are forty five additional examples under multiplication and twenty one in case of division that are not found in the Sanskrit source.

In the Tamil speaking region, in texts of the *Kanakkatikaram* corpus the local world of transactions remain as the focus, though the handling of arithmetic was more advanced from the primers in the enumeration of rule of three, magic squares, exhaustion problems, recursions and partitions. In a typical *Kanakkatikaram* text we have at least six distinct sections, classified according to the objects of computation. These texts primarily set out the rules of computation using different techniques. Normally, they are found to have sixty types or '*inam*' (in the sense of a 'genre') in as many verses. These verses enumerate techniques involving various kinds of measures related to land, gold, grain, solid stones, volumetric measures and a general section. For example, the section on land would deal with various ways to measure area of land of different dimensions, in both whole and fractional magnitude; estimation of total produce from a given area of land; assessment of yields, profit and so on. The section on gold would deal with computations related to estimation of quality of gold, calculation of price and combinations of mixture in the making of particular grades of gold. Since gold was also a unit of money, this section would deal with computations involved in transactions of money involving goods and labour. Verses dealing with grains for example would deal with techniques of conversion of measures, profit and loss calculations.

In a social sense, all such arithmetic representation with embedded cognitive aspirations, are characterized by a yearning to enable a person to be in control of a situation, to plan, estimate, assess and anticipate. Yet, the occasions were the normal day to day socioeconomic transactions (Kameswaran, 1998). However this was not specialized knowledge for specialists or for training specialists, even though there was constant yearning to move beyond the functional into realms of the fantastic. This further invokes a sense of how such prowess in computations was meant for public exposition, for display and for performance with a system of virtues associated with such abilities, imbibed in the *pathshala* practice of teaching and learning arithmetic (Subramaniam & Kameswaran, 1999).

Mathematics among the people

It is still not uncommon to find in several parts of the country, especially in rural areas, the circulation of mathematical problems as riddles and aphorisms among the people. The correspondence between what exists in the memory of the people, in certain realms of popular consciousness and their variants in the prosodic form recorded in texts is yet to be studied. However it does point to a common shared culture of problem solving among people. Such problems are often termed as recreational problems and there have been recent attempts to document them across cultures using classical texts of mathematics like that of Aryabhata (Singmaster, 2000).

The regional traditions of mathematics survive beyond the realm of texts with constantly changing variations over time, in particular with respect to the use of examples where one could also discern instances of social critique and satire. For instance, S. R. Sarma records a riddle from rural Andhra Pradesh which implicitly brings up the image of the 'clever' Brahmin, an image that is not too uncommon in parts of rural India.

> "15 Brahmins and 15 thieves had to spend a dark night in an isolated temple of Durga. The Goddess appeared in person at midnight and wanted to devour exactly 15 persons, since she was hungry. The thieves suggested that she consume all the 15 plump Brahmins. But the clever Brahmins proposed that all the 30 would stand in a circle and that Durga should eat each ninth person. Brahmins arranged themselves and the thieves in a circle...Durga counted out each ninth person and devoured him. When the 15 were eaten, she was satiated and disappeared, and only the Brahmins remained in the circle" (Sarma, 2011).

The problem was how did the Brahmins arrange themselves with the thieves in the circle? This riddle was also composed as a Telugu verse in a classical meter.

There are other varieties of riddles in the folklore. For example, a Bengali tribal woman not used to taking her husband's name when asked to provide the name of her husband, which was *shait*, meaning sixty in a dialect in Bengali would recite it as:

Tin tero diya barao
Noi diya milani karo
Mor soamir namiti aei
Par kore dao barit jai

This means if you multiply the number three by thirteen and add twelve and nine with the result, you see the answer is simply '*shait*', that is sixty. Or in the case of another example where children playing together count the number of players among themselves by reciting rhymes, where each word represents a particular number:

Yakor byakor tyakor shail
Kail porshu mongol bar
Kari gone majumdar
Dhaner aga naler shish
Khata doba unish bish

Here the word *yakor* represents the number one, *byakor* two and so on till *bish* meaning twenty (Sinha, 1995, p. 99). As recent as 2007-2008, field work in the Kaveri delta area in the coastal district of Nagapattinam in the state of Tamil Nadu provided an opportunity to record some of the riddles prevalent among the older generation in the villages, which interestingly are very similar to the kind of problems found in the *Kanakkatikarm* texts, discussed briefly above. The examples and the context have obviously has been changing over time, as can be noticed with the appearance of the cyclist in the following story:

> An old woman carried a basket full of lemons to sell in a nearby market. Then a person by accident hit her with his bicycle and all the lemons spilled out in all directions. The cyclist gathered all the lemons in the basket and asked the old woman, "how many lemons were you carrying in your basket?" She said, I am not sure about the total number of lemons, but I know one thing for sure. If I grouped the lemons in twos, one will be left behind. If done in threes, one will be left; if grouped in fours, one will be left; if grouped in fives, one will be left; if grouped in sixes, one will be left and if grouped in sevens nothing will be left behind. The cyclist replied, oh, in that case, I have gathered all the lemons you brought into the basket without a mistake and he left. How many lemons were there in the basket? (Answer 301) (Senthil Babu, 2008)

In the following example as well, one can see signs of these riddles changing examples and characters whereas the nature of the problem remain the same, pointing to the dynamic nature of transmission among people and strands of continuity in popular consciousness over time.

> A father had three children. Each one of them had some money and so did the father. One day, he called his eldest son, took the money that his son had, and put in the same amount from his own pocket. Out of the total money, he bought 4 rupees worth of books. Then he called his second son, showed him the remaining money he had, got the same amount from his second son, and bought 4 rupees worth of books for his second son as well. Then he called his third son, showed him the remaining amount, got the same amount from the son and bought four rupees worth of books for the third son as well. At the end of it all, he did not have a penny left with him. If that is so, how much did the father have in the beginning?

or in this case, it is more evident:

> A person went abroad and returned home after three years. A friend of his met him incidentally and asked, "How many passengers were in the ship when you traveled?" The person gave him this funny answer: Us, along with the same number as 'us', half of us, and half of that and yourself would make a 100 passengers. How many were there in the ship? (Answer: 36). (Senthil Babu, 2008)

Mathematics and work

Another realm that testifies to forms of transmission in the indigenous tradition is the one through work, involving artisanal communities and specialized craftsmen like carpenters, sculptors, goldsmiths, etc. Though the practice of apprenticeship in the various crafts and artisanal work varies with the nature of the profession and the community, the nature of the mathematical engagement in such work involves learning on the job. There are several texts which are like manuals for certain crafts both in the regional language traditions as well as in Sanskrit, from the past. There are very few individuals who could situate and understand the content of these texts but the fact that forms of these crafts and arts continue to thrive points to the strong ways in which forms of learning and knowledge transmission happen at work. Traditional stone and wood sculptors in south India, the bronze sculptors of Swamimalai in Tanjore, artisanal workers like the carpenters and blacksmiths are certain groups which continue to operate and transmit specialized knowledge through work. Detailed anthropological studies on these communities and their way of work and learning as a continuous process are yet to be undertaken in the country. There have been recent attempts to initiate such studies as in the case of boat makers in West Bengal. These boat makers employ traditional techniques, build large – 50-60 feet in length – deep-sea fishing boats. Most of them have had very little schooling and cannot read or write. They work with minimal tools and without a blueprint (Mukhopadhyay, 2011).

The social context in which such varied forms of circulation and transmission happened was however ridden with deep hierarchies, primarily determined by caste. The pathshala culture was not aloof from this. Extreme forms of labour servitude, spatial segregation and social discrimination against the oppressed caste groups meant institutionalized forms of denial in access to education. But the functionality centered curriculum of the *pathshalas* and the *tinnai* schools was also a means to control labour by appropriating labouring practices as legitimate knowledge that the youth from upper castes should be trained in. The majority of the labouring and oppressed caste groups without access to institutional education like the pathshalas however have always remained familiar with the world of arithmetic practices – counting, weighing, measuring, estimation, assessment, etc., in their realms of work in primary production and services in agrarian and mercantile practices. The cultural context lacked a common culture of learning and discovery where young learners from all backgrounds could sit and learn together. The constitution and reproduction of knowledge in the culture involved a process of exclusion. In the process, institutions such as the pathshalas were deprived of a much enriched engagement with work, knowledge and culture of all sections of the people in the local society.

Mathematics in *'tols'*

This brings us to another set of institutions in the indigenous tradition, which were centres of higher learning reserved exclusively for the upper caste Brahmins. There was no graduated system of learning from the pathshalas into these Sanskrit colleges, called '*tols*'. They were independent institutions, where the pathshalas were meant for the trading and agricultural classes and the tols for the “religious and the learned classes” (Basu, 1982, p. 32). In these institutions both the teachers and the students were Brahmins where theology, metaphysics, ethics, law, astronomy, logic and medicine were taught. Such institutions were supported by different forms of patronage of the ruling classes, often through land grants of different kinds (Dharampal, 1983, p. 30-32; Basu, 1982, p. 32). These were again institutions that were widespread in different parts of the country, in Bengal, Deccan and the Southern regions in particular, with highly specialized centres of learning like Benaras, Nadia, etc. which attracted students from all over the country. These institutions were often closely related to temples and monasteries and their focus varied according to the region and sect.

These centres of learning seemed to have preoccupied themselves with the study of logic and computational astronomy, with respect to the learning of mathematics. It was an entirely different kind of engagement that probably demanded very different ways of organizing mathematical techniques like a predominant concern with the algebraic mode. But this was a tradition that had its ups and downs depending on the fortunes of the patronage on which the institutions were entirely dependent. Nevertheless there was a continuity in the tradition and this could have constituted the basis for what became a canonical tradition of ‘Indian mathematics’, in which the primary means of transmission as oral or written within the canonical tradition remains an open issue among historians of Indian mathematics (Yano, 2006; Filliozat, 2004). But locally, this tradition in its various forms also served well in feeding into a sustained ritual function and status of a community whose social roles were integral to the social order.

Along with the *tols*, there were also centres of Arabic learning where Persian was the medium of instruction for it was the language of the court in most parts of the country, as in Bengal. The Arabic schools in particular taught “numerous grammatical works, exhaustive courses of reading on rhetoric, logic and law, a detailed study of the external observances and fundamental doctrines of Islam; Euclid and Ptolemy in translation were not unknown; there were also courses in metaphysics and natural philosophy” (Basu, 1982, p. 32-33). At a larger level, the entire complex of indigenous educational tradition seemed to have taken on the function to ‘conserve custom, to organize and sanction the existing political and economic order and to provide philosophical and religious enlightenment to the ruling classes’ (Basu, 1982, p. 33).

The colonial encounter

This entire tradition of indigenous education came into contact with the colonial project of the British primarily through the early Christian missionary societies like the Tranquebar educational experiments and the Schwartz’s schools in Tanjore, Ramnad and Shivganga in the 1770s and 1780s. The Baptist missionaries Carey, Marshman and Ward at Serampore, the London Missionary Society and the American Methodists in Bombay were the early agencies (Basu, 1982, p. 4) to engage with the *pathshalas* and the *tols* through the process of making education as a means of missionary work in the colony. These, along with the early, hesitant attempts of the emerging company state, gave rise to some of the early critics of the indigenous educational tradition, who reframed the ethos of recollective memory of the *pathshala* pedagogy into that of mechanical memory, or rote memory. The early educational surveys in the provinces of Madras, Bombay and Calcutta and later in the Punjab not only enumerated the *pathshalas* by counting and classification but in the process constituted an 'Indian indigenous education' divesting the sheer diversity of the curricular practices inherent to the *pathshala* practice in the different regions of the country. Robbing them of their diversity and spontaneity (Acharya, 1996) and branding them as 'rote' institutions, they sealed the functionality of the *pathshalas* from within.

The erasure of the curricular diversity in the *pathshalas* was also conditioned by the necessities of founding a revenue establishment, which hitherto had no semblance of homogeneity across different regions. Standardization of assessment practices in the revenue administration and state building together with the political imperative of sticking to a liberal rhetoric of a 'civilizing mission' meant that there were always contending ideals of functionality in establishing a modern education apparatus in a colony. The former required a 'recruitable public' which meant a definite notion of functionality in education that would yield a cadre of highly useful, functional men to serve the state. The latter although apparently wanted to free education from any functionality to cultivate liberal minds through the process of the grand imperial civilizing mission ended up serving the former agenda. One of the most ironic aspects of this legacy unfolded in the early nineteenth century when the colonial project in the name of dislodging mechanical memory from the *pathshala* practice did exactly the reverse by bringing it right in to the centre of learning. The institutionalization of this new functionality also tied education with employment changing hitherto local-public evaluation into a new machinery of impersonal, objective and very private, individual mode of evaluation bringing to the fore definite ideas of merit and capability in a highly hierarchical societal order.

The detailed processes by which these aspects played out in the different provinces in the case of the teaching and learning of mathematics right through the nineteenth century awaits the attention of scholars. There are a few studies that we could rely on to provide us with some picture about the developments in the different provinces. In the early decades of the nineteenth century, colonial intervention in education was ridden with ambivalence towards what was characterized as oriental learning and the initiatives under what came to be known as public instruction involved a strategy of 'tactical accommodation' of components of the indigenous education. For example in the Poona Sanskrit College traditional subjects like *jyotish*, *vedant*, *ayurveda*, *Nyaya*, *Dharmshastra,* etc., were taught to 'maintain a goodwill among the Hindus'. In Calcutta

Sanskrit college students were required to study mathematics through Indian classical texts. The mathematics contained in these texts were deemed to be 'very ingenious', yet inferior to European mathematics, while in the same institutions the Sanskrit translation of Hutton's mathematics was also used in the classroom, so that the 'students will be exposed to their own sciences along with a more advanced one' (Tiwari, 2006, p.1270). In the case of pathshalas too, in Madras presidency in Thomas Munro's first scheme to intervene in indigenous education, modern European sciences were taught at the district level Collectorate institutions, whereas the '*tinnai* curriculum', especially the *Encuvati* was retained in the *tashildari* schools established by the company administration.

There were also several concurrent teaching experiments as in the case of the Benaras Sanskrit college where *Nyaya* was taught along with European way of studying logic. Another instructor, Pandit Bapu Deva also taught astronomy, mathematics and mechanics. In fact, there was an institution for village schoolmasters - the *pathshala* teachers, at the Benaras college with an aim to introduce both European knowledge as well as Hindu texts used in the village and the teshil schools. This probably led to a phenomenon centred in Benaras where several works in Western mathematics and astronomy were written in Hindi and Sanskrit by the pandits of Benaras even during the second half of the nineteenth century (Tiwari, 2006, p.1274).

But the most interesting feature in the early initiatives in education at the beginning of the nineteenth century was in the realm of pedagogic innovations that were developed through a process of transmission involving the colony and Europe simultaneously (Tschurenev, 2008). One such innovation was the monitorial system of education. We provide a brief account of this process of transmission using the south Indian case involving the *tinnai* schools and the manner in which the missionaries engaged with them. The important features of the missionary engagement with the *tinnai* were, their negotiation with the local necessities of legitimacy and credibility, bargaining with the company administration to provide leverage in employment for their students and setting up their strategies of teaching and learning conditioned by these features. More importantly they were inadvertently carriers of a system of pedagogy, blessed by the Episcopal authority of the English church, which during the early decades of the nineteenth century was still the dominant player in elementary education in the whole of England. Here is the story of a system of pedagogic knowledge, which traveled from India, became one of the primary means to popularize elementary education in England, and came back to India, all along, going through continuous attempts at modification and improvement. This was called the Madras System of Education, or even sometimes the Malabar school system.

The Bell – Lancaster pedagogy and the *pathshalas*

Rev. Andrew Bell was the company chaplain in the Egmore Male Asylum in Madras in South India, during 1789-96. During his tenure, he encountered, or rather, discovered the *tinnai* pedagogy during one of his morning rides. Even though the available accounts in England make it out as a dramatic discovery of writing on sand (for reasons of economy) and for its monitorial system, in effect, Rev. Bell did two things:

- The actual working of the *tinnai* pedagogy was reframed into a set of principles based on Christian value system
- Internally, the memory mode of learning in the *tinnai*, using mutual instruction was reconstructed with respect to reading, writing and arithmetic, the three R's, as perceived in England.

Bell practiced his experiments in the Egmore asylum till he left for England in 1797, where he published his book on the Madras system in the same year. His arithmetic pedagogy, as spelt out in his book had very close resemblance to the *Encuvati* mode of learning in the *tinnai* schools. Every number had to be resolved into its component parts, which are simply half periods, consisting alternately of units and thousands. With respect to the four cardinal operations, he said, let the elementary parts be perfectly learned in classes of short, easy and frequent lessons, repeated as often as necessary. But before proceeding to actual operations, make the learning of tables perfect, so that, students could themselves construct addition, subtraction tables along with multiplication and division.

There ensued a major controversy between Bell and Joseph Lancaster, a dissenter and a Quaker, who claimed to have invented the monitorial system of education independently. The entire affair turned out be messy and engaged several levels of arbitration, including the Church of England and the Princess of Wales (Salmon, 1932). Without going much into the details of the controversy, the various missionary experiments in the first half of the nineteenth century in south India came up with a system of arithmetic teaching whose salient features were:

1. Plan of mutual instruction with monitors guiding the students
2. Memory and its practice central in reading and arithmetic, with loud recital, simultaneous vocalization and visualization while writing
3. Memorization began with the writing down of figures in sand tables
4. Centrality of tables, with the difference being (in relation to tinnai) that the four operations of arithmetic taught in relation to each other by simultaneous construction of addition and subtraction tables, and multiplication and division tables

5. Memorization proceeded along with the construction of tables
6. Memory tested step by step by the monitor during the construction of the table and while associating with operations
7. The actual operations however now had to be worked out on slate, and by using columns, making the slate, the central device.
8. Keys, or guidebooks to the monitors becme essential.
9. The general plan of arithmetic teaching became standardized as: 1. Combinations of figures, 2. Addition 3. Compound Addition 4. Subtraction 5. Compound Subtraction 6. Multiplication 7. Compound Multiplication 8. Division 9. Compound Division 10. Reduction 11. Rule of Three 12. Practice
10. Emulation, rewards, steadiness of application become the normative values for students to imbibe.
11. The teaching of catechism fitted in perfectly in the monitorial and memory based system

However much the argument was against rote learning, the learning of arithmetic now proceeded by learning a given set of rules, practicing these rules followed by "reduction" (i.e., problem solving). This is in contrast to the *tinnai*, where the starting point was the learning of numeration through the *Ponnilakkam* and the *Nellilakkam* by the simultaneous construction of the number series, which would then lead to the *Encuvati*. Along with this, problem solving was trained, testing associative memory, through oral means. The pride of each student, who would create his own manual on palm leaves would come after learning numeration and the tables.

Both Bell and Lancaster provided a major impetus to the spread of popular education in England, where education had been confined to the classical learning in the public schools and the two universities for the elite whereas the poor went to the Sunday schools, which were conceived as a means to counter the emerging radicalism of the working class. William Whewell and Augustus De Morgan, in their own ways pioneered certain reforms in the field of mathematics education. For Whewell, the teaching of mathematics was indispensable to liberal education, as 'discipline of the reasoning power will enable persons to proceed with certainty and facility from fundamental principles to their consequences', wherein the best means to educate men in reasoning would be the study of mathematics. Mathematics was about reasoning by practice superior to the study of logic, which was reasoning by rule. Learning arithmetic was like fencing or riding, a practical art, cultivated by habitual exercise (Whewell, 1836). In 1831, De Morgan published his work on mathematics education, *On the Study and Difficulties of Mathematics*, as part of his initiative to reform British education from the dominant hold of the classical education scheme of the Cambridge, Oxford kind, through his London based, Society for the Diffusion of Useful Knowledge, which published new textbooks and manuals for the teaching of Arithmetic and Algebra in the 1830s. The University of London, to which he belonged was part of a political initiative to move away from the dominant Oxbridge mode of classical education for the elite. He critiqued the Bell-Lancaster system and literally lambasted the pedagogy: this system, to him, broke down arithmetic into a multitude of rules, many of them so unintelligible that they could be 'Hebrew'. Pupils were not expected to understand the reasons for rules but merely to be able to apply them. Teachers were scared to teach such principles, for to do so required knowledge and understanding. Therefore, it was much easier to teach rules and various books of worked out solutions to avoid any troublesome questions. As a result, after several years of working meaningless and useless questions by the slates-full, the student left school as a 'master of few methods, provided he knows what rule a question falls under'. According to De Morgan, Arithmetic teaching should commence with a clear explanation of methods of numeration, illustrated by reference to other systems besides the decimal and supported with the use of counters. The Bell Lancaster system condemned the majority to the rote learning of half digested gobbets of information (Howson, 2008, p. 86). In effect, memorizing the rules to carry out each case, a student would only have to recognize which case a given problem falls under and apply the appropriate rule, a game of manipulation based on matching problems to rule (Phillips, 2005, p. 105-133). But even in the 1830s, when this critique was being developed in England, its public school system was very much rooted in the classical tradition, which was meant to educate politicians, civil servants, clergy, army-men and administrators of the empire (Howson, 1981).

However, the Bell-Lancaster system came to India much earlier. The first evidence of it could be found in the noted Burdwan Plan of 1818, which was a general plan of instruction for the Indian schools under the auspices of the missionaries. Bishop Reginald Heber went on a grand Episcopal tour, with the determination to push Bell's system of education in the provinces of Bengal and Bihar. In the South, Bishop Middleton started a school under the Bell model in 1819, followed by Fr. Rhenius of the CMS in the schools of Tirunelveli around the same time. The localizing of the Bell pedagogy in South India through the missionaries, brought up a scenario where within the same Tamil missionary institution, one had the Bell system in operation, with Tamil schoolmasters; and in other sections, a different Tamil schoolmaster would be teaching Tamil arithmetic of the Encuvati mode, using palm leaves. What exactly was happening to the two modes of memory, the Bell-Lancaster mode and the tinnai mode, within the confines of the same institution, with the native schoolmasters is difficult to figure out. But the missionary strategy evolved out of this experience, which clearly stuck to 'native ciphering' in its Tamil schools, with Tamil schoolmasters till they were ready for their European arithmetic. In their English medium schools in urban centers for the landed elite, they continued to teach European arithmetic, primarily using European textbooks, which were used in the military seminaries, including Walkingame, Charles Hutton, Maclarin, etc. Among these, Hutton's Course of Mathematics was the single most influential arithmetic textbook that generations of army-men and civil servants were trained on, in the seminaries of Woolwich and the like.

Some of the mathematics school books written by the missionaries like the one by Mary and Harle in Chinsura was unanimously promoted by the early School Book Societies like the Calcutta School Book Society which also gained popularity among the Bengali *pathshala* teachers as the demand among the native schools in Bengal was for English where "students were eager to acquire account keeping skills in both the old and the new English forms of arithmetic and they wanted to practice an accurate, elegant handwriting" (Tschurenev, 2011). In the case of the Madras School Book Society the first arithmetic textbook in Tamil by Ramasamy Naicker was published which classified the learning of arithmetic under the four operations as distinct from the Tamil mode of traditional classification based on *Ponnilakkam, Nellilakkam* and the *Kulimattu*. Along with Ramasamy, there were also in active circulation, Walkingame's Arithmetic and Charles Hutton's Course of Mathematics, two volumes which themselves represented the old and the new in the British arithmetic curriculum. The Bombay Native Education Society under the leadership of Capt. George Jervis adapted several of the contemporary British textbooks like Hutton, Bonnycastle into Marathi and Gujarati textbooks and published full fledged translations of some of the English textbooks. The entire two volume Course of Mathematics by Hutton was translated by Jervis as *Shikshamalla* (Bombay Native Education Society, 1831). Through all such attempts, Notation and Numeration, the four simple rules of arithmetic, compound rules and working of problems based on the rules, the rule of three and fractions constituted the syllabus, and the elementary mathematics curriculum deviated little from this set course for the next hundred years.

By the mid nineteenth century, with the combination of the missionary engagement and the company's intervention certain features of mathematics teaching became significant. It had set the elementary arithmetic curriculum to the four simple and compound rules along with the rule of three. Though professedly against memory, in practice it seemed to have perpetuated it as wasthe case in contemporary England. Arithmetic was about memorizing tables and manipulating numbers to a set of rules. The normative values associated with arithmetic became, perseverance and steadiness of mind, which was not yet found in the Indian student. However, it was believed that the anxiety of an English education would ultimately inculcate such values in them, through the learning of modern arithmetic. The use of textbooks became integral and indispensable. Along with them, practicing problem solving in the four operations became stuck to slates, or to pen and paper, and a few genuine attempts to introduce mental arithmetic all became mere supplements to the slate. But the elementary mathematics teacher largely was on his own. It was primarily left to him to deal with these new textbooks and the new mode but with the same old Indian student, without any training or assistance. It appears that this parallel coexistence of the *pathshala* mode and the textbook centered, slate centered arithmetic was well established by the the 1850s, and nothing much changed in their respective practices for at least, the next three decades. And since then, any encounter between the Indian student with modern mathematics, in its institutional avatar, was characterized as mechanical memory.

Indian initiatives

While this process was playing itself out during the first half of the nineteenth century during the making of the colonial state, production of textbooks acquired a very important place in the sphere of negotiation between the contending ideals of mathematics education. Vedanayagam Shastri, trained under the German missionaries in Tanjore attempted to reform and rewrite the traditional Tamil *Encuvati*, the table book used in the *pathshalas* in such a way that the process of constructing arithmetic tables become transparent to the student. He called it *En Vilakkam*, meaning number explanation as opposed to *En Cuvati*, meaning number text – from the fixed static number text to the dynamic and process centred number explanation as tables. Master Ramachandra's significant pedagogic initiative in Delhi led him to deduce an alternate way of solving simple problems in calculus, as finding the maxima-minima. In his project what was normally a problem of differential calculus was "brought within the possibilities of pure algebra", a project that was nourished by algebra as a "cultural metaphor", "designed to rejuvenate and update the supposedly algebraic disposition of the Indians" (Raina and Habib, 1989, p. 2083-84).

The arena of textbooks would continue to become a site for negotiations throughout the nineteenth century, especially at the turn of the century when it became the primary means of a nationalist engagement to intervene in education. Writing textbooks in the vernacular as a nationalist endeavour involved two identifiable tendencies. There were conscious attempts to resurrect the traditional texts and to publish them without allowing any single trace of the British or the European elements to enter the textbooks, as in the case of the publication of the *Encuvati* table books in the Tamil speaking region in the 1920s in Tamil, in the various small towns. The other tendency was to look for grounds of convergence between the two traditions, while attempting to write textbooks. The famous nationalist scholar Gopal Krishna Gokhale wrote Marathi textbooks for arithmetic where he used '*upapattis*' while introducing the four operations using Marathi numerical notation through out. There were distinct changes to textbook writing practices during the turn of the century where there seemed to have been a conscious attempt to avoid fraction tables, for instance. By the 1930s, one could discern a certain standardization in the production of arithmetic textbooks into chapters organized on the basis of the four operations, omitting the compound operations and retaining the rule of three (Subramaniam & Kanhere, 2011).

Appropriating the *pathshalas*

The dominant story however from the mid nineteenth century onward was the consistent attempts by the colonial administration to convert the *pathshalas* of the various regions into modern schools, apart from parallel attempts to institute their own set of institutions teaching European sciences. The Woods Despatch of 1854 paved the way from the early filtration theory to mass education by the colonial administration resulting in the establishment of a massive educational bureaucracy with a strong inspectorate system consisting of ranks of school inspectors whose primary job was to convince the *pathshalas* to adapt to a modern curriculum. This policy of conversion in practice resulted in setting up of a colonial educational bureaucracy, heavily centralized, which sought to extend its control right into the classroom. Textbooks became the vital medium of this control and curriculum prescription for eligibility of grant in aid, the guiding frame of authority. Minimal enrolment forced innovations within the grant in aid, on a continuous basis, like the salary grants system, the payment by results system or the combined system of both salary and results, all meant to convert the *pathshalas* into the modern fold (Shahidullah, 1996).

The *pathshalas* however survived undisturbed, but for a few towns where the inspectors would reach in their extensive tours (provided it was an upper caste neighborhood), where the village elite (often of the same caste as that of the inspector) would determine the possibility (or otherwise) of the school to convert (Senthil Babu, 2011). An entire series of public examinations in the colonial state's consistent effort to create a recruitable public came to be instituted from the early 1840s, which was also modeled on the British tests for civil services. These texts with their rigid prescription of the curriculum had a crucial role in the making of the first generation of educated middle class. But the mass production of qualified people for government jobs had to be soon kept under check, which resulted in further grades of public examinations and high rates of failure.

Within about ten years of the grant in aid experience, concerns were raised about the curriculum from various quarters, primarily led by the missionaries. The curriculum was too difficult for an age group of 7-15, with the movement of one grade to another now entirely decided on the basis of written examination rather than a scheme of 'liberal' education, as the British intelligentsia in the metropolis imagined. The rule of textbooks consolidated itself: it was just that you had vernacular translations of English textbooks instead of writing new textbooks as seen in the early period. Colenso, Barnard Smith and Bradshaw were the three arithmetic textbooks that were in contention, and repeated offers were made for better translations. Already by the 1870s, a well formulated critique of the curriculum, examinations and the rule of textbooks evolved and the most significant evil was found to be mechanical memory.

Assimilating the *pathshalas*

Evidence was collected from several members involved in the business of education during the Education Commission of 1881, popularly known as the Hunter commission which gave its Report in 1882. Various representatives who presented their views to the commission, across ideological and caste equations, argued for the inclusion of the *pathshala* arithmetic into the official grant in aid regime, if the government was at all serious about expansion of elementary education. In case of Madras where we have examined in detail the process of the engagement with the *tinnai* schools, scrutiny of the various arguments in the Education Commission reveal interesting arguments about the *tinnai* arithmetic curriculum. The results grant system, aimed mainly at the elementary schools, they argued, would only succeed, if the *tinnai* arithmetic becomes part of the official curriculum. And suddenly, in the submissions to the commission, arguments about relevance of the curriculum to local contexts, the necessity of skills to enable children for better participation in life, that education should also be for purpose of life and not merely for promotion of higher standards or for employment - were all voiced during the committee's proceedings. The tinnai master's proficiency in that mode of arithmetic was recounted, celebrated, almost with a sense of nostalgia. Ironically, this nostalgia came not from the educational establishment but from the employers: the railways, the chambers of commerce, the banks, revenue, public works–in short the employers were lamenting how the first generation of employees who came out of the traditional system of education were so efficient in all trades, especially in book keeping and mathematics. They lamented that "the present generation of high school and middle school graduates despite going through the grind of multiple public examinations, which do help in providing certification for us to employ on public standards, can't do simple addition properly" (Evidence, 1882). Others went to the extent of arguing that the British better realize that they are dealing with a complex civilization which the indigenous society is; they just cannot do with a cut and dried curriculum, imposed from without catering to the varied wants of an entire country; all one could innovate upon were 'refined ways of torture' in the name of examinations and evaluation and so on. There were also strong arguments that the community should decide the course of instruction. The Principal of the Presidency College of Madras, Mr. Duncan explicitly stated that the business of teaching deductive logic and arithmetic should be left to the natives as they were good at it (Evidence, 1882).

The result of such arguments in the various provinces due to the Education Commission reflected in several policy measures on the part of the colonial state in the last decades of the nineteenth century. The specific ways in which they figured in practice in the *pathshalas* and other kinds of elementary institutions awaits detailed study. However in the case of Madras presidency, there was a clear shift in coming to terms with the resilient traditional institutions by consciously trying to assimilate elements of their curriculum into the official mode. The traditional arithmetic now got assimilated into the official curriculum as 'mental arithmetic' along with slate centered practice of learning the four basic arithmetic operations and the rule of three. The tables of the *Encuvati* now became part of the new textbooks including the conversion tables of the various weights and measures into the modern English measures, while the use of Tamil numerical notation was entirely done away with. Rechristened now as 'bazaar mathematics', mental arithmetic became a regular adjunct to the rule based problem solving in practice and reduction. It was not recollective memory as the very mode of learning. Memorization was not trained and honed as interpretation. Memorization became an aid in learning arithmetic. It was not prudence that was the preferred virtue as in the case of *tinnai* arithmetic, but speed and diligent following of rules that would get results, became the normative value for students to imbibe.

The state monopoly over publishing textbooks was given up in favour of private enterprise, which resulted in a flourishing native textbook market bringing in a new generation of clerical and teaching professionals as textbook writers. They only had to follow the grant in aid based guidelines while attempting to write new textbooks. The grant in aid guidelines now also integrated public service examinations with the school curriculum by assigning portions to be covered in various subjects for particular examinations, as shown in Table 1 below. Along with generating a new culture of arithmetic practice attuned to the now well set mode of examination centered evaluation for the sake of employment, rather than for the sake of a general education, a new mode of functionality became institutionalized in the learning of elementary arithmetic by the turn of the century. This also in many ways sealed off the school arithmetic curriculum and pedagogy from various modes of learning and engagement with the world of arithmetic outside the school.

In this process, a curriculum of arithmetic mixed enough to make it legitimate in front of the local public along with some space for the rural schoolmaster to teach came into being. But the rate of failures was increasing in various examinations and the number of drop-outs in them were on the rise. The increasingly visible and vocal Indian intelligentsia, thanks to the emerging print world, started talking about the tyranny of exams, the rule of textbooks and growing unemployment leaving a lasting legacy of issues in the sphere of elementary teaching of mathematics for subsequent decades.

Table 1: Arithmetic Curriculum in Madras Presidency, 1881 (Evidence, 1882)

Name of the examination	**Class and Marks**	**Arithmetic Standards Required**
The Higher Examination for Women	Compulsory; Maximum 90 and Minimum 30	The four simple and compound rules, reduction, vulgar and decimal fractions, simple and compound proportion, practice, extraction of square root, interest

	Optional (Mathematics); Max 80 and Min 20	Euclid – The first two books with easy deductions Algebra – Addition, Subtraction, Multiplication, Division, Involution and evolution of algebraic quantities and simple equations with easy deductions
The Special Upper Primary Examination	Compulsory; Max 80 and Min 26	Four simple and compound rules, reduction and vulgar fractions *(English figures must be used, and the candidate must be acquainted with the principal Indian weights and measures)*
Examinations under the Results system	First (Lowest Standard); Max 16	Notation and Numeration to four places of figures, Simple addition of numbers of four figures in five lines. (*English figures must be used in this as well as in the higher standards)*
	Second Standard; Max 24	Notation and Numeration to seven places of figures. Multiplication table to 12 times 16. Four simple rules
	Third Standard (Vernacular); Max 32	Easy questions in the compound rules and reduction, restricted to Indian weights and measures, and money tables published by the DPI Easy mental arithmetic restricted to the simple rules
	Fourth Standard: Max 48	Miscellaneous questions in the compound rules and reduction, easy questions in vulgar fractions Mental arithmetic applied to bazaar transaction
		In Vernacular schools the questions will bear exclusively on the Indian tables published by the DPI, *including the native multiplication table of integers and fractions marked A, and the table used in native bazaars marked B.*
	Fifth Standard, Max 56	Simple and compound rules, reduction, vulgar and decimal fractions Mental arithmetic applied to bazaar transactions
	Sixth standard; Max 48; now with the head Mathematics	Arithmetic- as for the fifth std, with the addition of Practice and Simple Proportion Euclid – Book I, to the end of the 16^{th} proposition

The Middle School Examination (Seventh Standard)	Branch D: Arithmetic: 110 marks	The compound rules, Reduction, Vulgar and Decimal Fractions, Practice, Simple and Compound Proportion (English figures must be used and the candidate must be acquainted with the Indian weights and measures and the English tables of money, of Troy weight, of Avoirdupois weight, of Lineal, Square and Cubic measures, and of Time)
	Branch E: Mathematics, 90 marks	Euclid, Book I (50 marks) Algebra, to the end of Fractions (40 marks) Symbols permitted by the Madras University may be used.
The Upper Primary Examination	Branch B: Arithmetic (compulsory) 40 + 10	a) To work miscellaneous questions in Reduction, the Compound rules and Vulgar fractions b) Mental Arithmetic applied to bazaar transactions
Lower Primary Examination	Arithmetic, 40 marks	To work sums in the first four rules of arithmetic, simple and compound, including easy miscellaneous questions

The *pathshalas* continued to exist well into the first decades of the twentieth century independent of the colonial educational apparatus before they were rendered irrelevant or assimilated into it. Going to school and passing examinations acquired distinct yet different cultural meanings for different classes of people. Studying for employment however continued to remain a dominant concern and only strengthened over the years. The new nation state after independence was concerned with institutionalization of a modern general secular education but even a highly limited version of the idea of fundamental right to education took about six decades to materialize as a piece of legislation, giving rise to a series of social and political conflicts still centred around questions of access and denial. That of course is a different history.

Conclusion

We have tried to narrate the story, from available evidence, of the indigenous traditions of engagement with the learning of mathematics. In telling the story we have pointed out the strong functionality that resides at the core of curricular and pedagogical practice, and the social basis for it. The story of the colonial encounter could actually be reconstructed as a history of transition of the indigenous elementary institutions into the modern ones, the existing institutions of learning were not only vast in number but had a sustained, resilient presence through the colonial era. The encounter highlights the inherent contradictions of the colonial project. The resilience of indigenous institutions was based on a social resource, namely the public perception of what a relevant curriculum should be in the local school. Training in arithmetic was considered essential for participation in local economic transactions. Set against this was the colonial empire building exercise with its agenda of homogenization, which set out to counter and liberate the pathshala curriculum from its functional ethos, but the pathshala functionality refused to yield. What this left behind was a legacy of mechanical memory as the dominant mode of learning arithmetic, whereas *arithmetic* was not a goal of the *pathshala* but computational ability in context.

The notion of numeracy as defining the modern condition almost synonymous with the idea of literacy also tended to project one particular idea of what it means for a people to learn the world of numbers. Fifty years later, social movements that set out to 'make people literate' discovered remnants of the indigenous pathshala tradition inside the people who were anyway actively engaged with the world of mathematics in diverse ways. Social practices continued to enrich this tradition and helped it survive in many forms (Rampal, Ramanujam & Saraswati, 1999).

Such a historical study raises several questions for the modern social enterprise of universal mathematics education. Can the rich diversity of social and cultural practices be accessed for the purposes of mathematics teaching / learning in school? Does the fact that in this realm the practice of mathematics is embedded in functional terms raise a conflict with the modern articulation of goals of mathematics education, or can these be harmonized? How does a democratic practice of mathematics education address local but public perception of what a relevant mathematics curriculum should be?

While the answers to these and other similar questions are unclear, it must be acknowledged that the diversity of people's ways of engagement with mathematics still remains largely inaccessible to children in school. At the least, mathematics education systems will have to begin by recognizing this fundamental fact from history.

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4. Transforming the Elementary Mathematics Curriculum: Issues and Challenges

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Introduction

Mathematics is often referred to as the 'killer' subject and in India a large number of children fail or drop out before completing elementary school because they cannot cope with the demands of the curriculum. The Report 'Learning Without Burden' (Government of India [GoI], 1993) had pointed out that children were in fact not 'dropping out' but were being 'pushed out', owing to the 'burden of non-comprehension', as a result of an irrelevant curriculum, distanced from the lives of the majority, and often rendered 'boring and uninteresting' by outdated teaching strategies. This shift away from conventional 'deficit theories', which attribute children's inability to learn to some 'deficit' in their mental abilities or their home background, has led to a critical review of the curriculum and the traditional teaching learning process based on rote memorisation of facts. The National Curriculum Framework (National Council for Education Research and Training [NCERT], 2005a) guided the development of new curricula and textbooks based on how children actively construct knowledge, rooted in social and cultural practices. It recommends breaking down of rigid boundaries that encapsulate school disciplines, especially at the elementary stage, and calls for consciously removing the sharp dichotomies that exist between the knowledge of school and that of the child's home and community.

The Right to Education (RTE) Act for children aged 6-14 years mandates "learning through activities, discovery and exploration in a child centred and child friendly manner" by "making the child free of fear, trauma and anxiety" (Section 29, GoI, 2009), and has major implications for mathematics in elementary school (Classes I-VIII). This Act also posits that assessment must be done in a continuous and comprehensive manner and bans any mechanism of selection for the purpose of admission into schools, or the conduct of competitive Board examinations at the elementary stage. The traditional pattern of examinations has been extremely selective, based on contested notions of mathematical 'talent', and has not only intimidated children but has also dissuaded creative teachers, since their efforts to encourage sense making tend to get obliterated by the focus on procedural questions devoid of meaning and contextual relevance.

Goals for Elementary Mathematics

The Position Paper of the National Focus Group on the Teaching of Mathematics (NCERT, 2006a) calls for a shift from achieving 'narrow' goals to 'higher' goals, from mathematical content and procedural knowledge to processes and learning environments that promote abilities for mathematization, which invite participation, and offer *every* child a sense of success. According to Sfard (2008), a 'participationist' vision of learning mathematics, unlike the conventional acquisitionist approach, acknowledges that learners begin by participating in collective mathematical discourses, of the home, community, or school, and progressively learn to individualise the discourse, as they communicate mathematically with themselves. The challenge of designing curricula for schools as diverse and iniquitous as are in India, is therefore daunting, to ensure representation of diverse discursive mathematical practices, through critical pedagogies that enable democratic participation (Rampal, 2010). For, even the early math signifiers of 'more' or 'less' can have different meanings for participants positioned in particular ways. As has been noted by Walkerdine (1990), mothers of working class families frequently using the contrastive pair 'more/*no more*' rather than 'more/less', while strongly regulating consumption of basic commodities, tend to cause children to have strong negative associations with the term 'more', which are generally not understood by curriculum designers, who might naively assume mathematics to be 'culture neutral'. Moreover, even the ways people construct and use mathematics as part of their lives, where to materially survive some cannot afford to go wrong in a computation, while for others it may afford a theoretical or even a recreational engagement, can relationally redefine the classic 'concrete/abstract' distinction.

In the rights framework, it is recognised that quality is indeed tied to equity, which can be achieved only when there are high expectations from and meaningful opportunities for *all* children to perform well. Indeed in several countries the curricular goals for mathematics place the 'equity principle' high in priority, recognising that this school subject has traditionally played the role of 'gate keeper' and contributed to high inequities in performance and participation, especially of underserved, ethnic and disadvantaged groups, who are marginalised from the way school mathematics is structured. It has been seen that when the learning environments available to diverse groups are transformed, through more realistic, relevant and meaningful contexts, mathematical attainments become more equitable for all children, who show a greater sense of self esteem and confidence (Boaler, 2008; Nasir and Cobb, 2007). This equity framework challenges deeply entrenched beliefs in society that only some 'talented' students are capable of learning mathematics, and calls for a democratic restructuring of the purpose and nature of the school curriculum, as well as of what constitutes mathematical 'talent'.

The last few years have seen efforts in India to address this major challenge, and syllabi and textbooks have changed to some extent, though much of the task is still in progress. However, it is crucial to recognise that within the present state of resource starvation of most of our elementary schools, textbooks remain the only educational materials available for children. There are no concrete objects, games or manipulatives in the classroom, and, even though NCF 2005 lays stress on this, teachers most often do not recognise the significance of such materials and processes for math learning. Curricular reform thus involves layered negotiation, within and outside the system, from policy documents to classroom practices, in states and at the national level, involving several key players such as administrators, teachers, parents and the media, to change mindsets about how children learn and how that may be assessed, and what basic provisions are essential to make schools conducive for that learning to happen.

The Position Paper (NCERT, 2006a) proposes that mathematics teaching and learning should promote a multiplicity of approaches, crucial for liberating school math from the tyranny of the one right answer, found by applying the one algorithm taught. This focus is meant to make math enjoyable and challenging, through activity based learning processes of problem solving, estimation, optimisation, use of patterns, visualisation, representation, and mathematical communication, which play an important role in removing the fear of mathematics. There is also an attempt to draw upon the rich cultural resources of everyday and folk mathematics – for measurement, estimation, and understanding of shapes, symmetries and aesthetics - through contextual examples from art, architecture and music.

As part of the National Curriculum Framework 2005, the syllabus developed by NCERT laid out the following curricular areas in a progressive grid of concepts for the first five years of school: Geometry (shapes and spatial understanding), Numbers and numbers operations, Money, Measurement (length, weight, volume, time), Data handling, and Patterns (see Figure 1).

The syllabus recommends activities and exercises that span children's life experiences across the curriculum of different subject areas, where extensions of the activities are also part of the main text materials. It underscores that mathematics is a way of thinking and reasoning and textbooks must use children's local interests and enthusiasm for developing concepts, through an interactive approach that gives space for a child to articulate her reasons for choosing a certain strategy. Problem posing is acknowledged as an important part of doing math and calls for opportunities in the textbooks and classrooms for children

IN MATHEMATICS AT PRIMARY STAGE

Class III	Class IV	Class V
Geometry (16 hrs.) SHAPES & SPATIAL UNDERSTANDING • Creates shapes through paper folding, paper cutting. • Identifies 2-D shapes • Describes the various 2-D shapes by counting their sides, corners and diagonals. • Makes shapes on the dot-grid using straight lines and curves. • Creates shapes using tangram pieces. • Matches the properties of two 2-D shapes by observing their sides and corners (vertices). • Tiles a given region using a tile of a given shape. • Distinguishes between shapes that tile and that do not tile. • Intuitive idea of a map. Reads simple maps (not necessarily scaled) • Draws some 3D-objects.	**Geometry (16 hrs.)** SHAPES & SPATIAL UNDERSTANDING • Draws a circle free hand and with compass. • Identifies centre, radius and diameter of a circle. • Uses Tangrams to create different shapes. • Tiles geometrical shapes: using one or two shapes. • Chooses a tile among a given number of tiles that can tile a given region both intuitively and experimentally. • Explores intuitively the area and perimeter of simple shapes. • Makes 4-faced, 5-faced and 6-faced cubes from given nets especially designed for the same. • Explores intuitively the reflections through inkblots, paper cutting and paper folding. • Reads and draws 3-D objects, making use of the familiarity with the conventions used in this. • Draws intuitively the plan, elevation and side view of simple objects.	**Geometry (16 hrs.)** SHAPES & SPATIAL UNDERSTANDING • Gets the feel of perspective while drawing a 3-D object in 2-D. • Gets the feel of an angle through observation and paper folding. • Identifies right angles in the environment. • Classifies angles into right, acute and obtuse angles. • Represents right angle, acute angle and obtuse angle by drawing and tracing. • Explores intuitively rotations and reflections of familiar 2-D shapes. • Explores intuitively symmetry in familiar 3-D shapes. • Makes the shapes of cubes, cylinders and cones using nets especially designed for this purpose.
Numbers (42 hrs.) NUMBER SEQUENCE UPTO 1000 • Reads and writes 3-digit numbers. • Expands a number w.r.t. place values. • Counts in different ways - starting from any number.	**Numbers (40 hrs.)** NUMBERS AND OPERATIONS • Writes multiplication facts. • Writes tables upto 10 × 10. • Multiplies two and three digit numbers using lattice algorithm and the standard (column) algorithm.	**Numbers (40 hrs.)** NUMBERS AND OPERATIONS • Finds place value in numbers beyond 1000. • Appreciates the role of place value in addition, subtraction and multiplication algorithms.

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Syllabus for Classes at the Elementary Level

Figure 1: An illustrative page from the Primary school syllabus

to create a variety of problems for their peers and others. It also states that textbooks be written in a language that the child would normally use and understand, with creative visuals, comic strips, cartoons, narratives, stories and other interesting texts. It emphasises that it is not appropriate to begin with definitions and explanations but that concepts and ideas need to be arrived at by observation of patterns and exploration, before children can articulate in their own words. This is a significant shift from the traditional approach but has not yet been fully incorporated across the school curriculum, and even across the country.

Primary textbooks: Cross disciplinary themes

The National Curriculum Framework 2005 had recommended breaking down of the rigid boundaries between different subjects, to help develop a more holistic approach of learning from the child's environment and culture. This was attempted in the NCERT textbooks at the primary stage, within the three subject areas of language, math and environmental studies (EVS), where the latter integrates science and social studies. Thus, for instance, the language textbooks took up several themes that related to the EVS syllabus, while the EVS chapters drew upon historical travelogues, poems, dramatisations and other genres of written and oral literature.

As will be discussed in some detail here, a theme such as 'mapping', which traditionally comes under geography and through a perfunctory 'skills' approach, of only reading and drawing maps, without actually progressively developing children's understanding of several underlying concepts, was consciously introduced across the subjects of language, EVS and math. Research has shown that the conventional abstract maps used in geography textbooks are neither found engaging nor are understood by children who are confounded by the symbols and lines shown, and cannot relate those to any physical features of the locations indicated. On the contrary, colourful pictorial maps made especially for young children, which help them easily interpret features such as rivers, forests, mountains along with other characteristics of a place, make map reading much more meaningful and interesting, which they can undertake with little facilitation (Sunny, 2006).

While developing the new math and EVS textbooks, and the related Sourcebook (NCERT, 2008), some research on children's' understanding of mapping was also undertaken, to get deeper insights into children's thinking and interpretative processes. An interesting observation was that even privileged children (aged 8-10 years), studying in well known schools and achieving high marks in their exams, drew maps of their own locality showing locations in a linear arrangement (Map II and III), though the actual road formed a closed shape. When asked, on site, why they made a road from the top of the page to the bottom, they confidently explained that this was because while walking on that path they always went straight ahead and never turned back! Thus the experience of walking on the road shaped the map they made, without their being able to imagine it from an aerial perspective. It was a 13 year old girl in the locality who drew the closed loop for the road (Map I) and, unlike the younger children, showed some understanding of what the place would look like from 'above' (NCERT, 2008), though the orientation and scale of her map was still impressionistic and very different from that of the cartographer.

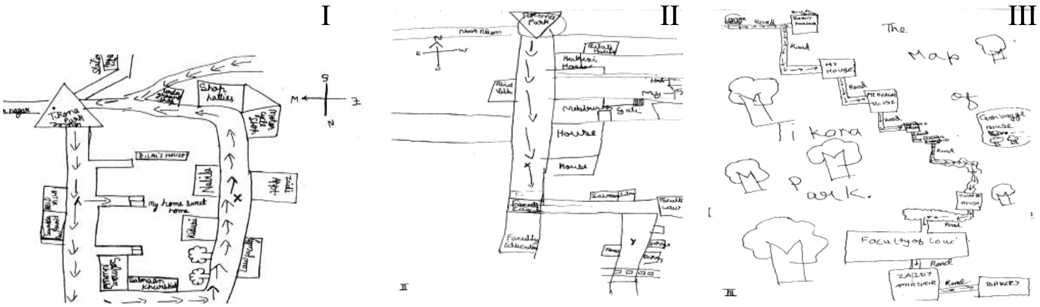
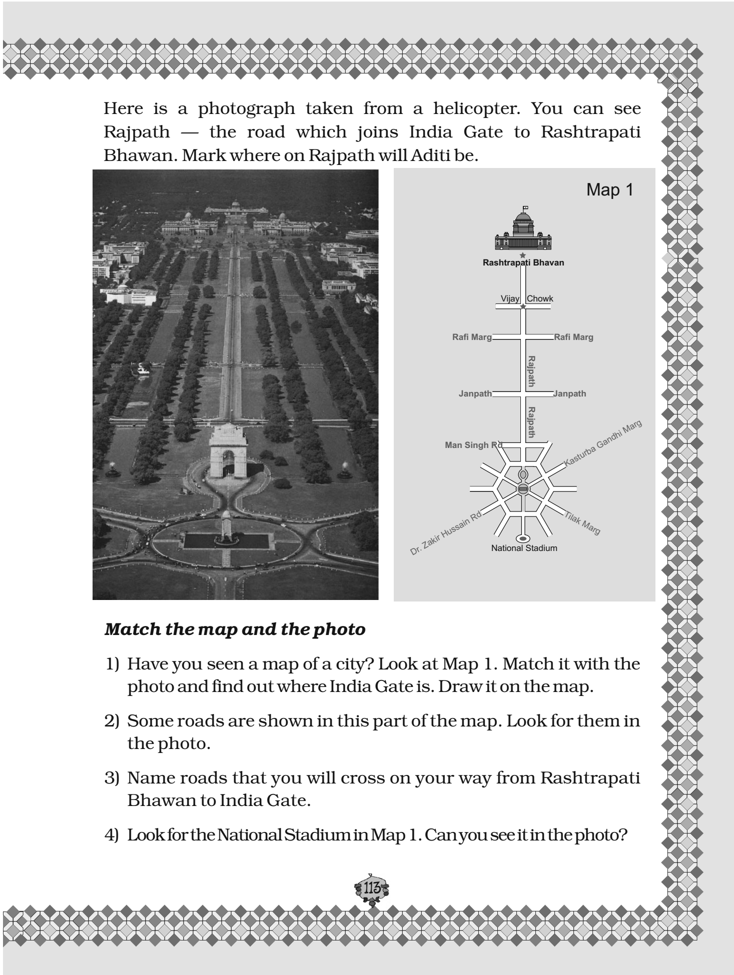


Figure 2: Map I (age 13 yrs), Map II (age 10 yrs), Map III (age 8 yrs)

Indeed it has been found that adults, in general, and even university students are unable to make sense of local maps to navigate themselves across their city, lamenting that school had not helped them at all in this respect. Indeed, recognising the inadequacy of the conventional school approach and noting the real-life relevance of developing an understanding of 'mapping', the primary math books focussed on it even more than what the syllabus had indicated. Across the math textbooks for classes II-V, several chapters were progressively developed on concepts of projections and perspective, aerial views of a site or object from different distances, spatial orientation, directions, scaling, representation, etc. Beginning with simple representations using iconic and pictorial maps related to specific contexts, through narratives, such as children finding their way to the beautiful monument of Taj Mahal, the concept gradually progressed to abstract schematic representations. A chapter was carefully developed in class V to encourage the comparison of an iconic map with an aerial photograph, of a well known location in New Delhi, namely, India Gate on Raj Path, which most children get a chance to hear about or watch on television on the occasion of the Republic Day Parade (see Figure 3).

The math books dwelt in detail on developing concepts related to mapping, and also used several creative formats, such as travelogues or a diary for a historical monument, visuals and pictorial maps, as well as games on the pattern of a treasure hunt. Simultaneously, the EVS textbook developed the ideas of mapping by exploring and 'reading' a historical fort, or through a chapter 'Sunita Williams in Space', based on the true experiences of a NASA astronaut of Indian origin, who eloquently described her thoughts as she looked at the earth from space, and even poignantly questioned why no 'lines' or boundaries could be seen between India and its neighbouring countries. This is also to address persistent misconceptions even among university students who continue to believe that different kinds of physical lines are etched on the surface of the earth, similar to the lines in maps, indicating state and national boundaries.



Here is a photograph taken from a helicopter. You can see Rajpath — the road which joins India Gate to Rashtrapati Bhawan. Mark where on Rajpath will Aditi be.

Match the map and the photo

1) Have you seen a map of a city? Look at Map 1. Match it with the photo and find out where India Gate is. Draw it on the map.
2) Some roads are shown in this part of the map. Look for them in the photo.
3) Name roads that you will cross on your way from Rashtrapati Bhawan to India Gate.
4) Look for the National Stadium in Map 1. Can you see it in the photo?

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Figure 3: Comparing a map with a photograph (Class V textbook)

Special thematic chapters: Authentic contexts

The primary math textbooks incorporated several real life narratives within chapters and also developed special thematic chapters to deal with varied issues of work, entrepreneurship, heritage, craft knowledge, history of pre-historic cave paintings, etc. Research suggests that when problems are rooted in the contexts of learners, students can demonstrate much deeper mathematical understanding and their performance is also higher on traditional mathematical measures (Hmelo-Silver, Duncan, & Chinn, 2007; Nunes, Schliemann & Carraher, 1993; Saxe, 1988).

Chapters were thus conceived of in two ways–while most focussed on concepts progressively developed as indicated in the syllabus, a few special thematic chapters used contexts that invoked and integrated concepts already learnt. For instance, chapters such as *Building with Bricks*, *The Junk Seller*, and *The Fish Tale* were thematic units that integrated concepts of shapes, numbers, measurement, money, patterns, etc. through authentic contexts. Some of these were developed to locate math in a socio-cultural framework, through issues emerging from the lives of craftspersons engaged in masonry, brickwork design and brick making, junk collectors and sellers, and fishworkers, boatsmen or fish sellers.

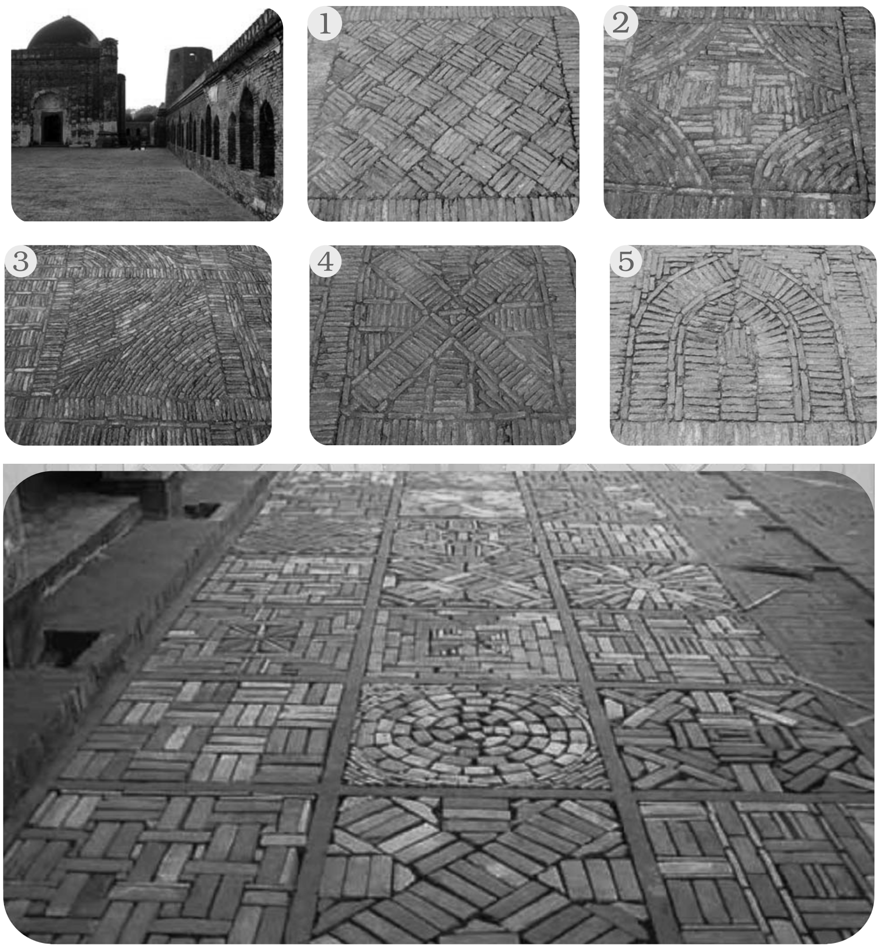


Figure 4: Brick patterns on the floor of a tomb and in the school courtyard (Class IV textbook)

Building with Bricks in Class IV begins with the true instance of a school being built by local masons, who are taken to visit the nearby mosque to observe the amazing variety of floor patterns built by their ancestors three hundred years ago (see Figure 4). They come back inspired and make their own brick designs for the school courtyard.

With modern bricks different from the older thinner ones, the masons generate different symmetries and patterns, which students are encouraged to analyse. The unit goes on to measure a brick, to study its faces, see its projections and how (as an example of a cuboid) it can be represented in two dimensions. It prompts students to analyse other brick designs in traditional architecture and uses some examples from the work of Laurie Baker (though without naming him), a well known Gandhian architect who devised low-cost environment friendly buildings, before the chapter travels to a kiln where 'hot and fresh' bricks are being made. Besides understanding the process through visuals students also work with contextually large numbers, and are first introduced to the number 'one hundred thousand', by relating it to the number of brick kilns in the country. This process of thinking of large numbers, relatively and in comparison with familiar contexts and orders of magnitude, is adopted throughout the books where, for example, they connect 100 with the scoring of a century by a famous cricketer, or are asked to recall in which contexts they have heard of a 'lakh' (one hundred thousand). These connections help in the maturation of number sense. Thus for example, "Asking students questions, such as, 'How long does it take to count to 1,000?' or 'Have you lived more or less than 1,000 days?' provides them an opportunity to think about 1,000 in a personal context, thus helping them better understand the size of 1,000 in a variety of contexts" (McIntosh, Reys and Reys, 1995, p. 216).

The Junk Seller is based on the true story of a young woman Kiran, who had, against all odds, of living in a poor, highly patriarchal rural society in the state of Bihar, managed to set up her own enterprise in the capital city. In her voice, it narrates her struggle, her early dislike of math in school and her acknowledgement of how it is now an integral part of her present vocation, which has indeed helped change her life and the situation of her family. Through this visual narrative with on-site photographs, the unit deals with her loans, her junk sorting and selling, hiring of collectors, recycling of materials, etc. It challenges several prevailing notions of gender and mathematics, the stigma of 'dirty work' as it relates to certain castes and their supposed low position in society, and the traditional emphasis on the *mahapurush* or 'great male leader as role model', while it also inspires young women with a sense of 'social agency' to develop their entrepreneur abilities to transform lives. Interestingly, this focus on cultural relevance and real life contexts caught the public imagination, and leading national newspapers and TV channels, that followed the development of the new textbooks through 2006-2008, reported on the primary math texts. Full page lead stories, with headings such as "NCERT's Bold New Experiment brings Maths Closer to Life" began with "Ever thought you could study geometry from brick patterns on the walls of a tomb in Murshidabad? Or arithmetic from a junk-seller in Patna? Well that's what the new Class IV math textbook by NCERT is all about: maths and real-life" (Mukherji, 2007).

The socio-cultural perspective

Mathematics is problematically perceived to be 'culture neutral' and school textbooks tend to homogenize the textual content as well as visual representation, so that these are further distanced from the lives of most young children. For instance, chapters on the concept of 'time' in textbooks across countries are often found to deal only with clocks and calendars, describing various units and devices of measuring time, and informing the child that a day is made up of 24 hours, an hour of sixty minutes, while a week is 7 days, and so on. However, if learning about 'time' is viewed through a socio-cultural framework, a richer canvas can be drawn, to begin with the contexts in which children already know about time, and gradually scaffolding their understanding through progressively more challenging tasks and investigations.

We briefly discuss here how the Math Magic Class III (NCERT, 2007) chapter on *Time* attempts to do this; it begins with asking children to correct some funnily jumbled up time markers used in a visual story, depicted in a folk art *pattachitra* style (which traditionally uses cyclic representations for time). Drawing upon children's common sense and real life observations and estimations of, for instance, how long it takes for a fruit to fall from a tree, for yogurt to set, for a litre of milk to boil, or a baby to come out of her mother's stomach (a deliberate attempt at subversive humour), it leads them to think of several processes that elapse in different orders of time–in years, months, hours, minutes or seconds. Moreover, being asked to find out how long it takes a potter to make a pot, a weaver to weave a *sari*, or someone to knit them a sweater, also helps children value the labour that goes into it (see Figure 5). Cultural narratives–about celebrating 'a thousand full moons' as is done in some parts of the country when a person completes eighty years, or about the 'time line' of a woman's life, where she recalls how after the death of her husband she fought against his scheming relatives who declared her a 'witch' to snatch her land– have been incorporated through exercises that bring diverse lived experiences into the classroom. This also departs from the approach of curricular 'infantilisation' or 'Walt Disneyfication' (Giroux, 1994), which believes that children should be protected from the 'harsh realities' and injustices of the real world, and therefore easily resorts to the comforting contexts of comic cartoon characters.

Indeed, conservative curricula allow at best a tokenistic approach where 'celebratory' multicultural representations are limited to viewing diversity through the lens of the essentialised 'other', without critical engagement about issues of 'difference', discrimination and dominance. In fact, when some upper class teachers wondered if

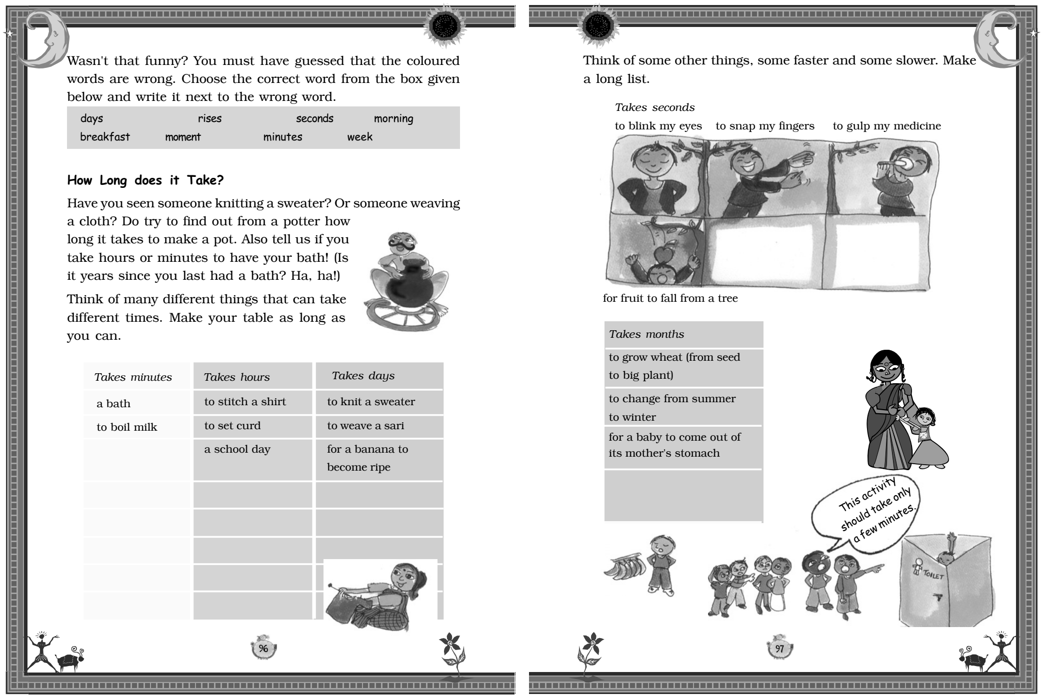


Figure 5: Finding how long it takes (Class III textbook)

'witches' should be discussed with young children, they were reminded that when no questions were asked on the suitability of a Harry Potter book or even about viewing violent cartoons about witches on television, then why was the cursory mention of a real person falsely declared a 'witch' so problematic. Instead, wasn't it important and inspiring for children to know that there were serious attempts to resist this exploitative practice against women, still prevalent in some areas? Moreover, transacting a mathematics curriculum in the socio-cultural framework "to read the world" also requires what Freire (1970, p. 62) calls "problem posing pedagogies", as distinct from problem solving ones, so that education "involves a constant unveiling of reality ... that strives for *critical intervention* in reality". It requires distinguishing between using mathematics in real world settings, usually limited to shopping, travelling, working or building, from those that ask students to critically investigate issues of injustice, through a sense of collective social agency (Gutstein, 2006). In fact, Lubienski (2000) has cautioned that "pseudo-contextual" problems found in most mathematics textbooks tend to disadvantage students from certain social backgrounds. In particular, they add a layer of challenge for students who fail to tease out tacit assumptions inherent in "school" mathematics problems which differ from those they encounter in life. At times, constructivism can also be misinterpreted to do much of the same, problem solving in a trivialized manner. The "general notion that problems can be given ready-made to students is highly questionable. Instead, teaching through problem-solving implies acknowledging that problems arise for students as they attempt to achieve *their* goals in the classroom. The approach respects that students are the best judges of what they find problematic and encourages them to construct solutions that they find". (Cobb et al, 1995, p.222)

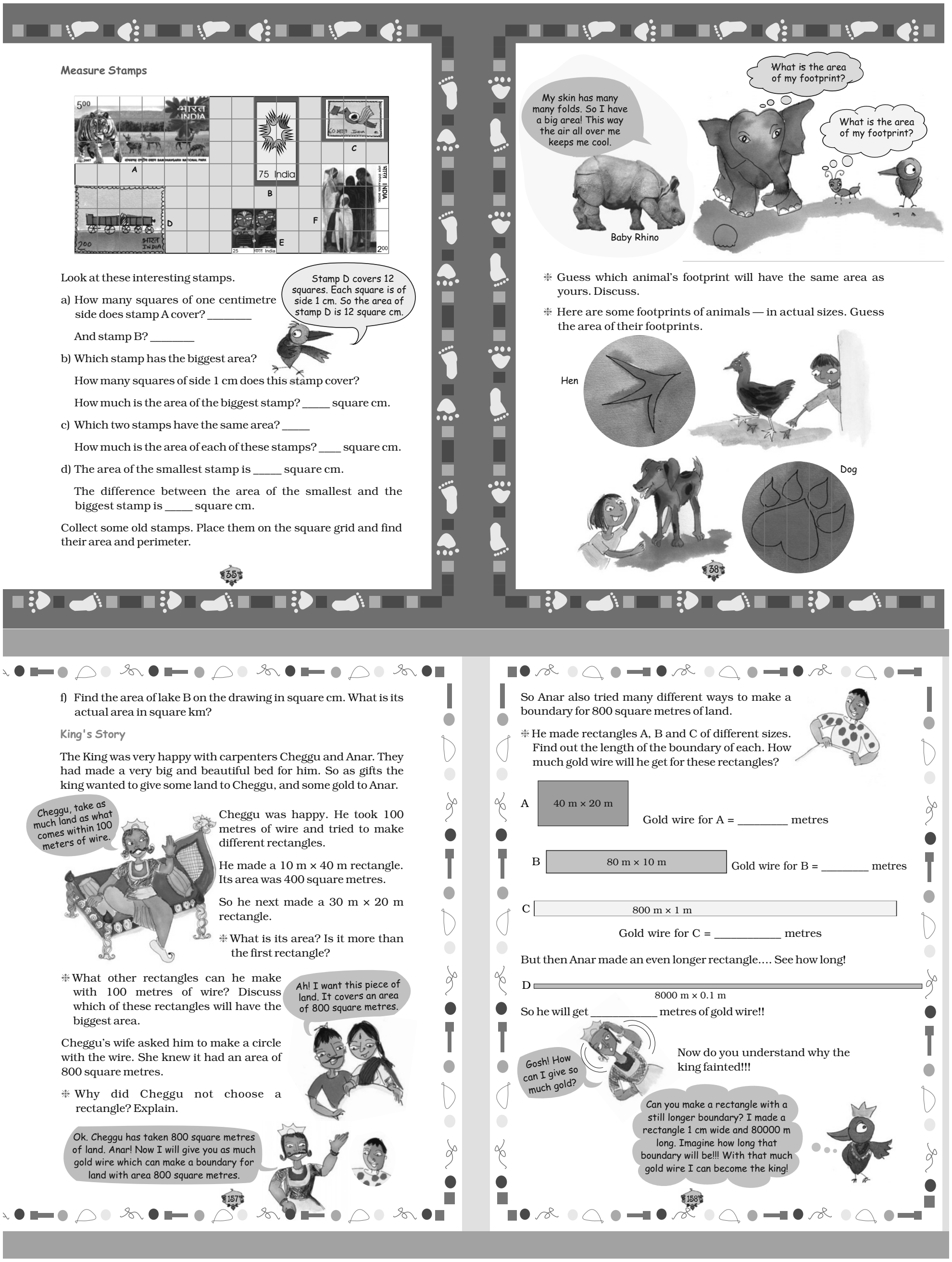
The chapters in the primary textbooks use more conversational language appropriate for children and attempt to bring in humour both in text as well as through visuals. Definitions and terminologies are avoided and children are encouraged to understand concepts through engaging with the narratives and exercises given and the activities to be undertaken. The artists were chosen from among creative illustrators of children's books, who were sensitive to different genres of folk and child art, and they worked closely with the writers to design the page as a visual text, which could be processed by children in a non-linear manner.

In general, there is a great deal of emphasis on estimation and the use of mental algorithms, with examples from folk and street math, which the majority of children are engaged with in their lives out of school (Rampal, 2003a, 2003b; Rampal, Ramanujam & Saraswathi, 1998). Unlike conventional textbooks, several chapters in these primary textbooks deal with shapes, symmetries and patterns, of which some are meant to help develop algebraic thinking before formal algebra is introduced in the upper primary school.

Patterns of assessment

In an attempt to shift away from the conventional pattern of math exercises which resort to an often meaningless drill at the end of a chapter, without allowing students to understand the purpose and the context in which computations are to be done, the present textbooks incorporate diverse exercises and activities into the design of the chapters. Thus for instance, as discussed earlier, the chapter on Time proceeds with the children having to fill in the time line given, to make one for themselves, and to complete tables mentioning activities that take minutes, hours, months, etc. Some chapters were in fact replete with assessment exercises and activities, as may be seen, for instance, in the following examples from chapters on area (see Figure 6).

Even for the chapters on numbers and operations, an effort was made to develop the number sense of children through diverse cultural contexts. It has been seen that providing rich situated activities help stimulate a deeper understanding of 'number sense', which is related to a student's ideas about numbers and strategies to work with them, and which does not necessarily happen even among those highly skilled at algorithmic computations that involve use of learned routines and procedures (McIntosh et al, 1995).



Measure Stamps

Look at these interesting stamps.

a) How many squares of one centimetre side does stamp A cover? _______

And stamp B? _______

b) Which stamp has the biggest area?

How many squares of side 1 cm does this stamp cover?

How much is the area of the biggest stamp? _____ square cm.

c) Which two stamps have the same area? _____

How much is the area of each of these stamps? ____ square cm.

d) The area of the smallest stamp is _____ square cm.

The difference between the area of the smallest and the biggest stamp is _____ square cm.

Collect some old stamps. Place them on the square grid and find their area and perimeter.

⁜ Guess which animal's footprint will have the same area as yours. Discuss.

⁜ Here are some footprints of animals — in actual sizes. Guess the area of their footprints.

f) Find the area of lake B on the drawing in square cm. What is its actual area in square km?

King's Story

The King was very happy with carpenters Cheggu and Anar. They had made a very big and beautiful bed for him. So as gifts the king wanted to give some land to Cheggu, and some gold to Anar.

Cheggu was happy. He took 100 metres of wire and tried to make different rectangles.

He made a 10 m × 40 m rectangle. Its area was 400 square metres.

So he next made a 30 m × 20 m rectangle.

⁜ What is its area? Is it more than the first rectangle?

⁜ What other rectangles can he make with 100 metres of wire? Discuss which of these rectangles will have the biggest area.

Cheggu's wife asked him to make a circle with the wire. She knew it had an area of 800 square metres.

⁜ Why did Cheggu not choose a rectangle? Explain.

So Anar also tried many different ways to make a boundary for 800 square metres of land.

⁜ He made rectangles A, B and C of different sizes. Find out the length of the boundary of each. How much gold wire will he get for these rectangles?

A 40 m × 20 m Gold wire for A = ________ metres

B 80 m × 10 m Gold wire for B = _______ metres

C 800 m × 1 m

Gold wire for C = __________ metres

But then Anar made an even longer rectangle.... See how long!

D 8000 m × 0.1 m

So he will get __________ metres of gold wire!!

Now do you understand why the king fainted!!!

Figure 6: Exercises and activities (Class V textbook)

Thus, inventing different strategies to deal with numbers, using common benchmarks for quick comparison and estimation, reviewing one's answer using often unconscious metacognitive processes of reflection, to change strategy where required, are all aspects of number sense which develop as part of a child's learning process, through exercises and tasks suitably situated in contexts.

However, these new patterns of assessment in the math textbooks do not necessarily lead teachers to shift away from the mechanical algorithmic approach that has been used in classrooms. Since the teaching of math has conventionally been thought to require only the chalk and talk method, with no effort to conduct activities or to use manipulatives, teachers have generally not been trained to use the process approach for assessing learning. Indeed, conventionally the aim of assessment in mathematics is to check if every child has got the canonical 'right' answer, using the 'right' algorithm, where the question of noting the process of thinking or doing a task does not arise. Therefore, for such changes in assessment to actually happen in classrooms, and indeed to also transform examination systems, will require a longer and more difficult struggle in India, with sustained work with teachers, education officials and even parents.

Indeed, during a decade of curriculum renewal efforts in the state of Kerala, the only state in the country to have achieved near universal elementary schooling and literacy, but still concerned about issues of quality and equity in education, a major change was effected in the pattern of examinations, closely tied with reforms in textbooks and pedagogical practices. Significantly, every question or 'evaluatory activity' as it was called, was viewed as a learning activity, in congruence with the activity based approach of classroom transactions. Questions based on straight arithmetical computations were generally avoided and an attempt was made to locate each in an authentic context where children could relate the operations to actual real life situations. Some examples are presented here, taken from a study undertaken to document the changes in the examination pattern of Kerala, and how these had been achieved through intensive and sustained teacher development workshops, interactions with parents and also community mobilization at the level of the local government (Rampal, 2002).

Examples of Exam Activities

In the following evaluatory activity from a Class IV examination, a child undertakes an optimization exercise for the purchase of some school related items within a given budget (translated from Malayalam language)

Class IV Exam - Evaluatory Activity

> *The cost of each item is given for each shop. To find out from which shop it is cheaper to buy materials for the school. Also to find how much it will cost to buy for 50 children from*

the cheapest store.

One child's answer:

Buying from Ramya stores is cheap. Computation done as follows:

Jaya Stores	Ramya Stores	Uma Stores
160	155	165
24	25	21
6	9	5
190	189	191

If one buys from Uma store it will cost Rs. 191/-
If one buys from Ramya store it will cost only Rs. 189/-
If one buys at Jaya store one child needs Rs. 190/-
189 x 50 = 9450
For 50 children it will come to Rs. 9450/- at Ramya store

A similar approach can be seen even in the formal State Scholarship exam held after Class VII, where a question was posed for the computation of compound interest, in the context of how different banks actually advertise loans. The aim was also to give space for students to talk about the constraints within which their families are compelled to take loans and the problems they may have faced, either through the system of money lenders or even banks. This was clearly a departure from the routine practice of handing this out as an unquestioned problem of computation, where context was only to invoke a real life 'setting' with a contrived or limited purpose (Dowling, 1998), without acknowledging the cultural conflicts or other social implications of 'real' lives.

A prior discussion was meant to be an integral part of the activity, with time allocated for it. Facilitators or teachers conducting the exam were given directions about the nature of questions to be asked before presenting the problem, in order to place children in the specific context and frame of mind, and to reduce the usual anxiety of being examined.

Class VII Scholarship Exam Evaluation Activity: Time 1 hour

Problem Solving (Compound Interest)

As a non-evaluatory activity the facilitator and the children participate in a **discussion**
-What are the different banks in your place?
-Have you ever been to a bank? For what?
-How is a bank useful to us?
-For what all needs do they give loans?
What are the interest rates for these different types of loans?

Presentation: The facilitator presents the problem in the form of a story.
Dileep decided to build a house. He did not have enough money with him. He had to collect Rs. 2 lakhs more for this purpose. Just then he came across two advertisements in

a newspaper

Problem

Dileep wanted to take a loan of Rs. 2 lakhs. Which bank will be more beneficial for him? Why? He will be able to repay this loan after 3 years. What will be the total amount he has to pay to the bank then?

1. Dhanasree Bankers (with illustrations as in an actual advertisement)

Attractive Housing Loans!

Amount	Interest Rate
Upto 2 lakhs	12%
Above 2 lakhs upto 5 lakhs	13%
Above 5 lakhs	14%

2. Sreelaxmi Bankers

Grand reduction in interest rates!

- Interest only Rs. 150/- for Rs. 1000 for a year (upto 2 lakhs)
- Interest only Rs. 160/- for Rs, 1000 per year (above 2 laksh and upto 5 lakhs)
- Interest only Rs. 170/- for Rs. 1000 for one year (above 5 lakhs)

Assessment reform is the biggest challenge and even though the new curriculum and textbooks developed at the national level have been adopted or adapted by several states, math examinations continue to remain largely unchanged. This has more to do with how teachers and administrators view math and its learning, and the overt importance given to routines, drills and algorithms. For instance, the recent term exam for class VIII in Delhi schools posing the following questions reflect a deep inertia and inability to depart from this conventional format:

- Find the square root of 1296.
- Find two rational numbers between 3/5 and 3/4.
- I borrowed Rs. 12000 from Jamshed at 6% per annum at simple interest for 2 years. Had I borrowed at 6% compound interest, what extra amount I would have had to pay?

Challenges to realising the goals of Elementary Mathematics Education

Though a lot of forward looking and significant changes have happened at the level of curriculum development and textbook writing, there are several challenges to be met to realise the goals of elementary education in mathematics. Some of these have to do with equipping teachers to cope with the demands the new curriculum places on them. There is also a need to review the new curriculum, particularly at the upper primary level, in order that it has something valuable to offer students who suffer the impact of socio-cultural and economic disparities, at home and in school, that limit their opportunity to benefit from the new curriculum.

Teacher Education and Teacher Professional Development

As discussed above, at the primary level the curriculum situates the learning of mathematics in diverse contexts. This is a radical departure from the prevalent approach to school mathematics still dominant at the upper primary level across the country, that lays a strong emphasis on learning standard algorithms and applying them in given situations. The change at the national level is slowly leading to change at the state level. However, one of the major challenges is in the area of teacher professional development so as to realise the envisaged change in the classroom and this is a formidable task for a variety of reasons. Foremost among them is that it calls for a change in the mindset of mathematics teachers about what constitutes the learning of mathematics. Trained and socialised in traditional behaviourist approaches, for most of the teachers school mathematics is synonymous with teaching and learning standard algorithms, practising them again and again by solving several problems (with minor variations if at all) and learning to apply these algorithms in word problems. Given this, the demands that the present NCF approach and the primary school textbook place on them is manifold. The NCF requires them to encourage children to evolve their own strategies for solving problems and also creatively guide children's intuitive strategies towards formal algorithms. To be able to do this they need to update their subject matter knowledge and pedagogic knowledge substantially as many of them, particularly those from rural areas and educationally backward states, have not had the privilege of acquiring quality education in mathematics. Another important factor is enabling teachers to acquire the skills required to manage and transform the classroom space from that of passive listening to participative learning, thus essentially foregoing the usually formidable 'authority' of the math teacher. Finally, to be able to realise the goals of equity and social justice concerns in mathematics education and to ensure that female students and those coming from rural, poor, marginalised caste and tribal backgrounds, get the benefit of the new mathematics curriculum, teacher preparation needs to go far beyond enhancing their subject matter knowledge and pedagogic content knowledge.

Drawing from Darling-Hammond's theoretical model, Bartell (2011) refers to "Self, Society, Students and School" as four key factors that guide teachers in learning to teach for social justice, where "self" includes teachers "reflecting on how their beliefs about teaching and learning are influenced by the cultural, historical, and economic contexts in which they grew up" (p.4), "society" includes an understanding that "teaching, learning, and schooling are linked to economic, political, and social power structures in society" (p.4), "students" include an understanding of them "in non-stereotypical ways while acknowledging and comprehending the ways in which culture and context influence their lives and learning" (p.4) and based on their evolving understanding of the above, "school" requires them to "develop and enact practices that support their students" (p.5). Each of these is significant in the Indian context and needs to be addressed in teacher education, as the majority of our learners come to school with deprivation of one kind or the other, with "ruined foregrounds" (Skovsmose, 2007).

Teachers have very low expectations from and deep sociological biases about these marginalised students. The normative notions that they carry about the learner–as someone for whom school is central, with near perfect attendance at school, and whose parents arrange to provide personalised attention–may not tally with the learners they encounter in the classroom (Pappu & Vasantha, 2010). In particular, there is very little that is made available to teachers to cope with and build on the differences students bring to the classroom. Without serious measures to sensitise teachers, provide them support to cope with the range of difficulties marginalisation brings to classrooms and to help them appreciate the NCF approach to address equity and social justice concerns in mathematics, the new curriculum and textbooks alone cannot deliver what they seek to do. This calls for a strong teacher education programme committed to addressing each of these issues. Unfortunately, teacher education institutions have been undervalued, neglected and made invisible in the larger academic scenario, with very little contribution from the higher education sector, including research institutes. There is an urgent need on the part of the state to place a strong emphasis on teacher education, to consciously locate it within the academic and research agenda of universities and institutes, to revitalise teacher education institutions and revise their curriculum, in order that they provide the required in-service and pre-service training for teachers to transact the new mathematics curriculum.

The new National Curriculum Framework for Teacher Education (National Council for Teacher Education, 2009) seeks to address the challenges of quality and equity concerns in teacher education by laying a strong and equal emphasis on three areas of teacher education, namely 'foundation of education' 'curriculum and pedagogy' and 'school internship'. Some states have already started revising their D.Ed, and B.Ed curricula in accordance with the new framework. However, there is also a strong need to enhance the status of teachers and to see teacher agency as central to realising the reform agenda. In the absence of teacher autonomy, and without rationalisation of the current teaching load on them, where they are often expected to teach large numbers of students in multi grade situations, it is unrealistic to expect change to happen soon, without the consonant support mechanisms. The challenge is also in reaching out to and convincing middle class parents who ignore concerns about children's learning in view of their own aspirations about future choices of employment. In connivance with the school management, these parents often scuttle any progressive change, by adopting textbooks (and even shabby guidebooks with solved answers) to maintain status quo through the traditional approach.

The Upper Primary School - Curriculum design and implementation

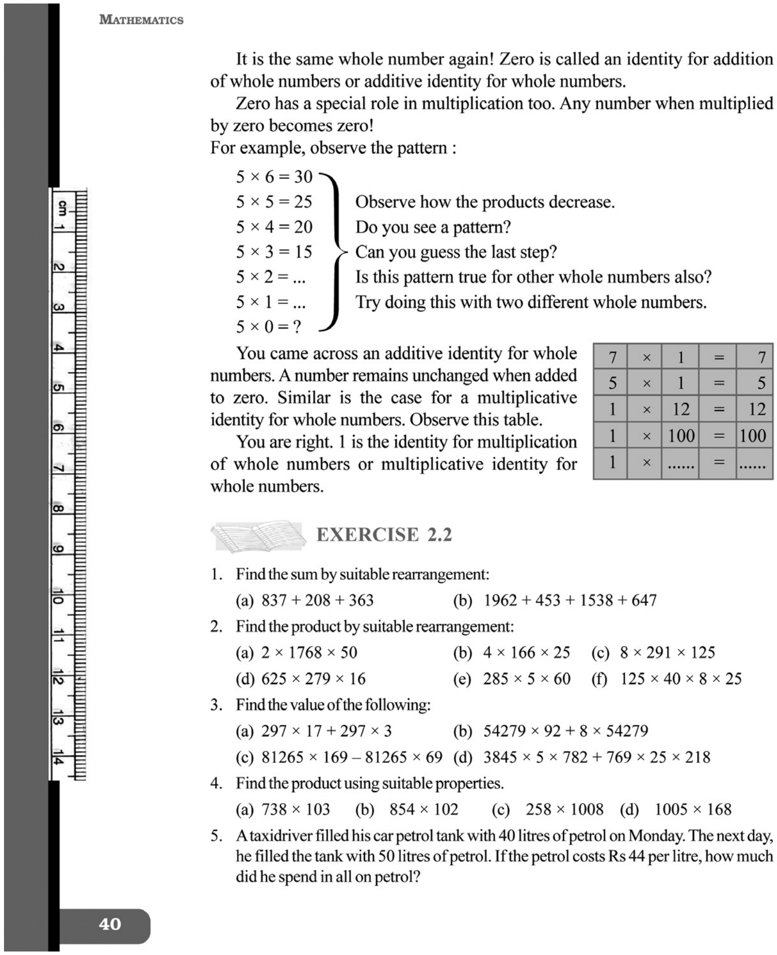
A major challenge that goes beyond teacher professional development is more fundamental and has to do with designing a curriculum at the upper primary level. The NCERT syllabus for upper primary mathematics states that the emphasis is on “the need to look at the upper primary stage as the stage of transition towards greater abstraction, where the child will move from using concrete materials and experiences to deal with abstract notions”. Though there are significant differences between the approaches followed by different states, the curricular content of mathematics at the upper primary level remains similar, with a strong focus on imparting disciplinary knowledge.

Typically the upper primary curriculum deals with rational numbers and their properties, algebra, geometry, data handling and ‘commercial mathematics’. Numbers and number systems deal with natural numbers, integers and their properties, decimals, rational numbers, their representation on the number line, factors, multiples, exponential notation, prime factorization of numbers, ratio, proportion, percentage, finding square roots and cube roots, including the algorithm for finding square roots. In addition, some state boards introduce fractional exponents and also finding the approximate values of square roots of 2, 3 and 5. Negative numbers, usually introduced early in class 6, are known to be a problem area.

More generally, there have been questions about the appropriateness of teaching number systems in anticipation of algebraic structures that they might encounter in future, rather than drawing students’ attention to some of the properties that numbers satisfy and how they can be exploited to simplify calculations (see Figure 7).

Algebra begins in class 6 with the introduction of symbols to stand for numbers. While the NCERT textbook limits itself to writing linear polynomial expressions and equations in one variable at the class 6 level, some state boards introduce higher degree polynomial expressions in two variables and the operations of addition and subtraction on them, ending with solving linear equations in one variable in class 6. By class 8, most of the boards cover the three operations on higher degree polynomials in two variables, factorization or division and algebraic identities.

Geometry is also formally introduced at this level. Traditionally, students have been struggling with notion of proof and the actual proofs of geometrical statements, which are mostly memorised. The new syllabus postpones formal proofs in geometry to the secondary stage, and instead lays emphasis on verification and reasoned justification. In geometry and measurement, the content begins with giving some idea of what geometrical objects such as points, lines and line segments mean, then moves onto polygonal figures, the notion of angle, using the geometry kit to measure angles, construct triangles, quadrilaterals, perpendiculars and angle bisectors, properties of triangles and quadrilaterals, circles, three dimensional objects namely spheres, cylinders, cones and



MATHEMATICS

It is the same whole number again! Zero is called an identity for addition of whole numbers or additive identity for whole numbers.

Zero has a special role in multiplication too. Any number when multiplied by zero becomes zero!

For example, observe the pattern :

$5 \times 6 = 30$
$5 \times 5 = 25$
$5 \times 4 = 20$
$5 \times 3 = 15$
$5 \times 2 = ...$
$5 \times 1 = ...$
$5 \times 0 = ?$

Observe how the products decrease.
Do you see a pattern?
Can you guess the last step?
Is this pattern true for other whole numbers also?
Try doing this with two different whole numbers.

You came across an additive identity for whole numbers. A number remains unchanged when added to zero. Similar is the case for a multiplicative identity for whole numbers. Observe this table.

7	×	1	=	7
5	×	1	=	5
1	×	12	=	12
1	×	100	=	100
1	×	......	=	......

You are right. 1 is the identity for multiplication of whole numbers or multiplicative identity for whole numbers.

EXERCISE 2.2

1. Find the sum by suitable rearrangement:
   (a) $837 + 208 + 363$ (b) $1962 + 453 + 1538 + 647$
2. Find the product by suitable rearrangement:
   (a) $2 \times 1768 \times 50$ (b) $4 \times 166 \times 25$ (c) $8 \times 291 \times 125$
   (d) $625 \times 279 \times 16$ (e) $285 \times 5 \times 60$ (f) $125 \times 40 \times 8 \times 25$
3. Find the value of the following:
   (a) $297 \times 17 + 297 \times 3$ (b) $54279 \times 92 + 8 \times 54279$
   (c) $81265 \times 169 - 81265 \times 69$ (d) $3845 \times 5 \times 782 + 769 \times 25 \times 218$
4. Find the product using suitable properties.
   (a) 738×103 (b) 854×102 (c) 258×1008 (d) 1005×168
5. A taxidriver filled his car petrol tank with 40 litres of petrol on Monday. The next day, he filled the tank with 50 litres of petrol. If the petrol costs Rs 44 per litre, how much did he spend in all on petrol?

40

Figure 7: Discovering and using number properties (Class VI textbook)

pyramids. The formula for finding the perimeter, area and volume are either arrived at or given straight away depending on the school board. Apart from these three major themes, there is some exposure to data handling (pictographs, frequency tables, chance and probability, measures of central tendency, graphs) and commercial mathematics (profit and loss, simple and compound interest).

The present NCERT textbooks in some sense demand a more serious engagement with the discipline of mathematics and we need studies to understand how the curricular demands are met even in the best of schools. The position paper on mathematics justifies the kind of mathematics taught at the upper primary level by arguing that it develops logical thinking and a training in following an argument to its logical conclusion, but there is a need to investigate how far this justification holds good and what hurdles there are to such learning.

A major cause for concern however is that state governments are slowly adapting the NCERT mathematics curriculum at the upper primary level, in spite of the socio-cultural differences in the student population. Most rural children and those from socio economically marginalised backgrounds study in government run schools where the state curriculum is in force. A large majority of first generation learners from the marginalised sections enter upper primary school without achieving appropriate learning in primary mathematics. As mentioned earlier, the issue is complex as it involves teachers' beliefs about the ability of the learners from rural or urban poor families and oppressed castes and their perceived need to learn mathematics, and calls for special training to teach children who have no support at home. While it is hoped that the newly enacted Right to Education Act for all children to get equitable quality education till they complete elementary school may bring in some positive change, the present learning levels have been known to be poor.

It has been found that written mathematics poses serious challenges for deprived children, even though they may have some competence in oral computations and problems situated in contexts meaningful to them. Class 6 children are not able to write three digit numbers, and many cannot perform subtraction or carry out division. Most of them have practically no understanding of fractions and how to use operations on them. Symbols like ¾ and ½ may pose difficulty for them, though they know that the three quarters is one quarter (*pav*) more than half (*aadha*) and so on. In one classroom interaction with 50 socioeconomically marginalised students of grade VI, few could make sense of an expression like 95 ÷ 5 =?. But they could make sense of the problem when placed in a context: 'If we distribute 95 rupees equally among 5 children how much would each get?' A few who managed to solve it did so by inventing their own symbol system to distribute and add up each child's share. In another classroom trial it was observed that owing to poor training at primary grades, most class 6 students belonging to similar backgrounds had to be taught division afresh and they preferred the partial quotients method over the standard division algorithm (Khemani & Subramanian, 2012). In other words, at the upper primary level, these children bring into the classroom some knowledge of numbers learned from their everyday life experiences but very little from the previous five years of learning at school. The disconnect between these children's encounter with numbers and the demands of school mathematics is something that is well studied and reported (Nunes, Schliemann & Carraher, 1993; Lave & Wenger, 1991) and our experience with children from marginalised backgrounds resonates well with these studies. In general making sense of abstract number problems and algebraic expressions remains a major challenge for them as their training in mathematics has not enabled them to acquire these skills.

Mathematics at the upper primary level is premised on the ability to read and write numbers, and make sense of arithmetical expressions as a starting point towards algebra. As children are not equipped to cope with this, classroom transaction gets reduced to children copying meaningless symbols from the blackboard, or from commercially available guidebooks in which the problems are worked out. Such classrooms where students cannot make sense of arithmetic expressions are not singular but fairly typical of classrooms catering to students from socioeconomically marginalised sections, or from rural background. They constitute a significant part of the student population. There are no studies demonstrating how negative numbers and algebra can be taught to such students in a meaningful way. The field experience of developing alternative approaches for the teaching of algebra and negative numbers for children in diverse rural contexts has shown that the challenge of imparting the existing curriculum at the upper primary level are indeed daunting.

Classroom interventions and close interactions with children show that most of them from lower socio-economic backgrounds have not learnt the use of the geometry kit. Angle measurement poses serious difficulty for many children. There are no suitable qualitative studies that focus on how children learn geometry and the challenges they face in different school situations, yet there is empirical evidence to believe that the content in geometry does not get transacted at all satisfactorily. A typical classroom transaction in mathematics could amount to the teacher working out a problem on the blackboard and the students copying with no comprehension. The upper primary classroom thus continues to alienate and remain effectively inaccessible for a large section of the socially marginalised student population and also for many from the urban middle classes (Subramanian, Umar & Verma, In Press).

Teaching and learning of algebra and negative numbers remains a major challenge even in urban middle class settings. There has been some research exploring alternative approaches for the teaching of algebra by bridging the gap between arithmetic and algebra using 'terms' (Banerjee, Subramaniam, & Naik, 2008). There have been several attempts to explore alternatives to teach negative numbers and some attempts to use 'geogebra' to teach geometry in the urban middle class context. However, these trials have not been replicated in other sites and have also not been incorporated in textbooks.

In other words, in spite of the fact that the new upper primary mathematics curriculum is designed to enable children to explore, experiment and acquire reasoning skills, it remains impractical for a large number of classrooms in India. One way to address the situation would be to retain the upperprimary mathematics curriculum as it is and examine how to improve the quality of primary mathematics for students from marginalised sections.

While a section of mathematics educators might believe that the problem lies only with the quality of the transaction in school, both at upper primary and primary, there are those who believe that the existing situation compels us to critically examine why we teach what we do at the upper primary level. In fact, mathematics educators concerned with social justice issues have engaged with similar questions (Ernest, n.d.; Gellert & Jablonka, 2007; Gates, 2001; Skovsmose, 2011; Greer, Mukhopadyay, Powell & Barber, 2009). Arguing that a narrow definition of mathematics guided by what mathematicians practice does not represent the multiplicity of mathematical practices in varied cultural contexts, they call for a critical appraisal of the aims of teaching mathematicians' mathematics and ask how mathematics at the upper primary level could be redesigned to enable the learner to critically engage with the socioeconomic and cultural reality in which they are placed. This suggests, for instance, that to place algebra in context, it would be useful to begin with problem situations that have an immediate relevance to the learner, rather than teaching algebraic representations and symbolic manipulations first and then applying them in contrived situations that the learner generally cannot relate to. Similarly, it would also be meaningful to introduce projects that call for data collection and analysis as a means for learners to understand diverse social realities and develop social agency.

While social justice concerns is one of the central issues in Indian education and hence also in mathematics education, the official stand as expressed in the National Focus Group Position Paper is categorical that there will be no differentiated curriculum as it could result in excluding the already excluded by denying them a fair chance to enter the mainstream (See Chapter 1, this volume).

A major challenge therefore is to revisit the upper primary mathematics curriculum to make it inclusive for all children, by reviewing notions of what constitutes meaningful mathematics learning and also to ensure that teachers use appropriate constructivist pedagogies to make classrooms active learning spaces for all. This will require serious critical engagement with the discipline of mathematics and curricular research that explores alternatives while also providing better models of teacher education and professional development, as has been indicated in this paper. It is hoped that in the coming years the National Initiative in Mathematics Education (NIME) would gain momentum, to involve a wide network of persons from different areas of work, in order to address the challenges delineated here and to achieve further progress in this direction.

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5. Mathematics up to the Secondary Level in India

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Background

India has placed great emphasis on educating all its children, since independence. Seeking a more just and equitable society, the Constitution of India is committed to providing to all children, opportunities for developing their capabilities and maximizing their learning in their areas of interest. Providing mathematics education is an integral part of India's commitment to universalization of education. Mathematics is a part of our general education and all children have to study mathematics till class 10.

Universalization of education was not an easy task for India at the time of independence. Large regions in the country did not have schools, schools that existed lacked infrastructure and the commonly held perception was that school education is not useful for all. Since then various initiatives of the government have led to a remarkable improvement in access to schooling and various studies show that demand for good schooling is not restricted to only certain groups of people today (PROBE Team, 1999). Significantly, the 86th constitutional amendment declared education a fundamental right of every Indian child in 2002, and the Right to Free and Compulsory Education Act (RtE) in 2009 gave further teeth to the idea of every child being educated up to the age of 14 (i.e. elementary school level) by making it justiciable. Today, primary schools exist within a kilometer of every child and elementary schools, every three kilometers. Access to secondary schools however, may require children to travel up to ten kilometers. While considerable progress has been made in providing schooling facilities to all children, children's learning remains a tenuous area. Various studies undertaken by government and private agencies in primary and elementary classes are evidence of very poor learning levels among children in both Language and Mathematics (Education Initiatives, 2010; Pratham, 2005-2010; NCERT, 2008). Children have difficulty in 'reading texts with understanding' and 'expressing their thoughts in writing'. Understanding of mathematics in primary classes is largely limited to 'procedural or rote-based learning' and in fact falling averages as we move from the primary to the elementary classes indicate an increase in the level of incomprehension for children (Education Initiatives, 2010).

In this paper, we will present how mathematics education, up to the secondary level, is conceptualized by our policy and curricular documents, textbooks and within the classroom. We will focus on the developments after 2005, but will spend some time discussing the journey. We will end with the challenges that exist for mathematics education on the road ahead.

Organization of secondary education in India

Education is a part of the federal framework of governance in India and so both the centre and the state governments enjoy authority in this area. The National Council for Education, Research and Training (NCERT) is the apex body for advising the central and state governments on school education. NCERT along with its state level counterparts–the State Councils of Education, Research and Training(SCERTs) are involved in various tasks like educational research, curriculum renewal, textbook creation, creation of supplementary material for children and teachers, pre- and in-service training and publications for teachers and children.

The country also has two national level boards of secondary education- the Central Board of Secondary Education (CBSE) and National Institute of Open Schooling (NIOS), the former being popular. All states of the country also have their own official boards of secondary education. Apart from these, one private board of secondary education also exists–the Council for the Indian School Certificate Examination (CISCE). More recently some international boards of secondary education are also coming into India. The secondary boards in India are in many cases responsible for the development of curricular expectations, syllabus and teaching-learning materials at the secondary level as well as reform in examination and evaluation practices. In a few instances they are also responsible for in-service teacher training.

In recent years, there has been an increase in the role of the NCERT and the SCERTs in processes of curriculum renewal and textbook development and boards are focusing on improving assessment processes and mechanisms.

It must be kept in mind that while there are official agencies like NCERT, SCERT and boards of secondary education that produce textbooks of mathematics, there are no restrictions on private publishers bringing out materials for these classes. There are many national and international publishing houses that bring out books; however, the expectation is that these be in line with the national and state curricular documents. Private schools largely form the market for these books.

The vision for mathematics education

The vision with which mathematics has been placed in the school curriculum has evolved over the years. In the 1950s and the 1960s, India developed its mathematics education as a step towards industrialization and scientific research. The Kothari Commission was set up for thinking comprehensively about education in India during this period and published its report in 1966. The report underlined the need for mathematics and science in school as well as in higher education; it emphasized the importance of children learning mathematics for the development of science and technology and for industrial growth. To quote from the report, “One of the outstanding characteristics of scientific culture is quantification. Mathematics, therefore, assumes a prominent position in modern education. Apart from its role in the physical sciences it is now playing an increasingly important part in the development of the biological sciences” (Government of India–Ministry of Education, 1966, p.181). The 1968 and 1986 National Policies of Education spoke in the same tone as the Kothari Commission report and the 1986 policy states that “mathematics should be visualized as the vehicle to train a child to think, reason, analyze and articulate logically. Apart from being a specific subject, it should be treated as a concomitant to any subject involving analysis and reasoning” (Government of India–Ministry of Human Resource Development, 1998, p.29).

The system of Nai Talim (New Education) that had emerged in the 1930s and 1940s from the thinking of various people like Dr. Zakir Hussain and Gandhi working towards building responsible, capable and educated Indians also realized the importance of mathematics. However, it viewed mathematics in terms of its use for the day-to-day requirements of people. The emphasis was on ensuring that calculations necessary for the survival of the child in the circumstances in which she was growing were learnt. The Zakir Hussain committee stated: “Knowledge of mathematics is an essential part of the curriculum. Every child is expected to work out the ordinary calculations required in the course of his craft work or his personal and community concerns and activities.” In this sense, the Kothari Commission widened this very concrete, tangible and narrow purpose of teaching mathematics.

Respecting the distribution of the areas of jurisdiction between the centre and state governments on matters of education, the National Policy of Education, 1986 clearly states that “the national system of education will be based on a national curricular framework which contains a common core along with the other components that are flexible”. The national curriculum framework brought out by the NCERT in 2000 gave some idea of the content in the syllabus and the kind of teaching process to be followed.

It felt that the teaching-learning process must heed the context of the child and their 'zone of proximal development' and learners should be able to relate the mathematics in their textbooks to their life experiences. This led to the idea of the mathematics lab and use of more and more concrete illustrations and activities in classrooms of mathematics. Under central government supported schemes, teachers and teacher educators made a lot of effort to develop activities and games that would somehow be linked to the teaching of mathematics.

The period of the 1990s and early 2000s was also the period when Minimum Learning Levels (MLL) formed the basis for the curriculum and textbooks and NCF 2000 also asked for their proper implementation. The idea of MLL arose from the need to provide equitable education to all children across India. It itemized learning of language, mathematics and environmental studies in the primary classes into small chunks/ competencies that all children were expected to achieve. Assessment and evaluation was also based on these small chunks. To be measurable these competencies had to be in the form of observable behaviour demonstrated by the child when she received the requisite inputs. This formulation of MLL also paid no heed to the time and space that children needed for concept building. There was a great deal of opposition to this and various alternative formulations were built. These included work by some organizations outside the government framework, some of them being partnerships with public institutions, like Eklavya in Madhya Pradesh, Homi Bhabha Centre for Science Education in Maharashtra, Vidya Bhawan Society in Rajasthan, Suvidya in Karnataka, School Mathematics Program of the Centre for Science Education and Communication of Delhi University, etc. These organizations had worked directly with various government schools and developed their own curriculum, syllabus and textbooks in this process. The experiences and ideas of these organizations have helped in giving shape to the National Curriculum Framework 2005. In fact, the upper primary textbooks produced by the Delhi state in 2000 were also a partnership between SCERT, Delhi and Vidya Bhawan Society, Rajasthan.

In the exercise undertaken by Delhi SCERT, many conceptual areas were re-organized and books made less loaded, complicated calculations eschewed and many areas elaborated. Topics such as surds, complicated proofs, stocks and shares, dividend calculations, income and sales tax were not included. The textbooks also attempted to use language and pictures as devices to communicate mathematics and were based on the argument that a book for the student should be at the level of her comprehension. Another important change initiated was the creation of a complete mathematics book instead of a textbook divided into sections. This subsequently led to spiraling and developing inter-relationships between various mathematics concepts. There was, however, no consensus on removing relatively tedious algebraic expressions, fractional number calculations, theorems and definitions in geometry, etc. There was a fear that the state syllabus would lag behind that of other states across the country. It was difficult for many to accept that it was pointless to load the program with tricks and algorithms to solve particular problems or for the child to do tedious algorithmic manipulations with numbers, algebraic quantities or geometric figures.

All this was part of the wisdom that fed into the emergence of the next National Curriculum Framework in 2005.

Mathematics education and the National Curriculum Framework (NCF) 2005

The National Curriculum Framework 2005 along with its Position Paper on Teaching of Mathematics published in the subsequent year provides direction to school and teacher education all across India, currently.

The main goal of mathematics education emphasized in NCF 2005 is 'mathematization' of the child's thought and processes. In doing so the document visibly enlarges the vision of school mathematics taking it beyond areas of obvious utility in daily life to enriching a student's scope of thought and visualization and in turn her ability to relate to the world and to mathematics. It talks about teaching mathematics not just for utilitarian purposes and recognizes mathematics as an important part of the development of the human mind, as an addition to the human ability to absorb, visualize, logically understand, build arguments, prove statements and in a sense interact with and deal with the world. To quote the document, "Developing children's abilities for mathematization is the main goal of mathematics education. The narrow aim of school mathematics is to develop 'useful' capabilities, particularly those relating to numeracy – numbers, number operations, measurement, decimals and percentages. The higher aim is to develop the child's resources to think and reason mathematically, to pursue assumptions to their logical conclusion and to handle abstraction. It includes a way of doing things, and the ability and the attitude to formulate and solve problems." (NCERT, 2005, p. 42)

The curriculum at the primary stage emphasizes concrete experiences (objects and day-to-day life experiences) in the progression towards mathematical abstraction. It gives due place to non-number areas of mathematics like space, visual patterns and data handling. While discussing number areas it recommends development of number sense including number patterns and de-emphasizes the algorithm, encouraging children to develop their own methods of solving problems. It curbs the need to teach children bigger and bigger numbers to prevent "overloading the child's cognitive capacity which can be better used for mastering the logical skills at the earlier stages" (NCERT, 2006, p.15). It provides space to visualization, estimation and reasoning along with computation abilities. It also recommends building a stronger conceptual base for fractions and decimals and de-emphasizes operations with fractions at the primary level.

At the upper primary stage, concepts that children have learnt are re-visited in more abstract forms, are consolidated and are elaborated into denser ones. Arithmetic is extended to algebra and children are expected to express the patterns they are seeing through generalizations. A study of space is undertaken through a Euclidean study of triangles and quadrilaterals as well as through solid geometry. Here the emphasis is on mathematical thinking and visualization. Data handling is also an essential area at this stage.

The broad description of the purpose of mathematics for secondary classes includes consolidating and elaborating the conceptual edifice of the elementary classes and significantly building upon the ability to perceive rules and generalizations, formulate ideas with precision, the ability to make logical arguments and the ability to prove statements. The secondary school books build on the pattern recognition and generalizations in the elementary books and go on to problems that require proofs to be found. These are simple to prove and can be done using many strategies. The logical formulation and the arguments included in each step along with precision of presentation are of value to engage with the world more effectively.

In the context of universalization of education the position paper on mathematics importantly talks about the development of a mathematics program that would ensure that everybody learns mathematics and does not fear it. The document identifies children's inability to deal with the hierarchical nature of mathematics as one of the main reasons for their giving up on it and thus emphasizes that the progress of the syllabus should be such that children have sufficient time to develop the fundamental concepts and thereby do not feel afraid of moving ahead. It also strengthens the need for the mathematics program to be so designed that it takes into account the requirement of revisiting concepts. The concepts in the program are sought to be developed spirally, with each concept introduced and dealt with on many occasions to give repeated opportunities to the learner to absorb them. Another reason that the position paper points to while discussing the fear of mathematics is the manner in which that the "language of mathematics learnt in school is far removed from their everyday speech, and easily forbidding" (NCERT, 2006, p. 5). The curriculum therefore also expects that the language used in textbooks would be like that spoken by children in their daily life.

Understanding the importance of the relationship between language and learning the curriculum framework emphasizes that the mathematics classroom should be alive and interactive in which children should articulate their own understanding of concepts, evolve models and develop definitions. Following the recommendations of the position paper on mathematics developed as a part of NCF 2005, the subsequent books for the elementary and secondary classes provide various opportunities to the learners to formulate principles and solutions in their own words. They argue that this helps develop and consolidate conceptual frameworks. The program also emphasizes the role of dialogue among learners and argues for opportunities for children to discuss and make presentations as a group.

Another principle that the curricular document lays down is that learning mathematics is not about remembering solutions or methods but about feeling capable of and knowing how to solve new problems. It also realizes the importance of problem posing in mathematics. The twin tasks of solving and setting problems helps develop an understanding of the concepts and principles of mathematics.

Importantly, it also asks for a need to look at mathematics as a whole and not through water tight sections of arithmetic, algebra and geometry; in doing so it advocates making connections within these areas of mathematics. The understanding is also that better conceptual understanding of mathematics involves exploring the relationship that mathematics has with other subjects as well. The mathematics program therefore makes an effort to link itself to other areas and have problems that include concepts from them. The effort is to help learners get a wider and deeper sense of mathematics and make them confident of dealing with its basic ideas. In the mathematics program up to class X, relationships with the natural and physical sciences, economics, etc., have been explored.

NCF 2005 thus advocated a major re-look at the syllabus, textbooks, nature of assessment and more importantly the way mathematics was taught in classrooms.

The syllabus after NCF 2005

Here we will discuss the prominent changes in the NCERT syllabus for the elementary and the secondary classes. The syllabus has undergone some substantial changes in terms of areas as well as the nature of treatment.

The presentation of algebra, geometry and data handling have changed considerably following the NCF 2005. Algebra has changed from learning to handle complicated algebraic expressions, using algorithms and learning to match them to specific problems involving tedious calculations in de-contextualized settings. It has become instead an attempt to understand the idea of a variable, functional relationships and the use of letter numbers in different ways. The attempt is to now understand algebra as a generalization of many of the ideas that are intuitively learnt or seen as patterns and recognizing that all the ideas that are intuitive or obtained from patterns cannot be generalized. It is inextricably linked to learning the use of mathematical language, of symbols and brevity.

Geometry has changed from remembering theorems and proofs to understanding space and spatial relations. Theorems and proofs are a part of understanding shapes and their properties along with the ability to use mathematical logic. Geometry as a whole now relates to visualization, symmetry, representation of objects, projecting and mapping, plotting functional relationships. Its relationship with algebra and arithmetic is therefore much clearer and elaborate. Some examples of what is included now are: solid geometry as an exercise in 3D visualization and conveying the idea that Euclidean geometry is not the only form of geometry, change in the handling of solid shapes through formulas of surface area and volume to understanding them through nets, developing an understanding of edges, surfaces, vertices, etc., and the ability to imagine objects from different positions and perspectives.

Data handling has emerged as an introduction to statistics. The earlier view was that data handling cannot be initiated before secondary classes. The extension of data handling to collection, organizing and presenting data through pictograms, tables, bar graphs and pie charts has made it possible to introduce it much earlier. Data handling is no longer about only calculating representative values like mean, median mode but about understanding when we need to use which representative value.

There have been significant shifts in the secondary classes as well. For example the logarithmic and trigonometric tables have not been included in NCERT books indicating thereby that there is no expectation from the child to do complicated calculations using these. The details of commercial mathematics have been reduced and emphasis changed to helping children understand the underlying concepts of ratio and proportion and linking different examples of their use under one conceptual thread. The nature of geometry has changed from a lot of theorems and knowing their proofs to development of an understanding of concepts using their experience and helping them understand the notion of a proof and how to construct it. The extent of work expected on circles has been considerably reduced in order to deepen and widen conceptual ideas on polygons. For equations, the importance and meaning of roots through graphical representation and factorisation is emphasized. There is an effort to help students form an idea of functional relationships.

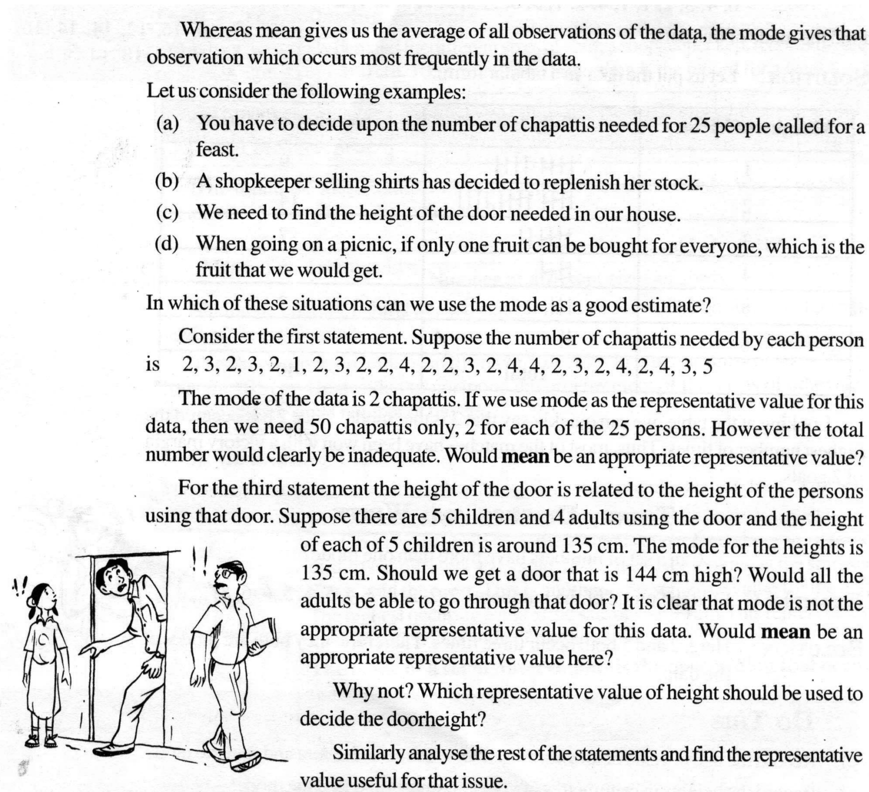
The expectation is thus that mathematics emerges as a subject of exploration and creation rather than as an exercise of finding old answers to old and complicated problems.

Textbooks after NCF 2005

If we look at NCERT books before and after NCF 2005 we see a marked difference in their presentation and appearance.

The principles that these books have utilized are –

- Textbooks should link the mathematics that children do in their textbooks to the mathematics they see and experience all around. Concepts should be introduced through situations of life in which they are placed.



Whereas mean gives us the average of all observations of the data, the mode gives that observation which occurs most frequently in the data.

Let us consider the following examples:

(a) You have to decide upon the number of chapattis needed for 25 people called for a feast.

(b) A shopkeeper selling shirts has decided to replenish her stock.

(c) We need to find the height of the door needed in our house.

(d) When going on a picnic, if only one fruit can be bought for everyone, which is the fruit that we would get.

In which of these situations can we use the mode as a good estimate?

Consider the first statement. Suppose the number of chapattis needed by each person is 2, 3, 2, 3, 2, 1, 2, 3, 2, 2, 4, 2, 2, 3, 2, 4, 4, 2, 3, 2, 4, 2, 4, 3, 5

The mode of the data is 2 chapattis. If we use mode as the representative value for this data, then we need 50 chapattis only, 2 for each of the 25 persons. However the total number would clearly be inadequate. Would **mean** be an appropriate representative value?

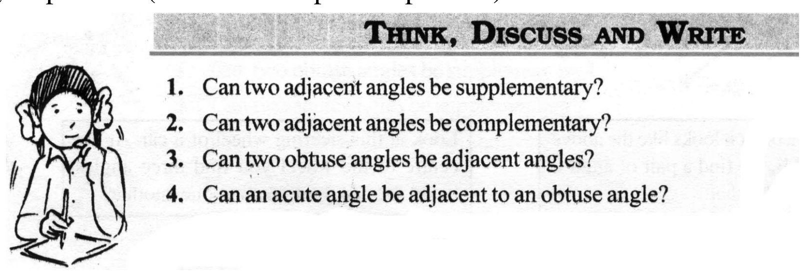
For the third statement the height of the door is related to the height of the persons using that door. Suppose there are 5 children and 4 adults using the door and the height of each of 5 children is around 135 cm. The mode for the heights is 135 cm. Should we get a door that is 144 cm high? Would all the adults be able to go through that door? It is clear that mode is not the appropriate representative value for this data. Would **mean** be an appropriate representative value here?

Why not? Which representative value of height should be used to decide the doorheight?

Similarly analyse the rest of the statements and find the representative value useful for that issue.

Figure 1: Data Handling, Class-7 NCERT Textbook

- Concepts and ideas should also be arrived at by observing patterns, exploring them and providing children opportunities to define them in their own words. Definitions and excessive terminology cannot be the beginning of concept formation.
- Concept building should be followed up with examples and exercises. Exercises should give practice (in both concepts and process) in various contexts. Exces-



THINK, DISCUSS AND WRITE

1. Can two adjacent angles be supplementary?
2. Can two adjacent angles be complementary?
3. Can two obtuse angles be adjacent angles?
4. Can an acute angle be adjacent to an obtuse angle?

Figure 2: Lines and Angles Class-7 NCERT Textbook

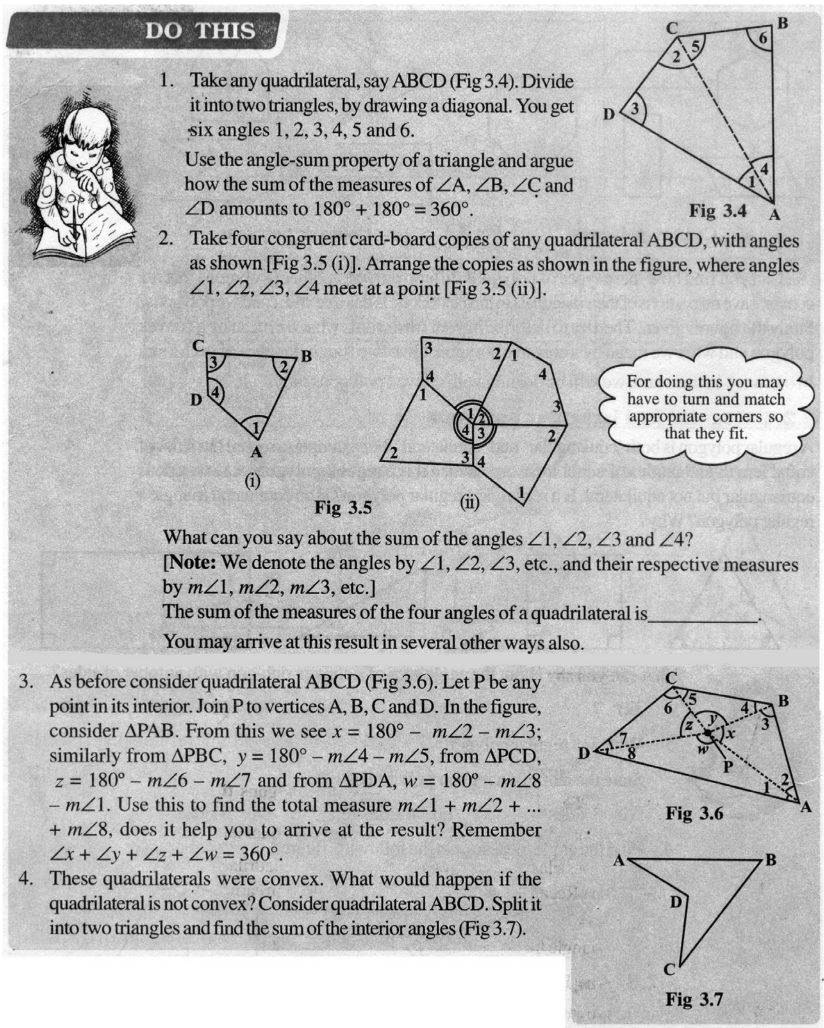
sive practice of algorithm should be avoided. These exercises should not only be placed at the end of a chapter but smaller ones should be present at different points when it is felt that some thought or practice is needed.

- Textbooks should be able to establish continuity with what children have previously learnt in the topic through a spiral arrangement.
- Wherever possible problems should be solved using more than one method. Children should also be encouraged to do the same and also come up with their own ways of solving problems.
- Problem posing is an important part of math and children should be encouraged create a variety of problems.
- Challenging questions should be provided at the end of each chapter.
- Textbooks should give space for collaborative learning and give space to children to work in groups and in pairs.
- Textbooks should use language which a child would normally speak and understand. As far as possible they should act as self-learning material for the student.
- Pictures should be used thoughtfully. They could be used to help the child in concept building and should also be used as background fillers to convey the idea that mathematics can be fun, can be done collectively and the math classrooms can be organized in many creative ways. The fillers also show that the book is for the child and she needs to think, solve problems and figure out ways.

Figure 3: NCERT class 7 and 8 textbooks

NCERT Class 7 and 8 textbooks

- Children are generally introduced to proofs in geometry. But it is important for them to understand that a similar process is followed for numbers and thus should be introduced to proofs in number theory too.
- All proofs need to be given in a non-didactic manner, allowing the student to see the flow of reason. Wherever possible more than one proof is to be given.



DO THIS

1. Take any quadrilateral, say ABCD (Fig 3.4). Divide it into two triangles, by drawing a diagonal. You get six angles 1, 2, 3, 4, 5 and 6.

   Use the angle-sum property of a triangle and argue how the sum of the measures of $\angle A$, $\angle B$, $\angle C$ and $\angle D$ amounts to $180° + 180° = 360°$.

   Fig 3.4

2. Take four congruent card-board copies of any quadrilateral ABCD, with angles as shown [Fig 3.5 (i)]. Arrange the copies as shown in the figure, where angles $\angle 1, \angle 2, \angle 3, \angle 4$ meet at a point [Fig 3.5 (ii)].

   (i) (ii)

   Fig 3.5

   For doing this you may have to turn and match appropriate corners so that they fit.

   What can you say about the sum of the angles $\angle 1, \angle 2, \angle 3$ and $\angle 4$?

   [**Note:** We denote the angles by $\angle 1, \angle 2, \angle 3$, etc., and their respective measures by $m\angle 1, m\angle 2, m\angle 3$, etc.]

   The sum of the measures of the four angles of a quadrilateral is____________.

   You may arrive at this result in several other ways also.

3. As before consider quadrilateral ABCD (Fig 3.6). Let P be any point in its interior. Join P to vertices A, B, C and D. In the figure, consider ΔPAB. From this we see $x = 180° - m\angle 2 - m\angle 3$; similarly from ΔPBC, $y = 180° - m\angle 4 - m\angle 5$, from ΔPCD, $z = 180° - m\angle 6 - m\angle 7$ and from ΔPDA, $w = 180° - m\angle 8 - m\angle 1$. Use this to find the total measure $m\angle 1 + m\angle 2 + ... + m\angle 8$, does it help you to arrive at the result? Remember $\angle x + \angle y + \angle z + \angle w = 360°$.

   Fig 3.6

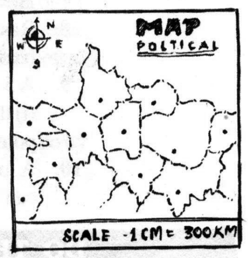
4. These quadrilaterals were convex. What would happen if the quadrilateral is not convex? Consider quadrilateral ABCD. Split it into two triangles and find the sum of the interior angles (Fig 3.7).

   Fig 3.7

Figure 4: Understanding Quadrilaterals, Class-8 NCERT Textbook

Mathematical concepts should be used in tandem with concepts of other subjects to build a deeper understanding of mathematics.

Example 5: The scale of a map is given as 1:30000000. Two cities are 4 cm apart on the map. Find the actual distance between them.



Solution: Let the map distance be x cm and actual distance be y cm, then

$$1:30000000 = x : y$$

or $$\frac{1}{3\times10^7} = \frac{x}{y}$$

Since $x = 4$ so, $$\frac{1}{3\times10^7} = \frac{4}{y}$$

or $$y = 4 \times 3 \times 10^7 = 12 \times 10^7 \text{ cm} = 1200 \text{ km}.$$

Thus, two cities, which are 4 cm apart on the map, are actually 1200 km away from each other.

5. A photograph of a bacteria enlarged 50,000 times attains a length of 5 cm as shown in the diagram. What is the *actual* length of the bacteria? If the photograph is enlarged 20,000 times only, what would be its enlarged length?

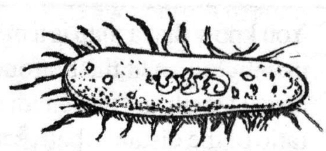


Figure 5: Direct And Inverse Proportions, Class-8 NCERT Textbook

- The texts and visuals should be sensitive to concerns of gender and equality.
- Anecdotes about the history of mathematics and achievements of mathematicians should be added make the subject interesting for children.
- The textbooks should also have some space to talk to the teachers about the design of the syllabus, the structure and presentation of the textbook including the exercises and on how to engage students.

As a part of the federal framework, states have jurisdiction to develop their own curriculum and textbooks keeping in mind the core areas laid out by the national curriculum. Thus, the impact of NCF 2005 is visible in textbook writing of a number of Indian states post 2005. Some experiences from textbook writing processes are mentioned here.

The state of Kerala has taken this spirit much farther, by taking up an extensive participatory process for curriculum formulation as well, in which literally thousands of mathematics teachers took part over a year. The new development of making textbooks available in electronic form, online or offline, has led to interesting possibilities unexplored hitherto. In the state of Kerala, software tools like Dr Geo and Geogebra have been integrated into these textbooks, so that students and teachers can use textbook illustrations interactively and dynamically, changing them as they grapple with meaning. Kerala's IT@School project has also offered linkages by showing the use of these mathematical tools in Science textbooks, thus offering children an opportunity to connect mathematics with physics or chemistry textbooks.

Andhra Pradesh has developed a curriculum framework and series of position papers including the position paper on teaching of mathematics. This paper was based on the NCERT Position Paper on Mathematics and also included some new ideas on the specific concerns of the state. A similar exercise was undertaken by the Bihar state when it wrote its curriculum framework, including ideas on mathematics education. While Bihar has developed its mathematics textbooks up to Class 8, Andhra Pradesh is in the process of doing so. In both the states the process of curriculum and syllabus writing and the development of textbooks has been a joint effort of SCERT functionaries, teacher educators, and mathematics teachers of the state and a non-governmental organization, which has been involved in the development of National Curriculum Framework 2005 and NCERT textbooks.

This process has been a challenging one and has been a learning experience for all those involved in it. It has been challenging for various reasons.

Probably the most important one is the understanding of government functionaries, be they teachers, teacher educators or administrators of education, about 'why mathematics needs to be taught which in turn determines their choice of what should be taught'. Exposed to years of working with an over burdened syllabus of mathematics aspiring to teach children bigger numbers and a taller mathematics, with an emphasis on computation, algorithms and the 'correct method' and most likely a similar experience in their own education, many functionaries find it very difficult to unburden the syllabus and emphasize the process of mathematical thinking. Even though the NCF is very clear on this issue, state functionaries continue to feel that reducing topics leads to loss of mathematical knowledge and children of their state are being deprived in this process. They also feel that such reductions will make their children unfit for various competitive examinations that they will take at the end of schooling.

Another very real challenge for this group is that of 'actually writing' keeping in mind that we have to help the learner engage with concepts, associate it to their life and at the same time develop a capacity to handle abstraction. While writing the group always finds it easier to give information to children, lay out definitions, solve example questions and then give long exercises. The decision of when to move away from concrete objects as aids in understanding towards more abstract conceptualizations is also one that textbook writers have to engage with. Spiraling through concepts also does not come easily as textbook writers often feel that once they have dealt with a concept in a particular class, they need to basically test children for understanding in the next one and no concept revisiting is required. Even while voicing the progressive principles laid out in the curriculum, textbook writers time and again fall back into their old writing styles and privilege mathematical knowledge over mathematical thinking. Giving space to children to work in pairs and in groups through tasks which have potential for collaboration is also an inculcated habit.

Another challenge is of building a healthy atmosphere of listening to feedback that other members of the group give about one's writing. The group of people who sit together also come with different experiences with different amounts of classroom experiences. Some are a part of the government structure and some are voices from outside. Writers also hold what they have written very close to their heart and find it difficult to take critical feedback and view their writing in the light of the set of logical principles that they have themselves laid down in the light of the curriculum. All these also present challenges but at the same time lead to richer and deeper discussions.

The task of the non-governmental partners in this endeavor has been to help the group retain focus on the principles of the NCF and help them write in a manner that children feel confident in approaching and continuing with mathematics. Importantly, it is also to build a healthy workspace where people are listening to each other's ideas and suggestions and owning the whole book and not only some chapters that they are involved in. This is a capacity building process for all involved. People learn a little bit more about mathematics and about mathematics teaching.

Clearly the goals of NCF 2005 are where we want to go. We are yet far from it and have earnestly started on the issues raised in the document. These are however very nascent attempts and the road ahead is long.

Challenges on the road ahead

The vision of mathematics education in NCF 2005 demands changes from the system and schools. It demands a change in the syllabi and textbooks and a change in classroom teaching and assessment. As we have discussed earlier, processes for the former have been initiated and stand at different levels of maturity in different states. However, the latter remains a formidable challenge. An appreciation of what NCF 2005 is saying requires extending the horizons of schools and linking them to the outside world and a different relationship between teachers and children including providing children with opportunities to explore, extend their mind and argue their stance. All these are very hard to achieve. There is little appreciation or acceptance of these principles in society, and among teachers and teacher educators, who are themselves struggling with their limitations in mathematical ability. Also there is little conviction that equitable learning is possible. The belief systems and prejudices about gender, caste, economic status and even cultural practices make mathematics teachers build classrooms differently from those expected in the NCF.

The biggest challenge for us is to change this attitude of teachers, parents and others to mathematics and why and how it should be taught. For most people "why mathematics education" still revolves around mathematics for calculations. Generally, the teacher believes that mathematics is about knowing solutions to problems and not about being able to understand what the concept means and about being able to think of ways of solving problems. The emphasis is on the 'correct answer' rather than on thinking of a variety of ways to approach the solution. Teaching, therefore, gets restricted to sharing solutions with students from either the textbooks or guide books, which offer short cuts and memory devices to children and are used widely especially in the higher classes. Teachers teach in a manner that is entirely de-linked from the experiences of children and participation by children is minimal. There is often even confusion between 'demonstration through concrete examples' and 'the proof of statements'. For the students, the classrooms largely consist remembering the definitions of mathematical ideas, axioms, postulates and solutions to problems or theorems and their proofs. Mathematics classroom, therefore, tends to become uninteresting for students. For most teachers, making mathematics interesting and vibrant is not possible because they themselves are often afraid of mathematics and consider it a subject for the privileged few who are capable and intelligent. 'Activity based mathematics teaching' and 'child centered' teaching are the buzz words, open to multiple interpretations and often get restricted to use of concrete materials for a few concepts in primary classes. Mathematics classrooms, in spite of NCF and the recent textbooks of NCERT remain didactic and assessments test calculations, algorithms, definitions and answers to 'difficult questions'.

Teachers who teach mathematics at the elementary and the secondary level are supposed to be graduates or post graduates in mathematics with a degree/diploma for teaching. In many cases, however, teachers with such qualifications are not available to teach mathematics. Mathematics is taught by teachers who are not very confident of their mathematics. Even in cases where mathematics graduates or post graduates teach the subject their conceptual understanding may be inadequate. Besides, their understanding of the nature of mathematics and attitude to it and its learning are very different from what is underlined in the NCF 2005. The lack of ability of teachers in mathematics is probably the result of their preparation at the school and the college level. It is also because of the inadequate time for pre-service training and the way classroom teaching for pre-service teacher education takes place. Given the large number of teachers in schools and the lack of avenues of for continuing their learning most teachers also do not remain in touch with what they have learnt. There is a strong need for such processes to be initiated that would enable teachers to become more confident and to continue to engage with them. There are, however, insufficiently many institutions and individuals capable of creating and implementing a process that would enable teachers to learn more mathematics and be more confident of their ability. In the Indian context, the lack of this institutional capacity to help teachers learn more mathematical concepts and more about mathematics is the biggest challenge. In India's effort towards universalization of mathematics education, these remain the most critical barriers. They affect the confidence and learning of children much more than the syllabus, textbooks, assessment and everything else put together.

A number of studies and experiences show that many barriers to schooling still exist. These include barriers for the girl child who is not allowed to go to the school after she has reached a certain age, generally the age of puberty. Many schools do not have boundary walls (52%) and separate toilets for girls (41%), and this takes schooling a step further away (NUEPA, 2009). The situation for the secondary classes is worse as the schools are farther from their homes and concerns about the security of girls, forces them to give up schooling. Another factor preventing girls from coming to school is the absence of women teachers in the higher classes. Access is not the only problem for girls and the general societal belief (also shared by teachers) is that the study of abstract ideas does not benefit girls and also that a girl's life priorities do not require her to take on anything as hard as mathematics and science. Frequently heard statements could be that "X is just like a boy, she is so good in mathematics". This attitude adds to the belief already implanted in them that they cannot learn mathematics.

There are also very strong prejudices about poor children and children from deprived social backgrounds. Some time ago almost all children in school were from the so called upper castes. The situation has changed today but a majority of mathematics teachers are still from the higher castes. Their belief is that the poor and lower caste children are not meant to learn mathematics and any sign of their disability is proof of their belief. It may not be hard to appreciate that such attitudes would also be present in children. Children from privileged backgrounds start with this advantage and that initial advantage is further strengthened by the belief of the system that only children from certain backgrounds can do abstract learning. This belief is in contrast to the commitment that India is bound to educate all its children and wants to teach mathematics to all children.

The NCF entails an expectation of a classroom that is interactive and inclusive and a teacher development program that not only builds the capability of the teachers for all this but also motivates them for this through mechanisms of sharing and scaffolding. At present various mechanisms for building the capabilities and interests of teachers are being evolved and include restructuring of pre-service courses of teacher education, strengthening of in-service training as well as strengthening of decentralized (cluster and block level) structures, seeking linkages between colleges of higher education and departments of education and teacher training colleges, etc. Attempts are also being made to reach ideas to the teachers through the use of ICT.

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6. The Senior Secondary Mathematics Curriculum

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Introduction

The National Curriculum Framework (NCF) 2005 in its position paper on the 'Teaching of Mathematics' (National Council for Educational Research and Training [NCERT], 2005) describes the higher secondary stage as the "launching pad from which the student is guided towards career choices." At this stage the student has to make a choice as to whether she will opt for the science, commerce or humanities stream. Clearly mathematics has an important role to play here in developing her skills so that she may pursue her chosen course. For curriculum makers the most difficult choice at this stage is between breadth and depth. Whether the curriculum should offer exposure to a variety of topics from various areas or limit the number of topics to develop competence in a few areas is an issue for debate. According to Thurston, "Instead, there should be more courses available……which exploit some of the breadth of mathematics, to permit starting near the ground level, without a lot of repetition of topics that students have already heard."

The NCF suggests that if breadth is chosen over depth, then the decision as to the extent to which the topics should be developed is a matter of serious consideration. The topics which have importance for mathematics as a discipline should be included and their treatment should be done at least to the extent that the student is able to see the relevance or utility of those topics in mathematics or in some other course of study.

Content and structure of the curriculum: Brief commentary

The higher secondary mathematics curriculum is dominated by Differential and Integral Calculus accounting for almost half of the content in class 12. Other topics include Matrices and Determinants, Vector Algebra, Three Dimensional Geometry, Linear Programming and Probability. These topics remain isolated and there are few instances where the linkages across these topics are highlighted. Also manipulative and computational aspects of these topics, rather than applications, dominate mathematics at this stage.

The syllabus of class 11 includes important topics like Sets, Relations and Functions, Logic, Sequences, Series, Linear Inequalities, Combinatorics, Trigonometric Functions, Complex Numbers, Straight lines, Conic Sections and Statistics. The striking thing about the class 11 syllabus, in contrast to that of class 12, is its large number and variety of topics. While many of these topics are rich in mathematical content their treatment is only done at a surface level. Also, since the Board Examinations at the end of class 12 tests only the topics of class 12, these topics remain neglected. NCF 2005 recommends that curriculum designers reconsider the distribution of content between classes 11 and 12.

The NCF 2005 position paper begins by stating that the primary goal of mathematics education is the "mathematisation of the child's thought processes" and the development of the "inner resources of the growing child." Mathematics empowers an individual to think logically, handle abstractions, generalize patterns and solve problems using a variety of methods. The document states that for children to acquire such integrated skills, a curriculum is needed that is "coherent and teaches important mathematics"; here, 'coherence' refers to the way the different strands of the curriculum reinforce one another and enable the student to apply concepts learnt in one strand to other strands, and to other school subjects such as science and social studies. Also, the mathematics taught in school should be 'important' in the sense that "teachers and students find it worth their time … addressing [the] problems, and mathematicians consider it an activity that is mathematically worthwhile." In this context, the document recommends that mathematics teaching at all levels be made more 'activity oriented' and student centred, so that students understand the basic structure of mathematics and learn how to think mathematically and how to relate mathematics to life experiences.

The NCF 2005 recommendations have been the driving force for revisiting and revamping the elementary school mathematics curriculum. But the recommendations have had little impact on the senior secondary curriculum. The textbooks as well as the content of the senior secondary curriculum have undergone very few changes over the years. Some topics have been removed while others have been added, but the approaches to the topics have remained the same. In the textbooks, chapters include an introductory note with some historical background, the basic concepts, theorems, results, examples and exercises. However, there are very few inputs in terms of applications. For example, in the chapter 'Application of Derivatives' the topics covered are almost the same from 1989 to the present: Motion in a Straight Line, Motion under Gravity, Rate of Change, Increasing and Decreasing Functions, Maxima and Minima, Rolle's Theorem, Mean Value Theorem, Tangent and Normal, and Differentials and Approximations. Here are two typical problems from the textbook:

> A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.
>
> A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without a lid, by cutting off a square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is a maximum?

The emphasis is largely on developing manipulative skills to solve problems, and there is relatively little on visualizing concepts and exploring applications. Also, many questions in the exercises are based on direct application of rules and formulae. For example:

> Find the angle between the straight lines $y - x\sqrt{3} - 5 = 0$ and $y\sqrt{3} - x + 6 = 0$.

The NCF 2005 also suggests that mathematical modelling be introduced at this level making it possible to include the applications of some of the mathematical concepts that students are learning at this stage. Increasingly, many countries across the world, such as USA and UK, have witnessed a growing collection of didactical research on including mathematical applications and modelling at the high school level and this has impacted the mathematics curricula in those countries. Studies have shown that the benefits of including mathematical modelling and applications in the curriculum are manifold. They can:

- Help relate topics taught in the classroom to situations outside the classroom.
- Highlight the relevance of mathematics as a discipline.
- Focus on applications which help to build the students' interest in the subject.
- Offer direction in career options.

The textbooks: An overview

In this section we provide an overview of the topics in the senior secondary curriculum and the way the topics have been dealt with in the NCERT textbooks. For our study we have considered the NCERT mathematics textbooks of class XII of three representative years (1989, 2000 and 2007) and the NCERT mathematics textbooks of class XI of four representative years (1988, 1995, 2002 and 2006). We start with Class XII.

Class XII textbooks

The following topics have been a part of the syllabus through the years:

- Functions, limits and continuity
- Matrices and Determinants

- Continuity and Differentiability
- Application of Derivatives
- Integrals
- Applications of Integrals
- Differential Equations
- Vector Algebra
- Three Dimensional Geometry
- Probability

Chapters on Mathematical Logic, Correlation and Regression, Computing and Numerical Methods were part of the syllabus till 1989 and then removed. The chapter on Mathematical Logic included subtopics on mathematical statements and truth values, the use of Venn diagrams in logic, conjunction, disjunction, conditional statements, biconditional statements, truth tables and applications to switching circuits. These topics were reintroduced in 2003 in a chapter called 'Boolean Algebra' which included Boolean algebra as an algebraic structure, principle of duality, concepts of conjunction, disjunction, conditional statements, biconditional statements followed by truth tables and applications to switching circuits. In 2005 Boolean algebra was removed from the syllabus. The chapter on Numerical Methods dealt with basic numerical analysis and included methods for approximating the solutions of polynomial equations using successive bisection and the Newton-Raphson method. For solutions of systems of equations, the Gauss elimination method and the Gauss-Seidel iterative method were discussed.

The NCERT textbooks were revised in year 2000. The syllabus was divided into parts A, B and C. Part A (70 marks) was compulsory for all students, part B (30 marks) was for students of the science stream, and Part C (30 marks) for students of the commerce stream. Calculus accounted for nearly 50% of the syllabus and was included in the compulsory part. Part A also included the following topics: Matrices and Determinants, Probability, and Boolean Algebra. Part B included Vectors and Three-Dimensional Geometry. Part C included topics related to Commercial Mathematics (Partnership, Bills of Exchange) and Linear Programming.

In 2005 the textbooks were again revised based on the recommendations of the NCF 2005. The revised textbooks appeared in 2007. The three parts A, B and C were removed, and the topics were now sequenced in the following manner

- Relations and Functions
- Inverse Trigonometric Functions

- Matrices
- Determinants
- Continuity and Differentiability
- Application of Derivatives
- Integrals
- Applications of Integrals
- Differential Equations
- Vector Algebra
- Three Dimensional Geometry
- Linear Programming
- Probability

There was however no major change in the approach to dealing with these topics in the revised textbooks. For example, if we look at the chapter on Application of Derivatives we find the book of 2007 had problems and exercises similar to that of the previous years. In the section on Maxima and Minima the following problems have been appearing through the years:

> Find two positive numbers x and y such that their sum is 35 and the product x^2y^2 is a maximum.
>
> A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without a top, by cutting off squares from each corner and folding up the flaps. What should be the side of the squares to be cut off so that the volume of the box is a maximum?

In all the other chapters the content has largely remained the same in terms of concepts, explanations, solved examples and exercises.

From the year 2007, calculus was introduced in class XI. Thus the topic of limits was included in the textbook for class XI. Based on the recommendations of the NCF 2005 position paper, two chapters were added in the Appendix of the textbook for class XII.

> **Proofs in Mathematics:** This dealt with various types of proofs in mathematics, namely the direct and the indirect approach. In the direct approach, straightforward proof, mathematical induction and proof by exhaustion were discussed whereas in the indirect approach, proof by contradiction, proof by proving the contrapositive statement and proof by counter examples were discussed.
>
> **Mathematical Modelling:** This chapter highlighted the need and importance of mathematical modelling, the principles of modelling and steps involved in the

modelling process. It included examples from the topic of matrices, trigonometry and linear programming and ended with a paragraph on the limitations of the modelling process.

Class XI textbooks

Now we study the NCERT mathematics textbooks of class XI of four representative years (1988, 1995, 2002 and 2006). The following topics have remained in the syllabus through the years:

- Sets
- Relations and Functions (till 1995 this was combined with the chapter on sets)
- Trigonometric Functions
- Principle of Mathematical Induction
- Complex Numbers
- Quadratic equations
- Linear Inequalities
- Permutations and Combinations
- Binomial Theorem
- Sequences and Series
- Straight lines
- Conic Sections
- Statistics

Chapters on the topics Solution of Triangles, Inverse Trigonometric Functions, Linear Programming and Algorithms and Flowcharts were part of the syllabus till 1995 and removed later.

When the textbooks were revised in 2000, the class XI syllabus was divided into parts A, B and C just as was done with the syllabus of class XII. Part A (70 marks) was compulsory for all students, Part B (30 marks) was for students of the science stream, and Part C (30 marks) was for students of the commerce stream. Part A included all the topics mentioned above. A chapter on Mathematical Logic was added to this section. This included mathematical statements, logical connectives, truth tables, tautologies, logical equivalence, duality, algebra of statements and use of Venn diagrams in logic. Part B (for science students) included introductory chapters on Vector Algebra and Three Dimensional Geometry. These topics were dealt with in greater depth in Section B of the class XII textbook. Part C (for commerce students) included topics on Stocks, Shares and Debentures, Average and Partition Values and Index Numbers.

In 2005 the textbooks were again revised based on the recommendations of the NCF 2005, just as was done for Class XII. The revised textbooks appeared in 2007. The three parts A, B and C were removed, and the topics were now sequenced in the following manner:

- Sets
- Relations and Functions
- Trigonometric Functions (this included trigonometric equations but the subtopic on solution of triangles was removed)
- Mathematical Induction
- Complex Numbers and Quadratic Equations
- Linear Inequalities
- Permutations and Combinations
- Binomial Theorem
- Sequences and Series
- Coordinate Geometry (Straight lines, circles and conic sections)
- Introduction to Three Dimensional Geometry
- Mathematical Reasoning
- Statistics and Probability
- Limits and Derivatives (Calculus for the first time was introduced in class XI)

Two chapters were added in the appendix of the book.

Infinite Series: This included the subtopics on Binomial theorem for any index, Infinite geometric series. The topic of Exponential and Logarithmic series was moved to this chapter.

Mathematical Modelling: This chapter dealt with the concept of mathematical modelling which was extended in the class XII textbook.

Over the years there has not been any major change in the approach of dealing with the topics in terms of introducing or explaining the concepts or in the examples and exercises. For example, if we look at the chapter on Straight Lines we find that the subtopics covered are the same from 1988 to the present. These include equation of a straight line parallel to the axes, slope-point form of the equation of a line, two-point form, slope-intercept form, normal form, symmetric form, angle between two lines, condition of concurrency of three straight lines and translation of axes.

Also many of the questions in the exercises are based on direct application of the formulae and results presented in the chapters and have been repeated in the textbooks through the years. They appear to focus more on testing the student's manipulative skills. The following are some sample questions:

> Find the equation of the line perpendicular to $3x + 2y = 8$ and passing through the midpoint of the line joining $(5, -2)$ and $(2, 2)$.
>
> Find the angle between the straight lines $y - x\sqrt{3} - 5 = 0$ and $y\sqrt{3} - x + 6 = 0$.

Even the topic on Conic Sections has remained the same through the years. A diagram shows the sections of a right circular cone. This is followed by definitions, derivations of the equations for different conics and their properties. The problems and exercises are also based on the application of rules and formulae.

Missed opportunities in the Senior Secondary Curriculum

In this context we briefly comment on the "missed opportunities" in the syllabus. We ask, are there opportunities in the syllabus where without any further additions being made to the syllabus, much more can be done, where connections can be shown in a natural way, where technology enabled exploration is possible by the very nature of the topic? All this is asked keeping in mind the central tenet of the NCF 2005 document: "There is one main goal of mathematics education in school: the mathematisation of the child's thought processes." As noted by David Wheeler, "It is more useful to know how to mathematize than to know a lot of mathematics." When looked at in this light, numerous missed opportunities become visible; one finds numerous activities are possible which serve to unify different strands within the syllabus. The value of such activities is very great, because one of the traditional areas of weakness in our curriculum is the lack of attention given to connections between topics (and still less attention is given to connections across subjects). We quote one such example: the justification for the formula for volume of a pyramid: "V equals one-third (area of base) × (height)." This formula is made known to children when they are in the 9th or 10th standard and the presence of the factor 1/3 remains a point of mystery which never gets resolved. But if this topic is revisited when the students are in the 11th standard and have learnt a certain amount about finding a formula for the sum of the squares of the first n natural numbers, about proof by induction, and about limits, then this sets the stage for an illuminating activity in which we obtain the formula for volume by a graded sequence of steps, starting with dividing the pyramid into *n* slices of equal thickness parallel to the base (where *n* is large); finding the volume of each slice by treating it as a cuboid (some input is needed from the geometry of similar triangles here); summing the volumes (this is where the formula for $1^2 + 2^2 + ... + n^2$ is needed); estimating the limit computationally using a computer; and then actually determining the limit analytically. Finally one gets the known formula, and it is indeed a pleasure to see it emerge in front of our eyes. Finally one has the opportunity for demonstrating the correctness of the formula by an activity in the mathematics laboratory, in which we show how 6 congruent right pyramids of suitable size can fit together to yield a cube.

Given the value of shifting focus in the curriculum from content to process, it is important that we identify as many such opportunities as possible, because they bring many strands together and have great value in integrating concepts in a student's mind.

Assessment

Assessment in the Indian school education system is largely limited to the summative variety, and it is for the most part a device to measure cumulative learning: a device used to help teachers write reports and to help make pass/fail decisions. Thus there is little or no feedback into the learning process.

In few countries is it as true as in India that summative assessment in the form of a school leaving examination holds the key to one's future, in the sense of opening or closing doors of opportunity. The problem is of sufficient gravity that every single year there are suicides associated with it: children unable to cope with the disappointment and shame of failure, or with the fear of condemnation. Inevitably, the spectre of such assessment exercises a significant influence on the ambient educational culture, inviting poor educational practices and the creation of a powerful parallel education system called 'coaching'. Indeed, it invites criminal activity as well, through the leakage and sale of examination papers. It will be clear from these remarks that the school leaving examination is an extremely high stakes event.

School education in India follows a ten plus two system: ten years of compulsory schooling in which all students follow the same stream, followed by two years in which one chooses a set of optional subjects. These are grouped into streams: Mathematics, Physics, Chemistry (commonly known as 'MPC' or 'PCM'), Biology, Physics, Chemistry ('BiPC' or 'PCB') and so on.

The Central Board of Secondary Education (CBSE) and Council for the Indian School Certificate Examinations (CISCE) are national examination boards, and the better known schools in the country are associated with one or the other of these. CBSE follows the syllabus set by NCERT and uses NCERT textbooks, whereas CISCE sets its own syllabus, at both the 10th standard and 12th standard levels, and does not prescribe textbooks; schools are free to use textbooks of their choice. The academic standards of the two boards are comparable. The major difference is that CBSE has done away with the 10th standard examination, and has substituted it with the CCE system mentioned below. However the 12th standard examination continues in its original form.

Recently, the Central Board for Secondary Education has taken steps to bring in alternate assessment systems and has introduced a 'Continuous Comprehensive Evaluation' system (CCE for short; Continuous and Comprehensive Evaluation, 2011). It has prepared elaborate manuals for teachers on how CCE is to be transacted, and has held workshops on CCE methodology. The scheme certainly holds promise, but its long term effect on the academic culture of our schools remains to be seen.

Entry into colleges is decided either on the basis of the marks secured in the 12th standard or entrance examinations conducted by the respective colleges. Population pressures mean that entrances are a highly competitive process, particularly for prestigious colleges like the IITs (Indian Institutes of Technology) or AIIMS (All India Institute of Medical Sciences). This single fact has had a great influence on secondary school education — unfortunately, not a positive one; indeed, one that trickles down to the primary level. The entrance examinations of a few institutes have now become bench marks in the country. We shall look at the style of a few of these examinations later in this essay.

A situation peculiar to this country is the phenomenon of tutorial colleges ('coaching centres') which seek to prepare students for entrance to highly sought-after institutions. Some of these colleges are themselves highly sought after, and they have their own selection examinations, a situation which invites the possibility of an infinite iterative loop! One could laugh in good humour at the situation if it were not so wasteful of human energy. The methods used by these colleges amount to all-out drill, mastery of pattern recognition through analysis of past papers (a kind of reverse engineering set in an educational context), and reliance on huge memory banks. Over the last several decades these practices have gotten absorbed into the ambient educational culture of the country.

Comments from NCF 2005

The NCF 2005 document lists four core areas of concern: "(1) a sense of fear and failure regarding mathematics among a majority of children, (2) a curriculum that disappoints both a talented minority and a non-participating majority, (3) crude methods of assessment that encourage a perception of mathematics as mechanical computation, (4) lack of teacher preparation and support in the teaching of mathematics." It amplifies on the third point: "While what happens in class may alienate, it never evokes panic, as does the examination. Most of the problems cited … relate to the tyranny of procedure and memorization of formulas …, and the central reason for the ascendancy of procedure is the nature of assessment …. Tests are designed … to assess knowledge of procedure and memory of formulas. … Concept learning is replaced by procedural memory. Such antiquated and crude methods of assessment have to be thoroughly overhauled ….” It recommends the following: (1) Shift the focus of mathematics education from achieving narrow goals to higher goals; (2) Engage every student with a sense of success, and at the same time offer challenges to the emerging mathematician; (3) Change modes of assessment to examine mathematisation abilities rather than procedural knowledge; (4) Enrich teachers with a variety of mathematical resources.

The third and fourth points are of relevance here. With regard to the third point it adds: “Since the Board examination for Class X is for a certificate given by the State, implications of certified failure must be considered seriously. Given the reality of the educational scenario, the fact that Class X is a terminal point for many is relevant; applying the same … standard of assessment for these students as well as for rendering eligibility for the higher secondary stage seems indefensible. … [Given] the high failure rate in mathematics, we suggest that the Board examinations be restructured. They must ensure that all numerate citizens … become eligible for a State certificate. … Nearly half the content of the examination may be geared towards this. … However, the rest of the examination needs to challenge students far more than it does now, emphasizing competence and expertise rather than memory. Evaluating conceptual understanding rather than fast computational ability in the Board examinations will send a signal of intent to the entire system, and over a period of time, cause a shift in pedagogy as well. … These remarks pertain to all forms of summative examinations …. Multiple modes of assessment … need to be encouraged. This calls for … research and a wide variety of assessment models to be created and widely disseminated.” These are stirring words, and we hope they will be visited repeatedly by examination boards and state education departments in the years to come.

How examination boards handle assessment

In this section we study the way the CISCE deals with some selected topics and with problem solving in general.

Public examinations, high school level (class 10 – ICSE 2010)

(VAT computation) A manufacturer marks an article for Rs 5000. He sells it to a wholesaler at a discount of 25% on the marked price and the wholesaler sells it to a retailer at a discount of 15% on the marked price. The retailer sells it to a consumer at the marked price; at each stage the VAT is 8%. Calculate the VAT received by the Government from: (a) the wholesaler, (b) the retailer. [Question 6, ICSE 2010]

(Trigonometry) Without using trigonometric tables evaluate $\frac{\sin 35^\circ \cos 55^\circ \ + \ \cos 35^\circ \sin 55^\circ}{\csc^2 10^\circ - \tan^2 80^\circ}$

[Question 3(b), ICSE 2010]

Public examinations, high school level (class 12 – ISC 2010)

In this section we look at examples from one of the class 12 school leaving examinations.

(Matrix algebra) If the matrix $\begin{bmatrix} 6 & x & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{bmatrix}$ is singular, find the value of x. [Question 1(i), ISC 2010]

(Coordinate geometry in two dimensions) Show that the line $y = x + \sqrt{7}$ touches the hyperbola $9x^2 - 16y^2 = 144$. [Question 1(iii), ISC 2010]

(Differential calculus) Using a variable substitution, find the derivative of $\tan^{-1}\sqrt{\frac{a-x}{a+x}}$ with respect to x. [Question 5(a), ISC 2010]

(Integral calculus) Evaluate the following integral: $\int \frac{2\sin 2\theta - \cos\theta}{6-\cos^2\theta - 4\sin\theta}\, d\theta$. [Question 6(a), ISC 2010]

(Differential equations) Solve the differential equation $\csc^3 x\, dy - \csc y\, dx = 0$. [Question 1(x), ISC 2010]

(Boolean algebra) x, y, z represent three switches in an "ON" position, and x', y', z' represent the same three switches in an "OFF" position. Construct a switching circuit representing the polynomial $(x + y)(x' + z) + y(y' + z')$. Using the laws of Boolean algebra, show that the above polynomial is equivalent to $xz + y$, and construct an equivalent switching circuit. [Question 4(b), ISC 2010]

(Data analysis: Moving averages) Consider the following data.

Dates	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Sales	2	5	0	12	13	25	45	13	31	18	11	2	3	1

Calculate three-day moving averages and display these and the original figures on the same graph. [Question 15(b), ISC 2010]

(Mean value theorem) Using Rolle's theorem, find a point on the curve $y = \sin x + \cos x - 1, x \in \left[0, \frac{\pi}{2}\right]$, at which the tangent is parallel to the x-axis. [Question 3(a), ISC 2010]

(L'Hopital's rule for limits) Evaluate: $\lim_{x\to\pi/2}\left[x\tan x - \frac{\pi}{2}\sec x\right]$. [Question 1(iv), ISC 2010]

Note the 'all or nothing' nature of the questions. This is typical of most examinations in the country. Note also the question asked in Data Analysis. This area is very poorly represented in the Indian curriculum; the coverage is limited to computation of a few statistics, with little or no interpretation of the numbers. The 'numeracy' component of such courses is low. But a characteristic feature of most such examinations is the high level of manipulative ability.

Project work

Progress has been made regarding project work. Some examination boards have a component for investigatory project work, which students do under a teacher's supervision. The marks allocation is modest: 10% of the total.

An example of a shift in thinking at the national level is the award of the KVPY scholarship based on original work in science or mathematics. It is possible to get the scholarship through a regular examination mode, but the organizers recognize that this mode discriminates against certain kinds of students, and hence that a dual approach is needed at a national level.

Competitive examinations: Engineering colleges

Here we study some questions asked in post school entrance examinations. We limit ourselves to two such exams: the AIEEE or All India Engineering Entrance Examination, and the JEE-IIT or Joint Entrance Examination for the IITs. A glance of the questions reveals the high level of preparation needed to do well in the exams. The time availability needs to be kept in mind: no more than three minutes for a typical multiple choice question! The situation is complicated by the fierce competition and very large number of candidates, which imply that a difference of just one mark may account for many hundreds of candidates. In response, students use strategies based on pattern recognition and memorization of large numbers of solved problems. One can well imagine the effects of this kind of high intensity input when it is continued for a year or longer: the effects on conceptual understanding, and on the psyche of individuals.

Here are the paper formats: ('PCM' is short for 'Physics, Chemistry, and Mathematics'):

- AIEEE: 30 MCQs each in PCM (duration 3 hours).
- JEE: Paper I: 28 questions each in PCM; 8 MCQ, single correct choice; 5 MCQ, one or more correct choices; 5 comprehension type MCQs; 10 numerical answer (with a two digit answer); 84 questions in all (duration 3 hours).
- JEE Paper II: 19 questions each in PCM; 6 MCQs, single correct choice; 5 numerical answer (with a single digit answer); 6 comprehension type MCQs; 2 matrix column matching type; 57 questions in all (duration 3 hours).

We take a look at some problems from AIEEE and JEE papers.

(Combinatorics): The number of 3×3 non-singular matrices with four entries as 1 and all other entries 0 is: (a) 5 (b) 6 (c) at least 7 (d) less than 4. [Question 71, AIEEE 2010]

(Coordinate geometry):

Statement 1: The point $A(3, 1, 6)$ is the mirror image of the point $B(1, 3, 4)$ in the plane $x - y + z = 5$.

Statement 2: The plane $x - y + z = 5$ bisects the line segment joining $A(3, 1, 6)$ and $B(1, 3, 4)$.

Then: (a) Statement 1 is true, statement 2 is false; (b) Statement 2 is true, statement 1 is false; (c) Statement 1 is true, statement 2 is true, and statement 2 is a correct explanation of statement 1; (d) Statement 1 is true, statement 2 is true, and statement 2 is not a correct explanation of statement 1. [Question 73, AIEEE 2010]

(Probability and combinatorics): Four numbers are chosen at random (without replacement) from the set $\{1, 2, 3, ..., 20\}$.

Statement 1: The probability that the four numbers when arranged in some order will form an AP is $\frac{1}{85}$.

Statement 2: If the four chosen numbers form an AP, then the set of all possible common differences is $\{\pm1, \pm2, \pm3, \pm4, \pm5\}$.

Then: (a) Statement 1 is true, statement 2 is false; (b) Statement 2 is true, statement 1 is false; (c) Statement 1 is true, statement 2 is true, and statement 2 is a correct explanation of statement 1; (d) Statement 1 is true, statement 2 is true, and statement 2 is not a correct explanation of statement 1. [Question 72, AIEEE 2010]

(Trigonometry): Let P and Q denote the statements

P: $cosA + cosB + cosC = 0$

Q: $sinA + sinB + sinC = 0$

If $cos(B - C) + cos(C - A) + cos(A - B) = -\frac{3}{2}$ then:

(a) P is true and Q is false (b) P is false and Q is true (c) both P and Q are true (d) both P and Q are false. [Question 64, AIEEE 2009]

(Calculus, derivatives): Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only root of $P'(x) = 0$. If $P(-1) < P(1)$ then in the interval $[-1, 1]$: **(a)** $P(-1)$ is the minimum and $P(1)$ is the maximum of P; **(b)** $P(-1)$ is not the minimum but $P(1)$ is the maximum of P; **(c)** $P(-1)$ is the minimum and $P(1)$ is not the maximum of P, **(d)** neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P. [Question 84, AIEEE 2009]

(Complex numbers): Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is: (a) 48 (b) 32 (c) 40 (d) 80. [Question 24, JEE 2009]

(Trigonometry): In triangle ABC with fixed base BC, the vertex A moves so that $cosB + cosC = 4sin^2\frac{A}{2}$. If a, b, c denote the lengths of the sides opposite to the angles A, B, C then: (i) $b + c^2 = 4a$ (ii) $b + c = 2a$ (iii) the locus of point A is an ellipse (iv) the locus of point A is a pair of straight lines. [Question 31, JEE 2009; this is a MCQ in which there is more than one correct choice]

(Integral calculus): Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1 - f'(t)^2}\ dt = \int_0^x f(t)dt$ for $0 \le x \le 1$, and $f(0) = 0$, then: **(a)** $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$; **(b)** $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$; **(c)** $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$; **(d)** $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$. [Question 23, JEE 2009]

(Vectors): If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are unit vectors such that $\left(\vec{a} \times \vec{b}\right) . \left(\vec{c} \times \vec{d}\right) = 1$ and $\vec{a}.\vec{c} = \dfrac{1}{2}$ then: (i) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar; (ii) $\vec{b}, \vec{c}, \vec{d}$ are non-coplanar; (iii) $\vec{b}, \vec{d}$ are non-parallel; (iv) $\vec{a}, \vec{d}$ are parallel, and $\vec{b}, \vec{c}$ are parallel. [Question 26, JEE 2009]

(Calculus, limits): Let $L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - x^4/4}{x^4}$ where $a > 0$; then: (i) $a = 2$ (ii) $a = 1$ (iii) $L = \frac{1}{64}$ (iv) $L = \frac{1}{32}$. [Question 30, JEE 2009]

Trick questions are also asked in such examinations: those for which one can get the answer through a trick peculiar to just that question, or those in which one can examine the multiple options given and quickly eliminate all but one of them, typically by some very elementary argument which barely does justice to the depth of the question itself.

The relevant question yet again lies in the social setting that lies behind these entrance examinations–the coaching culture, which is so numbing of human initiative, and the intense expectations placed on pupils by parents. The really sad part lies in the intention: the primary purpose behind these highly sophisticated examinations is, surely, to filter out and exclude, rather than to include and nurture.

Educational assessment is an exceedingly complex matter, for the issues it touches are so many, and of so varied a nature: from educational pedagogy to inherited societal problems such as caste barriers with which India is struggling. (These have held back the movement in society of whole communities within the country.) Yet, it is not an open problem in the sense that a mathematical problem may be described as 'open'. The primary difficulty, we feel, is a lack of clarity of educational vision. What is needed is not so much the subtlety of mathematical questions (not that this is not relevant, but it is far from being a primary issue) but the matter of quality of nurture and care that should inform and be the ultimate driving force behind assessment.

The question of technology

Mathematics has for years been the common language for classification, representation and analysis. Learning mathematics forms an integral part of a child's education. Yet, it is also the subject traditionally perceived as difficult. The primary reason for this is the significant gap between content and pedagogy. The last few decades have witnessed serious experimentation and research in mathematics education all over the world and there has been a shift of paradigm as far as mathematics teaching-learning is concerned. Mathematics education is being revolutionized with the advent of new and powerful technological tools. Because of these tools mathematics education can focus on problem solving and reasoning that empower students to explore, conjecture and reason logically. While traditional mathematics is often fraught with rote memorization of procedures, computational algorithms, paper-pencil-drills and manipulation of symbols, the use of technology encourages teachers and students to engage in deep mathematical thinking involving analysis, problem posing, problem solving and rich conceptual understanding.

Many countries are increasingly using technology in mathematics teaching and learning but this is not the case in India. Although integration of technology in schools is not uncommon in India, its use in mathematics teaching and learning in this country is not prevalent. In this section we list the challenges facing mathematics education in India and suggest ways in which technology can play a role to overcome these challenges.

The challenges facing school mathematics education in India may be broadly categorized under the following heads.

- Transaction of curriculum
- Inappropriate assessment
- Teacher preparation

Each of these can be significantly impacted by the appropriate use of technology. For example, technology can aid in the visualization of concepts, in exploration and discovery, in bringing the experimental approach into mathematics, in focusing on applications, in redefining the teacher's role, in helping sustain students' interest, in individualized grading and assessment, and in teacher outreach.

But while technology has profound implications for teaching and learning, it does not by itself supply solutions. The mere provision of technology in a class does not solve the problems faced in teaching. It is thus essential that serious research and experimentation go into the use of technology as an aid to teaching mathematics. Mathematics educators must consider in depth the possibilities created by computer software and handheld calculators.

Implementation of technology poses many challenges, the greatest being the socio-economic challenge. The priority of Government is to reach education to the masses. Technology must be cost effective and easy to deploy. The last few years have witnessed extensive use of computer technology in schools. However mathematics teaching continues in the traditional 'chalk and board' manner. Technology, if used for teaching mathematics, is primarily for demonstration purposes and does not involve the student actively. It is imperative that a mathematics curriculum be designed which integrates technology.

To successfully face the challenges in implementing technology in the Indian context, pre-service teacher education programmes must be designed where student teachers are taught mathematics using various technological tools. This will help develop new perspectives on integrating technology in their teaching-learning. In-service teacher training programmes must focus on changing teachers' mindset towards technology and helping them overcome their 'technological anxiety'. Technology must play a role in developing their pedagogical content knowledge. These professional development programmes must be held in a sustainable manner. This requires collaboration with technology solution providers who can provide ongoing support for the use of the technological tools in the schools. Students should be given adequate access to technology on a daily basis. Further, involvement of teachers on a large scale will require fundamental changes in teaching practices.

Mathematics laboratories are a medium through which students can explore and visualize mathematical concepts and ideas through the use of technology, and the potentialities of this medium need to be explored to its fullest extent.

In conclusion one may say that much work remains to be done if we are to effectively use the power and reach of modern technology in mathematics education in India. Perhaps the area of greatest challenge is teacher preparation: developing sustainable professional development programmes for teachers which not only enhance the skills of the teacher in terms of usage of various technological tools but also focus on improving their pedagogical content knowledge using technology. Another challenge is that the present curriculum does not readily lend itself to integration of technology. The goals of mathematics learning and assessment need to undergo a major shift in paradigm in a technology integrated mathematics curriculum. Also technology must be cost effective and easy to deploy in order to achieve large scale integration in schools and teacher education institutions. All this has tremendous implications in terms of infrastructure requirements. So a great deal of work remains to be done, but the benefits would clearly be enormous.

Concluding note: Some reflections

1. As mentioned earlier the content has largely remained the same over the years. The approach to dealing with the topics has also remained the same – to a much greater degree. Every chapter usually begins with a brief introductory note which sometimes includes a historical background of the development of the field, and then introduces the basic concepts of the topic. The chapter is divided into sections and sub-sections which deal with definitions, theorems, results, examples and exercises. This has been the format in which the topics have been written over the years.

2. However given the fact that the demands of mathematics education are changing, this needs to reflect in the mathematics curriculum and also in the way the topics are dealt with in the textbooks. The use of real world applications and modelling in dealing with concepts in various topics will help create a context for applying mathematical theory; this can act as a motivating factor. It will help highlight the beauty and relevance of mathematics as a discipline and at the same time focus on the usefulness of the topics being discussed. Indeed if the higher secondary stage is the 'launching pad' from which the student is guided towards career choices, the mathematics curriculum needs to provide inputs in terms of application to various fields of study such as the engineering sciences, biological sciences, economics and econometrics and even the social sciences.
   If applications have to be introduced, usefulness cannot be the only criteria for including a particular topic. The structure of mathematics is also critical in deciding which applications should be introduced in a particular topic, and in what sequence. In fact the applications should blend into the chapter enabling the student to
   - Learn new mathematical content;
   - Learn that mathematics applies to real problems;
   - Apply or practice the mathematics they have learnt;
   - See that applications are important because they lead to new mathematical problems and hence to the development of mathematics.

   Clearly the applications and modelling approach could be used to develop the concepts in various topics such as calculus, matrices and determinants and probability. Including material on mathematical modelling in the Appendix of the textbook does not appear to serve much purpose. Rather, a chapter exclusively on mathematical modelling could be included, dealing with models related to traffic flow, cryptography, optimization, genetics and so on. This will help students see how mathematics applies to various fields of study.

3. The senior secondary mathematics curriculum needs to have adequate emphasis on the understanding of mathematics as well as problem solving skills. Presently the emphasis seems to be largely on the computational aspects. Topics which form the foundation of Pure Mathematics courses at the undergraduate level may be included in the curriculum at an elementary level such as Group Theory. Similarly topics in applied mathematics such as Graph Theory, Game Theory, Markov chains and Numerical Methods may be included in the curriculum at an elementary level, and students given a choice to opt from among these topics. This will help align the senior secondary mathematics curriculum with the requirements of the mathematics courses at the undergraduate level.

   It is apt to say that the senior secondary mathematics curriculum needs to undergo a major shift in terms of structure and presentation of content.

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7. Curriculum and pedagogy in mathematics: Focus on the tertiary level

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Introduction

The aim of this essay is to present a critical overview of mathematics education at the tertiary level in India. 'Mathematics at the tertiary level' would typically refer to mathematics taught at the undergraduate and post-graduate levels and would also cover research degrees in mathematics. By focusing primarily on undergraduate mathematics education and by extrapolating from different experiences we hope to throw some light on curriculum and pedagogy in mathematics at the tertiary level in India.

Only a tiny fraction of the Indian population (1.2 billion strong) enters higher education – it is yet vast in numbers, and expanding rapidly. India is therefore faced with the triangle of quantity, quality and equality. The immediate twin challenges that the country faces are: how do we increase the percentage of the population that accesses higher education? And how do we improve the quality of higher education?

Undergraduate education is akin to a pivot or keystone that holds together myriad strands that contribute to society. It is the case that whether we consider a teacher at school or a lecturer at the University, a researcher or a manager, or anyone holding a white-collar job, the one common aspect they share regarding higher education is that of having had undergraduate education.

The degree at the undergraduate level paves the way for the future. It is usually carefully chosen keeping in mind interests, aptitude and career opportunities ahead. It comprises those years in one's life where one is moving from a system governed by 'restrictions' to one that is about 'making choices and decisions' that stay with the rest of one's life. It is also the first adult interaction that one has with education.

The French philosopher and mathematician René Descartes (1596-1650) said "Mathematics is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency." Mathematics is explicitly and implicitly present in many things of importance to society. Mathematics has a role to play in so many different fields: innovations in medicine, digital encryption, communication technology, modelling real life phenomena, predicting disasters, organisation of enterprises, business and transport to name a few.

Yet, in general there is not only a lack of awareness about the indispensability of mathematics but instead there is a marked tendency to 'ignore' mathematics. Indeed, it is fashionable to acknowledge publicly the deep-seated fear of the subject that stems from memories of bad experiences with mathematics usually encountered at school. It is important for mathematicians and mathematics educators to acknowledge and seek ways of changing this. It is neither good for the discipline nor society to be in a situation where possibly over 50% can only recount dislike and unhappiness associated with mathematics.

At the heart of mathematics education lies undergraduate mathematics education. It would be impossible to tackle any of the problems associated with mathematics education, at any level without intervention at the undergraduate level. After all, the harbingers of change, if there are to be any, will be the teachers, policy makers, the creators and imparters of curriculum and pedagogy. And each one of them will have been shaped by their undergraduate (mathematics) education. Hence it is necessary that we examine the doctrines that govern undergraduate mathematics education in India.

What institutions or courses comprise undergraduate mathematics education? What should be the aims and goals of undergraduate mathematics education? What is the state of undergraduate mathematics education in our country? Are our courses geared to meeting the stated goals and aims? These are some of the questions that we shall cover in the essay. We will also touch briefly on mathematics education at post-graduate and higher levels by looking at programmes and initiatives in India that try to strengthen mathematics at the tertiary level.

Undergraduate mathematics education

Before examining the different types of degrees under which mathematics is taught at the undergraduate level we should dwell on possible goals of undergraduate mathematics education. One would expect a student leaving with an undergraduate degree to have some knowledge about the society she lives in and to have a set of skills that include the ability to communicate, to work in a team and to use modern tools like computers and computer networks. This should be enhanced by mathematical knowledge and skills gained through courses done, and should result in the ability to apply mathematical techniques to analyse, model and solve problems.

A very important goal of undergraduate mathematics education is also to create a pool of students who would continue with studying mathematics at the postgraduate and research levels and consequently also enlarge the pool of people who will form the educators of the next generation.

Are these goals being met? If these goals were met then we would be creating not only a good pool of future mathematicians but also meet the demands of the public and private sectors, business and society. A general statement made by the corporate sector a few years ago was that only a fourth of graduates in India were employable. This is probably true for Mathematics graduates too. A critical review of curriculum and pedagogy at the undergraduate level is therefore essential.

From the past to the present

Mathematics is taught as part of many different undergraduate degrees in India. Mathematics education at the undergraduate level needs to therefore cover the many different programmes in science, engineering, commerce and social sciences. Understandably, the requirements of knowledge of mathematics vary considerably for different programmes and the curricula for respective programmes are framed accordingly.

The most intensive mathematics degree at the undergraduate level is the three or (four) year Honours programme in Mathematics. Typically, in such programmes two-thirds of the courses are from the mathematics discipline and a third would be from disciplines other than mathematics.

Questions that are related and must be explored given the interdisciplinary and multidisciplinary needs of society are: Is it possible to create and run multidisciplinary programmes that are equally intensive with mathematics as one, but not the only, focus of study? Is it possible for mathematics to be part of undergraduate programmes that are not focused on mathematics without such courses being entirely oriented towards utilitarian skills?

The next category consists of the Bachelors in Arts or Bachelors in Science Programmes that offer mathematics as one of two or three disciplines that a student has to study. The weightage of mathematics courses in these programmes would vary from a third to half. Students pursuing physics, economics or four-year engineering degrees at the undergraduate level would also do a fair number of mathematics courses. Mostly, the faculty involved in teaching these courses would be from mathematics departments. Students pursuing undergraduate degrees in 'commerce' (involving courses like commerce, accountancy, etc.) also have some courses in mathematics.

So what was the mathematics curriculum at the undergraduate level like about half a century or so ago: What were the books used by undergraduate students of mathematics? How many undergraduates with a mathematics background were produced then in India?

What kind of careers did they choose? Many of the topics that were taught then would be termed as classical and are no longer part of current curriculum. When did undergraduate mathematics curriculum reform take place in India? How were faculty members trained and prepared to handle the new changes? Particularly, what were the changes that paved the way for Analysis and Abstract Algebra to enter the curriculum and replace courses like astronomy or tensor calculus? We explore answers to these questions based on written replies to questions sent to some senior mathematicians.

Professor M. S. Raghunthan[1] is one of India's pre-eminent mathematicians. We posed questions to him over e-mail on his undergraduate education.

Reproduced below are the questions and answers of Raghunathan (MSR) about his undergraduate education. Some footnotes have been added to clarify and give more information.

Question: In which institution did you do your undergraduate degree and when?

MSR: I got my degree in 1960. I was a student of Vivekananda College in Madras.

Question: What did the undergraduate mathematics curriculum consist of then?

MSR: There was considerable variation in the curriculum in different universities especially at the MSc level. Here is roughly what I was taught in the BA (hons) course of the Madras University. In those days Maths was clubbed with the Arts in Madras. The "honours" course in the Southern Universities were different from those in the North. It was a 3-year course after "Intermediate"[2] while BA was a 2-year course. At the end of the course the Honours students wrote the same exams as the MA students but were awarded the BA Honours degree. It could be converted into an MA[3] a year later by paying a fee! The certificate actually said that the candidate is awarded the MA "by efflux of time"!

In the first year we were taught Differential Calculus a la Joseph Edwards[4]

[1] See http://www.math.tifr.res.in/~dani/msrfr.pdf for more on Professor Raghunathan.

[2] During that period in the South and possibly elsewhere in India, students spent 11 years at school followed by 2 years of 'intermediate' study at a college. The BA (Honours) described above was a three-year programme after intermediate and resulted in the successful candidates being able to convert their BA (Honours) to an MA degree. The BA was a 2-year degree and students wishing to pursue a MA after the BA would have classes with the second year cohort of the of the 3-year BA (Honours).

[3] This is in keeping with the tradition followed at Oxford and Cambridge. Even now at Oxford or Cambridge there are no taught MA courses. Instead a BA Honours can be converted to an MA after paying a fee and after waiting for several terms. This stems from an old tradition in 'Arts' where after a BA you were apprenticed to a Master with whom you honed your skills and hence reaching a new status of 'Master of arts' after a suitable period.

[4] Joseph Edwards, *Differential calculus: with applications and numerous examples*; *An elementary treatise*, Macmillan 1886.

(Ramanujan[5] studied this book; it was no rigorous treatment of the calculus - expanded functions happily in power series without talking about convergence even while different forms of "reminder after *n* terms" were given. It had also some differential geometry of curves in the plane. There was some Trigonometry (the syllabus was the contents of an "advanced" text book by S. L. Loney[6]). There was some algebra - Barnard and Child[7] covered all that we studied but had more. Then there was some Euclidean Geometry which went well beyond Euclid - things like Brocard[8] points, the kind of stuff, later generations do not know much about. In the second year we were introduced to Analysis (now things got rigorous) - Shantinarayan's[9] book essentially covered the syllabus. Hardy's[10] pure mathematics was also used. Then there were (particle) Dynamics and Statics courses through the second year. Loney's[11] advanced level books were used. There was a course on synthetic projective geometry (conic sections), but there was also "analytic geometry" - here the material covered is in the book by Askwith[12] - basically again all about conics, but dealt with through equations. In the final year we had solid geometry - that is three-dimensional "analytic" geometry, more analysis, rigid dynamics (Loney's book again) and some sophisticated stuff like Lagrange's equations and Hamilton's action principle, etc. There was a paper on "astronomy" which was essentially spherical trigonometry. There were "two special" papers which varied from college to college. Our college offered Hydrodynamics and Complex Analysis. There were other colleges that offered "Modern Algebra" in place of Hydrodynamics.

The BA "pass" course offered some of the above material eschewing anything rigorous! S L Loney had textbooks, which were not "advanced" in all the subjects mentioned above and that defined the curriculum for the BA.

In the North, there were places where (local) differential (which really amounted to tensor calculus) was taught and there were other subjects

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[5] See http://www-history.mcs.st-and.ac.uk/Biographies/Ramanujan.html for more on Ramanujan.

[6] S L Loney, *Plane Trigonometry*, Cambridge: At the University Press, 1893 (reprinted now by Michigan Historical Reprint Series)

[7] Barnard S and Child J M, *Higher Algebra*, Macmillan and Co Limited, London 1939.

[8] See http://mathworld.wolfram.com/BrocardPoints.html.

[9] Shanti Narayan, *A course in Mathematical Analysis*, first published in 1949. (A revised edition of this book is now publsished by S. Chand and Co, Delhi, India.)

[10] G H Hardy, *A course in pure mathematics*, first published in 1908 by Cambridge University Press. (Latest Reprint in 2002.)

[11] S L Loney, *Elements of Statics and Dynamics*, Cambridge: At the University Press, 1932.

[12] E H Askwith, *The analytical geometry of conic sections*, *A. and C.* Black, 1908.

like relativity and Electromagnetism that were also offered at the master's level.

Question: How many mathematics graduates would India have produced each year in that era?

MSR: Madras University perhaps produced some 100 to 150 MAs and about 500 BAs - that is a guess but not entirely baseless. There were probably about 20 universities in the country awarding degrees in Mathematics and a dozen offering MA.

Question: What kind of careers did mathematics graduates pursue in that period?

MSR: Teaching of course. A good number from Madras ended up in the accountant general's office. Some became clerks and the bright ones wrote the IAS[13], IPS[14] or Central Services exams. Actuaries was another.

Question: If the mathematics curriculum was very different, when did topics like Abstract Algebra and Real Analysis become part of the undergraduate curriculum in India? Was there any special effort made to train teachers to teach these new subjects?

MSR: I think in the North Real analysis was being taught in the BA classes of some universities. Abstract algebra was taught only at the master's level – if at all – in those days, even in the North. Abstract algebra began to be taught perhaps in the mid seventies.

Professor S. G. Dani,[15] another of India's eminent mathematicians, responded to our similar questions and reported that he completed his graduation in 1967 with physics as his major and mathematics as a minor subject. During his time, mathematics curricula were very narrow. In particular, courses like group theory, complex analysis, linear algebra, basic differential geometry were not included in the undergraduate syllabus. Like Raghunathan, Dani recalls the books by Loney and Shantinarayan as popular ones in those days.

We also get a fair idea of curriculum reform undertaken in the state of Gujarat (located in Western India) through 'History of Curricular reforms in Mathematics' by Professor M. H. Vasavada[16]. Below are excerpts from his piece. This has been produced almost verbatim

[13] Indian Administrative Service

[14] Indian Police Service

[15] Please see http://www.math.tifr.res.in/~dani/ and http://aimconf.webs.com/profiles/S%20G%20Dani.pdf for more on Professor Dani.

[16] After passing the MSc examination, Professor Vasavada first joined V P Science College, Vallabh Vidyanagar, and then the Department of Mathematics, M S University of Baroda, as a lecturer. In 1964, he went to USA under the Fulbright travel grant and joined the University of Wisconsin for his PhD degree. After getting his PhD degree in Functional Analysis under L C Young, he returned to India as a Reader

except for a few minor corrections and a few footnotes that have been added.

> The revision in syllabus came first at PG level and then at UG level. The logic was that the new appointees in the colleges, who got their MSc degree with the revised syllabus, would have already learnt the new material. The lead for PG reforms in Gujarat was taken by M. S. University of Baroda[17]. Dr. U. N. Singh[18], who had a PhD from University of Allahabad and a DSc from Sorbonne (Paris, France), started teaching Measure Theory and Lebesgue integration to MSc students as far back as 1958. Some teachers were also sent to TIFR[19] to learn new subjects like abstract algebra. Then research students under Dr Singh started working in modern branches like operator theory. Also some of the research workers who had gone to USA in the early sixties for their PhD started returning and took up assignments in various university departments. This made the transition from old to new at PG level smooth. By 1970, the PG mathematics departments in universities in Gujarat had revised their syllabi and started teaching new courses.
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> The change at the UG level was slow in coming. There were a large number of colleges and a large number of college teachers. But the summer programmes and the in service programmes for college teachers were organised by university departments with funding from UGC and by 1975, the courses at UG level also were revised. The main subjects taught at UG in the old syllabus were Classical Algebra, Calculus of functions of one and several variables, Real Analysis, Pure Plane Geometry, Analytic Geometry of two and three dimensions, Differential Equations, Statics, Hydrostatics, Dynamics, Astronomy and Electricity and magnetism.

Vasavada's account gives us a detailed picture of the old Syllabus that was taught at both the undergraduate level and the postgraduate level before syllabus reforms came about in 1975 for the undergraduate level and 1970s for the postgraduate level. He attributes the coming of 'modern topics' to the efforts of Professor U. N. Singh.

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in M. S. University. In 1972, he joined the Postgraduate Department of Mathematics of the Sardar Patel University at Vallabh Vidyanagar as a Professor of Mathematics and the Head of the Department and retired in 1996.

[17] See http://www.msubaroda.ac.in/.

[18] "In January 1958, Professor U.N. Singh, D.Sc. (Paris), was appointed as the Professor and Head of the Department. With this advent, the Department became quite active and members were sent for visits/ Ph.D. programme abroad, several summer schools in Modern Mathematics was organized. In fact, the American Mathematical Monthly in an article in 1966 lauded the curriculum of our department as an ideal curriculum. Professor U.N. Singh left the Department for Delhi University in 1966." Excerpt from the write-up on the Department of Mathematics, M S University Baroda.

[19] See http://www.tifr.res.in/index.php/en/ for more on Tata Institute of Fundamental Research.

Undergraduate Level (Old Syllabi)

Algebra: Numbers, Symmetric functions, Theory of Equations (Cubic and Biqudratic equations), Summation of finite and infinite series, Determinants. Book: Higher Algebra by Barnard and Child.
Calculus: Leibnitz rule for nth derivative, Partial differentiation, Singular points, asymptotes, envelopes, curvature, curve tracing, maxima and minima for functions of several variables, reduction formulae for integrals
Books: Calculus by Edwards; Differential Calculus by Shantinarayan
Analysis: Continuous and differentiable functions of one and several variables, series of functions, uniform convergence, Fourier series, Riemann Integration, Mean Value Theorems of differentiation and integration, Double and Triple integrals, Trigonometric, exponential and logarithmic functions
Books: Pure Mathematics by G. H. Hardy, Analysis by E G Phillips, Real Analysis by G S Mahajani, Calculus and Analysis by G K Hebalkar.
Pure Geometry: Projection, cross ratios, Perspectives, Harmonic Section, Involution, Conics
Book: A course of Pure Geometry by E H Askwith
Analytic Geometry of Two Dimensions: Polar Coordinates, General Equation of Second Degree.
Analytic Geometry of Three Dimensions: Lines, planes, conicoids.
Books: Coordinate Solid Geometry – Elementary Treatise by R J T Bell, Analytical Solid Geometry by Shantinarayan.
Differential Equations: Linear differential equations of second and higher order, systems of linear differential equations.
Books: Differential Equations by Daniel Murray.
Statics, Hydrostatics and Dynamics:
Statics and Dynamics of a particle, Elements of Hydrostatics
Books in all the three subjects were by Ramsey
Astronomy
Book: Astronomy by Smart
Electricity and Magnetism

Postgraduate level

Subjects taught:
Algebra: Continued Fractions, Differential Equations, Congruence modulo a number, Fermat's theorem and its Euler's extension, Indeterminate equations, Infinite products.
Book: Higher Algebra by Barnard and Child
Coordinate geometry of three dimensions
Book: Coordinate Geometry of Three Dimensions by R J T Bell
Plane Geometry: Conics, Recirocation, Inversion
Book: Plane Geometry by E H Askwith
Spherical Trigonometry:
Book: Spherical Trigonometry by Todhunter and Leathem
Higher Plane Curves: The study of curves in the plane represented by cubics and higher degree equations
Real Analysis
Complex Analysis
Books: Complex Analysis by E.G.Phillips

Also there were two optional papers, to be chosen from:

Functions of a Complex variable
Statics and Dynamics

Astronomy
Differential Geometry

Professor Dinesh Singh[20], son of U. N. Singh, and himself a mathematician of repute recollected the following information regarding his father's undergraduate and postgraduate years of study. U. N. Singh did his undergraduate degree and an MA in mathematics at Allahabad University in the 1940s. In the MA, U. N. Singh was already using books by Copson[21], Titchmarsh[22] and Hobson[23]. Books on Abstract Algebra by I N Herstein (*Topics in Algebra*, John Wiley and Sons, 1964), Birkhoff and Maclane (*A Survey of modern Algebra*, A. K. Peters, 1977) were beginning to be used in the 60s and 70s respectively. Additionally, books on Analysis by Walter Rudin (*Principles of Mathematical Analysis*, McGraw Hill, 1953) and Royden (*Real Analysis*, Prentice-Hall, 1963) were also being used. The era of 'modern mathematics' seemed to have arrived or had it? The answer to this can only be given when we consider the present syllabi. We do this in the next section.

An analysis of current mathematics curriculum at the undergraduate level

The mathematics curriculum taught in various programmes is presented in this section. This covers a gamut of programmes from those that have over two-thirds of their coursework consisting of mathematics courses to those in which a third or less consists of mathematics courses. The focus will primarily be on high weightage mathematics programmes.

The BA (Bachelor of Arts) and BSc (Bachelor of Science) programmes are of three year duration and generally they are run by different universities. These programmes with Honours or Major in mathematics are aimed at training the student for higher (graduate) studies in mathematics. On the other hand, mathematics curriculum in BA/ BSc programmes without Honours in mathematics (including Honours in another subject) caters to the need of a wide range of courses in sciences and social sciences. Similar is the case for the mathematics curriculum for BCom (Bachelor of Commerce) programmes run by the universities. There are several institutions, which have five-year integrated MSc programmes in mathematics which are aimed to train the student either for higher studies in mathematics leading to research or for industry. The Indian Institutes

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[20] See http://www.du.ac.in/fileadmin/DU/faculty/PDF/2906_02.pdf for more on Professor Dinesh Singh.

[21] See http://www-history.mcs.st-and.ac.uk/Biographies/Copson.html for more on Copson.

[22] See http://www-history.mcs.st-andrews.ac.uk/Biographies/Titchmarsh.html for more on Titchmarsh.

[23] See http://www-history.mcs.st-and.ac.uk/Biographies/Hobson.html for more on Hobson. Incidentally, E. W. Hobson was one of the mathematicians that Ramanujan wrote to in England seeking mathematical advice. This was before Ramanujan wrote his now famous letter to G. H. Hardy.

of Science Education and Research (IISER)[24], National Institute of Science Education and Research (NISER)[25] are examples of such institutions. The mathematics curriculum for these programmes is quite ambitious. This is also the case for the curriculum of the undergraduate programmes of institutes like the Chennai Mathematical Institute (CMI)[26] and the Indian Statistical Institutes (ISI)[27].

The mathematics courses in the undergraduate engineering programmes (of four years duration) conducted by a large number of engineering institutes/ colleges vary substantially across the institutions and according to the requirements of the programmes. For example, the mathematics courses in the programmes of the institutes like the Indian Institutes of Technology[28] and National institutes of Technology[29] are more rigorous than those of many of the state and private engineering colleges which are content with a working knowledge of mathematics.

Curriculum for BA, BSc, four-year BS and five-year Integrated MSc programmes in universities and institutes

Generally the Mathematics courses of both BSc and BA programmes (with Honours/ Major in Mathematics) are the same; the two programmes differ in the nature of the subjects a student chooses from in addition to mathematics, that is, whether from science or social sciences stream.

The BA/ BSc (Honours/ Major in Mathematics) curriculum of most of the universities include the following as compulsory courses:

1. Classical Algebra

Algebra of complex numbers, geometry in complex plane, de Moivre's Theorem and applications, roots of polynomials, Fundamental Theorem of Algebra (statement).

Theory of equations, relations between the roots and coefficients of polynomial equations in one variable, transformation of equations, Descarte's rule of signs, symmetric functions of roots, solution of cubic equation by Cardan's method.

Set, relations and functions, binary operations, Integers, division algorithm, Principle of Mathematical Induction, well ordering of positive integers, Fundamental Theorem of Arithmetic.

2. Linear Algebra

System of linear equations, real matrices, determinants and inverse of a matrix, row reduction and echelon form.

Vector spaces, linear span, linear dependence and independence of vectors, basis and dimension, quotient spaces and its dimension, rank nullity theorem, sums and direct sum of subspaces.

Linear transformations and their representation as matrices, the algebra of linear transformations, the rank nullity theorem, change of basis, dual spaces.

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[24] See http://en.wikipedia.org/wiki/Indian_Institutes_of_Science_Education_and_Research.

[25] See http://www.niser.ac.in/.

[26] See http://www.cmi.ac.in/.

[27] See http://www.isical.ac.in/.

[28] See http://en.wikipedia.org/wiki/Indian_Institutes_of_Technology.

[29] See http://en.wikipedia.org/wiki/National_Institutes_of_Technology.

Eigenvalues and eigenvectors, characteristic equation of a matrix, Cayley Hamilton theorem, minimal polynomial, characteristic and minimal polynomial of linear operators.

3. Calculus

Differential Calculus: Higher order derivatives, Leibniz rule, L'Hopital's rules.

Functions of several variables, level curves and surfaces, limits and continuity, first and higher order partial derivatives, tangent plane, directional derivatives and the gradient, extrema of functions of two variables, method of Lagrange multipliers.

Integral Calculus: Integration techniques, definite integrals, Improper Integrals, applications in finding areas, arc lengths and volumes of revolutions.

Double integral over rectangular and nonrectangular regions, triple integral, change of variables, divergence and curl, line integrals, Fundamental Theorem and path independence, Green's theorem, surface integrals, Stokes' theorem, Divergence theorem.

4. Differential Equations

Ordinary Differential Equations: First order equations, exact differential equations, integrating factors, Bernoulli equations, existence and uniqueness theorem, applications.

Higher-order linear differential equations, solutions of homogeneous and nonhomogeneous equations, method of variation of parameters, operator method; series solutions of linear differential equations, Legendre equation and Legendre polynomials, Bessel equation and Bessel functions of first and second kinds

Systems of first-order equations, phase plane, critical points, stability.

Partial Differential equations: First order partial differential equations; solutions of linear and nonlinear first order PDEs; classification of second-order PDEs; method of characteristics; boundary and initial value problems (Dirichlet and Neumann type) involving wave equation, heat conduction equation, Laplace's equations and solutions by method of separation of variables, initial boundary value problems in non-rectangular coordinates.

5. Analysis

Real line, field and order properties, Completeness property, Archimedean property, density of rationals, open and closed sets, closure, sequence and convergence, Cauchy's criterion, monotone convergence theorem, Bolzano-Weierstrass theorem, limit superior and limit inferior, series and convergence, Cauchy's convergence criterion, test of convergence of series with nonnegative terms, absolute and conditional convergence, alternating series, Leibniz test.

Limits of functions and sequential criterion, continuity, continuous functions on closed intervals, intermediate value theorem, uniform continuity, differentiability, Rolle's theorem and mean value theorems, Taylor's theorem, Taylor's series, Power series, radius of convergence.

Riemann Integral, the fundamental theorem of integral calculus, mean value theorems of integral calculus and applications, improper integrals and their convergence, comparison tests, absolute and conditional convergence, Abel's and Dirichlet's tests, beta and gamma functions.

6. Modern Algebra

Binary operations, groups, subgroups, normal subgroups, Lagrange's theorem, normal subgroups, quotient groups, homomorphism and isomorphism, isomorphism theorems, Cayley's Theorem, inner automorphisms, automorphisms groups, conjugacy relation, normaliser, centre of a group, class equation and Cauchy's theorem, Sylow's theorems and applications.

Rings, Integral domains, fields, subrings, characteristic of a ring, idempotent and nilpotent elements in a ring, principle, prime and maximal ideals, simple rings, definition and examples of vector space and its subspaces.

The above listing contains the topics, which are covered in BSc/BA courses in mathematics of majority of the universities. India being a vast country with many universities and institutes having BSc/BA (Honours or Major) programmes in mathematics, there are a lot of variations in the course curriculum for the programmes. The courses under the above heading in many universities go much wider and deeper.

Apart from the above courses, the programmes usually contain several other courses, some of them as elective/optional. A list of some of these courses would include:

i. Analytic Geometry of two and three dimensions
ii. Complex Analysis
iii. Mechanics – Statics, Dynamics, Hydrostatics
iv. Linear Programming, Optimisation Theory
v. Numerical Analysis
vi. Probability and Statistics
vii. Computer Programming
viii. Discrete Mathematics – Combinatorics, Graph Theory
ix. Number Theory
x. Mathematical Finance

A typical course curriculum for the BA/ BSc (General) and BCom programmes may include courses from:

1. Classical Algebra
2. Calculus
3. Analytical Geometry of two and three dimensions
4. Differential Equations
5. Modern Algebra
6. Numerical Methods
7. Linear Programming
8. Probability & Statistics
9. Computer Science & Programming
10. Discrete Mathematics
11. Mechanics

Normally, these programmes have five to nine courses in mathematics covering many of the topics listed above.

Several institutions, for example, Indian Institute of Science, Bangalore (IISc)[30], University of Hyderabad[31], IISERs, NISERs, IIT Bombay, and CMI, have four-year BS/ five-year Integrated MSc programmes which offer courses on Topology, Algebraic Topology, Manifolds, Functional Analysis, Galois Theory, Harmonic Analysis, Lie Groups, Fourier Analysis, Homological Algebra and Commutative Algebra at higher stages of their programmes, apart from basic courses from the above lists.

The University Grants Commission (UGC)[32] is the regulatory body of the Government of India that looks after the university education system of India. The UGC is invested with maintaining the quality of education imparted in Universities. To this end, it suggests uniform curricula for different programmes run by the universities. It carries out exercises from time to time for renewing and updating its model curricula. While the wisdom of having a uniform curriculum for all universities and institutes with various levels of resource abilities and requirements is debatable, the efforts of UGC work as an impetus for the universities to reflect on and review their respective curricula. The UGC, in an effort to improve standards decided on creating a model curriculum for the mathematics taught at the undergraduate level. The UGC recommended courses for BA/ BSc (Honours) in its Model Curriculum[33] of 2001 are as follows:

1. Algebra and Trigonometry: includes matrices, systems of linear equations, theory of equations; introductions to groups and rings; De Moivre's Theorem and its applications in trigonometry, etc.
2. Calculus: includes differential & integral calculus, ordinary differential equations of first and second order.
3. Vector Analysis and Geometry: includes Theorems of Gauss, Green and Stoke, and analytic geometry (conics and second degree equations, plane, sphere, cone, cylinder, conicoids, etc.).
4. Advanced Calculus: includes convergence of real sequence and series, continuity and uniform continuity, differentiability, mean value theorems, Taylor's Theorem; continuity of functions of two variables, partial derivatives, extrema of functions, etc.
5. Differential Equations: includes ordinary and partial differential equations, calculus of variations, variational problems with moving boundaries, etc.
6. Mechanics: includes statics and dynamics.
7. Analysis: includes Real Analysis (Riemann integral, improper integral; series of arbitrary terms, double series; partial derivatives, Schwarz and Young Theorems, implicit function theorem; Fourier series) and Complex Analysis (continuity, differentiability, analytic functions, conformal mappings, etc.) and Metric spaces.
8. Abstract Algebra: includes groups, rings, vector spaces (covering also inner products, orthogonalisation, etc.) and modules.

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[30] See http://www.iisc.ernet.in/ug/about.htm.

[31] See http://www.uohyd.ac.in/.

[32] See http://www.ugc.ac.in/ for more on the UGC.

[33] See http://www.ugc.ac.in/policy/maths.pdf for details.

9. Programming in C and Numerical Analysis: includes numerical integration and approximation; Monte Carlo Methods.
10. Probability Theory and Optimisation.

Each of the above was suggested as the topics for two courses to be spread over two semesters. Apart from these compulsory courses, several optional courses are suggested out of which a student is to select two. These are:

i. Principles of Computer Science
ii. Differential Geometry
iii. Discrete mathematics
iv. Mechanics (Dynamics of rigid bodies, hydrostatics)
v. Mathematical Modeling
vi. Application of Mathematics in Finance and Insurance
vii. Special Theory of Relativity
viii. Elementary Number Theory
ix. Combinatorial Number Theory
x. Computational Mathematics Laboratory

For BA/ BSc (General) programmes, UGC recommends compulsory courses as in (1-8) of the above list and the rest as optional courses along with courses chosen from (i-x) above.

The model curriculum document of the UGC also gives a comprehensive set of references for each of the courses. By and large, these are a mix of modern books some catering exclusively to an Indian audience while some are books that are used world-wide. However, an intriguing fact that one notices when one looks at the recommended books in detail is that some of the books followed in the 1960s during Raghunathan's undergraduate days like books on trigonometry, statics and dynamics by S. L. Loney and Shanti Narayan's Mathematical Analysis are still in use. So after more than 50 years, there is a lot of change in many quarters but none in some. The scenario clearly indicates that some core areas such as Calculus, Basic Algebra and Geometry have necessarily been part of the curriculum, irrespective of the time frame, and some old classics continue to prove their utility as quality texts. On the other hand, some new titles such as Contemporary Abstract Algebra by Gallian, Abstract Algebra by Dummit and Foote, Linear Algebra: a Geometric Approach and Topology of Metric Spaces both by S. Kumaresan, Real analysis by Carothers, etc., have proved their merit as textbooks for the respective courses.

The main reform in syllabi seems to have taken place in the 70s all across India. Since then however, the undergraduate curriculum in mathematics in most of the universities has not undergone any paradigm shift, either in approach or in contents. Many of the courses cover topics in both width and depth, but they are too compartmentalised. The relevance of the courses to other branches of science, technology or social sciences is not emphasised or demonstrated. Not only do they seem to be isolated bundles of knowledge far away from other areas, but they also lack in interactions among themselves. Therefore, the curriculum does not equip the student with applicability of mathematics in the scenario of modern scientific and technical developments.

Even the UGC recommendations seem to have failed in giving leadership in adapting to changing requirements in mathematics education at the undergraduate level in terms of applicability of mathematics on the one hand and the role of technology in mathematics education on the other. It is conspicuous that UGC's recommendation places the more applicable courses, namely, Programming in C, Numerical Analysis, Probability Theory and Optimisation as optional courses for BSc (General), whereas it retains all classical courses as compulsory.

Moreover, the use of Information Technology (IT) in mathematics seems to have by-passed the vast majority of such programmes across Universities in India. Even teaching the use of spreadsheet programmes, Computer Algebra Systems (CAS), etc., to aid in understanding and visualising mathematics, developing good programming skills to help model and analyse mathematics is certainly not part of the mainstream of undergraduate mathematics curriculum in India.

There are a few universities that have tried to buck this trend. For example, University of Delhi in its undergraduate mathematics curriculum has laid emphasis on applications of mathematics through mathematical modeling and use of tools like Matlab, Mathematica and Maple for studying different courses and also through the use of newer books that try to integrate the pure and applicable side of mathematics. Some of the newer Universities like Ambedkar University, Delhi (AUD)[34] and Shiv Nadar University (SNU)[35] are trying to create undergraduate mathematics curricula and use teaching methods that integrate applicability, help foster team work and give all students an opportunity to appreciate both 'pure mathematics' and 'applications'. Further, communication and presentation skills, computational skills, as well as linkages to other disciplines are also explored. There may be many more such initiatives that are not listed here. It would be interesting in the years to come to see if they succeed in filling the lacuna in the current programmes.

[34] See www.aud.ac.in.

[35] See snu.edu.in.

A brief analysis of current mathematics pedagogy at the undergraduate level

Probably, among all the teaching and learning aspects in India, the scenario of mathematics education at the undergraduate level is most alarming. In most of the colleges, 'teaching' means merely demonstrating the content, usually by 'stating and proving a theorem', and 'learning' is the same as the ability to reproduce such proofs. There are minimal interactions in the classroom with students taking a passive role. Students are seldom trained to work things out themselves. The objectives of teaching become merely to prepare the student for assessments, where she is expected to reproduce after rehearsing whatever the teacher has demonstrated. Classroom teaching and student's learning are reduced merely to preparation of the student for performance in such assessments. There is a limited scope of development of the mathematical abilities of the student. This in turn affects the very objective of preparing the student to meet the challenges in her future pursuit of higher studies or participation in industry and society.

Such teaching soon becomes counter-productive. Through the programme, the student is neither prepared with mathematical maturity for the pursuit of higher studies, nor trained with mathematical know-how for being capable of applying mathematics in other fields. On the contrary, intuition and original creativity of the student is lost in the mechanical teaching-learning process. The performance of students at national level tests like JAM[36], TIFR entrance, PG entrance tests at various universities, or for national level Postgraduate scholarships clearly reveals the fact that the teaching-learning processes at undergraduate level requires much more attention.

Assessment in the undergraduate programmes of universities is largely limited to summative assessments through semester end or annual examinations conducted centrally by the universities. Some of the universities allow limited formative assessment in the form of internal assessment at the college level. However, most research institutions have both formative and summative assessments for their programmes.

Usually, the university annual and semester end examinations assess mainly students' ability to reproduce the material provided in the text books, almost in a 'state and prove' fashion. In many of the universities, the process becomes further unchallenging because of the fact that the same set of questions are repeated in the examinations almost routinely over the years. The result is devastating. Often even a student scoring very high marks is unable to answer simple questions like giving definitions, examples or counter-examples. No easy solution exists but no solution will exist without well-trained faculty equipped to use teaching and assessment methods that will help meet the goals that every undergraduate degree should have.

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[36] See http://gate.iitkgp.ac.in/ for more details regarding JAM.

Given the scenario described above it is certainly a foregone conclusion that well trained faculty just does not exist in the proportions required. Though this is true for most parts of the country, the situation in some regions is much more alarming. For example, the entire North-East region is facing acute shortage of qualified teachers at all levels.

So how do we begin to create a pool of well-qualified, motivated faculty? We should first investigate what pre-service qualifications are required in order to be able to teach at the undergraduate level. In-service training, teaching methods that are broadly prevalent also need to be considered. The means and methods prescribed by UGC to improve quality of teachers and teaching at the undergraduate level are also analysed here.

Pre-service qualifications and training for teachers at the Undergraduate Level

The University Grants Commission (UGC) has the role of maintaining the quality of university education and sets the qualifications for faculty to be able to teach mathematics in the universities (public or private). The minimum eligibility for teaching mathematics at the undergraduate level and for a job as an Assistant Professor in post-graduate University departments, as prescribed by UGC, is a Masters in Mathematics with at least 55% marks and a National/State Eligibility Test (NET[37]/SET) certificate in Mathematics or allied subjects.

It is interesting to note that the NET was recommended as a solution in 1983 by a UGC committee in order to battle dropping standards amongst the teaching fraternity in higher education. The Committee under the chairpersonship of Professor RC Mehrotra was instituted by the UGC to review pay scales of teachers at Universities and Colleges and recommended the following[38] for the post of lecturer (Assistant Professor):

> i) Qualifying at the National test conducted for the purpose by UGC or any other agency approved by UGC.
> ii) Master's degree with at least 55 % marks or its equivalent grade.
> The qualifications should not be relaxed even for candidates possessing M.Phil/Ph.D. at the time of recruitment.

UGC and allied agencies have been conducting NET since 1989, but PhDs are now exempted from NET/ SET. At the same time the UGC has also stipulated rules regarding PhD degrees in order to improve the quality of PhDs.

The Committee in 1983 in fact went on to say that "the stipulation of M.Phil/Ph.D as an essential qualification for Lecturers had neither been followed faithfully nor did it necessarily contribute to the raising of teaching and research standards. In fact, it was of the view that, if at all, it had led to the dilution of research standards on account of the

[37] See http://www.ugc.ac.in/inside/net.html for more details regarding NET.
[38] Taken from http://www.ugc.ac.in/inside/net.html _review%20_mungekar.pdf.

rush to get a research degree in the shortest possible time".

The NET examination does work as a filtration system, however it has not succeeded in improving the quality of the entrants into the teaching profession at Universities. The quality of MPhils and PhDs is also something to worry about. The irony now is that there are coaching classes catering to candidates who wish to clear NET.

Most departments catering to post-graduate degrees would have faculty having a higher qualification than Masters and NET. Another point to be kept in mind is that the faculty involved in teaching at the undergraduate level in mathematics is usually separate from those involved in teaching mathematics at the post-graduate levels. The result is that the pre-service qualification of faculty involved in teaching mathematics at the undergraduate level is usually just Masters and NET or at most an MPhil degree in mathematics or a related field. Most faculty members involved in teaching at the undergraduate level have very limited exposure to research in mathematics. While it is possible for faculty members to improve their qualifications while in service, the majority of faculty members do not do so as they do not see any incentive in doing so. The only in-service requirement is for faculty to do two or three 3-week refresher courses. This is usually a mandatory requirement for faculty of public universities for purposes of career advancement or promotions. The quality of such courses is very variable, most serve to just provide a certificate to aid promotional aspects and do not in anyway 'refresh' the faculty.

Unless there are in-service incentives and disincentives in place that require faculty members to improve their qualifications one does not expect any radical change in the knowledge reposited with faculty or their own experience in terms of research. It is also unlikely that the assessment patterns or teaching methods will change without incentives-disincentives built in at the policy level.

Apart from improving qualifications which is crucial, faculty also need to be aware of how Information Technology (IT) can be used effectively to motivate and attract students. Being able to visualise and simulate mathematics using computers and software can add substantially to the learning experience provided that it is done with care. Programming skills would also enhance the ability of the student to analyse mathematical problems, model problems, and search for patterns. All these would then help to formulate solutions and could even lead to new research. This way of integrating the power of IT with mathematics would certainly require in-service training of existing faculty. There are also a large number of textbooks written for undergraduate students that look at teaching and learning of mathematics in a new way. These books are easy to read, link mathematical results with applications in the real world and also have a large number of problems and projects that aid in understanding the deeper concepts. They also invariably have net-based resources to help student visualise where possible and provide students with situations that aid hands-on investigation of the concepts involved through IT.

Assessment plays multiple roles: guiding the teacher on the manner in which students have learnt what has been taught, guiding the student on the extent to which she is making progress and guiding a future teacher and/ or employer on what knowledge and skills have been acquired. In the Indian context, the last role, with an emphasis on marks and grades, tends to become the primary focus for students as subsequent admissions or employment seem to be directly dependent on these. Since students place such premium on doing well in assessments, it should be turned into a vehicle that actually covers all aspects. It should be possible to create assessment scenarios, which make sure that grades and marks are linked to actual learning and ability to apply the concepts learned. Specifically, assessments should be used to guide a multi-tier/multi-stream approach to undergraduate education without attaching a stigma of failure to those in the slower streams (or a misplaced sense of achievement among students placed on faster tracks!).

Strengthening tertiary mathematics

At the tertiary level, there are several innovative programmes that impart training in mathematics. These help to strengthen the formal system learning mathematics at colleges and universities. Many of these activities are possible due to the support, financial and otherwise received from the National Board for Higher Mathematics (NBHM)[39].

The NBHM was set up by the Government of India under the Department of Atomic Energy (DAE), in the year 1983, to foster the development of higher mathematics in the country, to formulate policies for the development of mathematics, help in the establishment and development of mathematical centres and give financial assistance to research projects and to doctoral and postdoctoral scholars. NBHM functions essentially autonomously framing its own budget taking into account the funds made available by DAE.

The role of NBHM in improving mathematics at the tertiary level and other important activities that have helped students and faculty are considered here. These include, the efforts made under Mathematics Olympiads training, the Madhava Mathematics Competition, Mathematics Training and Talent Search (MTTS) Programme and Advanced Training in Mathematics (ATM) Schools. All of these activities are funded by the NBHM.

The aims and objectives of these activities, their salient features, the numbers covered through these activities, the impact these activities have had over a period of time are presented in this section.

[39] See http://www.nbhm.dae.gov.in/.

Mathematics Olympiads

Mathematical competitions have been held in India for a reasonably long time. Various organisations in different regions have been conducting competitions for school children on their own initiative. After the constitution of the NBHM, all these competitions were given a national coordination and the Indian National Mathematical Olympiad (INMO)[40] was started in 1986. Simultaneously, India also started preparation to send teams to the International Mathematical Olympiad (IMO)[41]. It sent its first team in 1989 to Germany and since then India has been participating consistently in IMO.

The foremost aim of the olympiad is to find mathematically gifted students among the huge population of India. Many talented students in our country are not even aware of their interest in mathematics. Mathematical Olympiad is aimed at spotting these talented children and aims to nurture their talent so that they can pursue a career in mathematics. In the backdrop of the peculiar socio-economic situation in India where professional courses are held in high esteem, olympiads help children find their real interest and some counseling helps these children to pursue mathematics. However, the Mathematical Olympiad also helps to find really gifted children who can compete with other children of the same age group from different parts of the world through IMO. The selection of a team to represent India in IMO is also the objective, but the primary objective is to nurture talent.

Since its participation in IMO in 1989, Indian contestants have been doing reasonably well in the IMO. So far 136 students have contested in IMOs during the last 23 years. Among them, nine students have won gold medals; fifty-five students have won silver; and fifty-one students have won bronze medals.

The main impact of the Mathematical Olympiad in India is the awareness it has brought. Students interested in mathematics find that a career in mathematics is not the last option. They learn that there are good institutions in India where they can study higher mathematics leading to very good career options in academic and research institutions, research establishments and industries. A large number of students who write INMO take up mathematics in their undergraduate studies. NBHM provides financial assistance to those who would like to pursue mathematics in India. Several students have also taken up allied areas like computer science and are pursuing theoretical problems in computer applications. Quite a few of the medalists have finished their doctorate and are now in good academic positions. Mathematical Olympiads have also helped in raising the standard at the school level. There are several teacher-training programmes to equip teachers in problem solving. These help teachers and in turn their students. With more awareness among the students, teachers too have to do their ‘homework properly’ and

[40] See http://en.wikipedia.org/wiki/Indian_National_Mathematical_Olympiad.
[41] See http://www.imo-official.org/.

prepare better. It is definitely the case that such activities increase the mathematical reasoning of the children.

Madhava Mathematics Competition

The success of Mathematics Olympiads provided a strong motivation for having a similar competition at the undergraduate level. The purpose of having a Mathematics Competition at UG level is multifold:

1. The competition would cultivate a culture of 'problem-solving' among the UG students.
2. The competition would provide a motivation for the interested students and teachers to go beyond syllabi and work on more challenging problems.
3. The competition would help in identifying the better undergraduate students and in turn would allow us to design the mechanism of nurturing them in a more systematic way.
4. The competition would generate interest and enthusiasm among the students and could help in attracting good students to Mathematics.
5. The participating institutions would be linked through the competition and a possibility of a meaningful interaction between them would evolve.

The competition is still in its early stages but it is rapidly growing as a national level competition for math undergraduates in India. The Centre for Postgraduate Studies in Mathematics, S. P. College[42], Pune organises the competition jointly with the Homi Bhabha Centre for Science Education, Mumbai.

A distinctive feature of the competition is organisation of a nurture camp for the prize winners. The camp is organised in the month of May at Bhaskaracharya Pratishthana[43], Pune. Though at present the competition is conducted in some parts of the country, it has been planned to hold it countrywide in near future.

Mathematics Training and Talent Search (MTTS) programme[44]

Mathematics Training and Talent Search Programme (MTTS) is a national level four weeks intensive summer training programme in mathematics. It has been being organised in every summer for the last 19 years (since 1993) in India. This programme was conceived and has been directed since its inception by Professor S. Kumaresan[45], now at

[42] See http://www.spcollegepune.ac.in/newsite/.
[43] See http://www.bprim.org/.
[44] See http://www.mtts.org.in/.
[45] See http://mathstat.uohyd.ernet.in/people/kumaresan/More-about-Kumaresan for more about Professor Kumaresan.

the University of Hyderabad. It is one of the most significant and successful mathematics training programmes and has made an impressive impact on mathematical scene in India over the years, especially at undergraduate and post-graduate levels. Each year about 180-190 talented students selected from all over India, undergo this training programme at three different levels at several centres across the country.

The programme aims to teach mathematics in an interactive way and to develop independent thinking in mathematics among the participants. To promote active learning, the teachers usually ask questions and try to develop the theory based on the answers and typical examples. At every level the participants are encouraged to explore, guess and formulate definitions and results. Moreover, the programme provides a platform for the talented students so that they can interact with their peers and experts in the field. This serves two purposes: i) the participants come to know where they stand academically and what they have to do to bring out their full potential and ii) they establish a rapport with other participants and teachers which help them shape their career in mathematics.

The MTTS Programme has made some significant impact in the scenario of mathematics education in India, especially at the Post-Graduate level. Out of a total of more than 2700 participants, about 300 students pursued higher studies. Many of the participants of the programme, who at present are engaged in teaching and research in the premier institutions of the country, acknowledge that their attitude towards mathematics was transformed by the programme. Apart from its contribution in research, the programme has also produced some good teachers for school and college education, though the number is small relative to the requirement for the country.

The success of MTTS has resulted in the organising of a similar training programme for teachers titled Pedagogical Training for Mathematics Teachers (PTMT) from the year 2012.

Advanced Training in Mathematics (ATM)[46]

There are a large number of doctoral students, postdoctoral fellows and faculty members in universities and various institutions of higher learning for mathematics in India. It was felt that there is a need for programmes that will help such scholars improve their knowledge base, and would also strengthen and broaden areas of research that this body of scholars can engage in. To this end the NBHM launched ATM Schools in the inter-related areas of algebra, analysis, partial differential equations, discrete mathematics, geometry, number theory and topology. The aim of this integrated training programme is to provide basic and advanced knowledge in these areas and emphasise their interconnections via a series of instructional schools.

A highlight of the instructional school is a series of lectures for a week by an eminent

[46] See http://www.atmschools.org/.

mathematician under the title *Unity of Mathematics Lectures*. These instructional schools are also held at different levels for different target audiences. Of particular importance are instructional schools devoted to imparting training to college and university teachers. So far over 750 college teachers and about 3000 PhD and MSc students have been beneficiaries of the training under ATM.

Conclusion

There are about 400 Universities and 18000 colleges (including Engineering and Polytechnics) in India where the teaching and learning of mathematics takes place. Out of around 2,000,000 students enrolled for undergraduate courses, about 400,000 students enroll for post-graduate courses. The estimated number of students pursuing post-graduation in mathematics is around 25,000. The number of students pursuing MPhil or PhD in mathematics is in the range of 800-1000 and there are about 30,000 teachers working at the undergraduate or post-graduate levels.

While the numbers involved are large, these make up only a tiny fraction of the populace. It is clear that for a country that is making great strides in many fields, the lack of an educated work-force will prove to be a huge speed-breaker.

An analysis of the past and the present shows us several things that India can be rightly proud of but at the same time cautions against any complacence. It is clear that the mathematics in the undergraduate mathematics curriculum has broadly kept up with international standards. Even the model curriculum of the UGC does have plenty of courses from what would be termed as 'modern mathematics'. However, the teaching-learning process at the undergraduate level is not even meeting what should be its minimum goals.

It is true that the average undergraduate mathematics student now has access to far more books, information and access to computers and computer networks. They also have a reasonably good curriculum to study from. The average qualification of a faculty member teaching at the undergraduate level is better than what was the case several decades ago. More women students are doing mathematics than ever before. In urban centres, half the mathematics class is usually women and this ratio improves further in taught post-graduate courses in mathematics. These are all positives that we can be justifiably happy about.

However given the decade we are currently in and the growing needs of our society and the needs of the discipline itself, unless we take strong ameliorative steps the rate at which we are improving is just not going to be enough. If we take a closer look we actually see the many gaps and lacunae that require immediate healing. There is a requirement to both work out long-term strategies and at the same time to also have good achievable short-term goals. Given the diversity and size of India there have to be a multiplicity of approaches rather than a single quick fix.

To sum up, the curriculum in most of the high weightage undergraduate mathematics programmes seem to be focused on fast-tracking young men and women to be research mathematicians. On average however much less than a fourth of undergraduate mathematics students actually decide to pursue an academic career in mathematics. Further the pedagogy and assessment patterns followed actually do not do much to foster or enhance the ability to think originally or to critically analyse and solve unseen questions. Thus on average the undergraduate programmes in mathematics fail in at least two important ways: one they are not really equipping and training the minority that plan to take up a career in mathematics in the manner they should; two, the majority are neither gaining any understanding of the role of mathematics in society nor are they learning the skills required by all in terms of communication, presentation, or the use of modern computer technology.

A solution to this is certainly possible. On the curricular front we need to create a syllabi that through its content, recommended books and resource material would make learning mathematics meaningful in more ways than one. Improved qualifications, focused in-service training for faculty particularly in terms of familiarity with programming and use of mathematical software, improved infra-structure, and well conceived schemes of both incentives and disincentives can create a pool of faculty members who are equipped to use innovative teaching methods to impart the curriculum. Further the schemes for strengthening tertiary mathematics need to be scaled up and need to spread to smaller towns and rural districts. Special attention also needs to be given to attracting more students and also more women students to research.

The existing hierarchies in education have created compartmentalised discrete structures that mitigate against continuous flow of information and ideas between different levels of mathematics. There also seems to be almost no data capturing the state of undergraduate mathematics education. There is no significant research undertaken about undergraduate or tertiary mathematics education. By and large the community of mathematicians and mathematics educators in India seem to inhabit separate worlds. This too needs to change. Improvement just at the undergraduate or tertiary level is not enough. The entire community needs to focus on improving mathematics education at all levels. Seminars, conferences and research can go a long way in creating the necessary paths that lead to a better understanding of the problems. It will also help in framing policy that will hopefully pave the way and provide the right setting for the solutions to take root.

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8. The preparation and professional development of mathematics teachers

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Context and background

The need for adequate preparation and professional development of teachers has been recognised the world over with the realisation that the teacher' conceptions and attitudes play an important role in the teaching learning process. This is reflected in Sowder's (2007) comment that " recognition of the need to change the way in which mathematics is taught and learned is international in scope". In India, the importance of the teacher was recognised by the Education Commission as early as 1964-66 which observed that "of all the factors that influence quality of education... the quality, competence and character of teachers is undoubtedly the most significant". Research evidence from other countries indicates that professional preparation of teachers is significantly related to students' achievement (National Mathematics Advisory Panel, 2008). Moreover, the recent reports of the international TEDS-M study indicate that rigorous maths instruction in schools and demanding university teacher preparation programs in countries like Taiwan and Singapore accounts for their teachers having better knowledge of mathematics and its teaching. (Tatto et al., 2012).

In this article, we will discuss the need for the preparation of mathematics teachers in India, the institutional arrangements that exist, the context that guides the priorities, efforts that have been made, the challenges that continue to limit gains, and possible ways forward. The discussion will largely focus on the mathematics teachers at the school level. There is recognition of the need for some kind of specialised training and preparation of teachers at the tertiary level of education beyond the regular university education. However, virtually no institutions or models exist to directly address the need for better mathematics education at the tertiary level, beyond the periodic "refresher courses" that undergraduate teachers are required to attend, which largely focus on enriching teachers' knowledge of the subject rather than on issues of teaching or learning[1] (See Chapter 7, this volume).

In India efforts have been on continuously over the decades to expand access to schooling for a rapidly growing population. As a consequence, the teacher education system has also expanded vastly but unevenly with some states still having inadequate infrastructure to train teachers. Of around 5.3 million "regular" teachers (i.e. excluding "para teachers") at the elementary level, roughly 80% have a teacher training qualification (Mehta, 2011). The pre service training is typically for elementary school for two years, and for secondary school for one year (often amounting only to 6-7 months of instruction). The curriculum and instruction time available do not provide enough opportunity for the student teachers to reflect on their experiences and prepare them to face the ground realities of teaching in a school (National Council of Teacher Education [NCTE], 2009). This adds to the pressure to provide in service education to more than 5.5 million teachers at the elementary level alone at regular intervals while teacher education institutes grapple with "lack of resources, infrastructure, training materials and professional expertise" (Walia, 2004). There is a need to recruit even more teachers as around 30 million children are still estimated to be out of school.

Recent years have seen the launching of a vigorous effort to universalise elementary education through strong legislation. The pressure created by the Right to Education (RTE) Act implemented in 2010, has led to the realisation that in some parts of the country, vastly many more teachers are needed than are employed at present, and the institutional infrastructure for teacher education in these regions needs to be rapidly expanded. This has created a situation where attention is focused on the urgent importance of teacher preparation. At the same time, the sheer magnitude of effort needed increases the pressure for short-term, patch-up measures which may weaken the system of teacher preparation in the long run. There is also a blurring of the distinction between in service and pre service teacher education. In several states, teachers without specialised teacher qualifications already teach in schools, but are now required by the RTE Act to obtain a diploma or degree in education within a specified time period. While this puts additional pressure on the system, it also creates sites where teacher students interact intensively with practicising teachers in an academic setting.

Besides the RTE Act, the other major contextual factor is the new National Curriculum Framework (National Council for Educational Research and Training [NCERT], 2005), abbreviated henceforth as "NCF 2005". The NCF 2005 advocates a shift away from a textbook centred rote learning approach, to one that emphasises the link between school

[1] Building on the success of the decades long MTTS programme (See Venkataraman, Sholapurkar & Sarma, this volume) the Pedagogical Training for Mathematics Teachers (PTMT) programme has been launched in 2012 under the MTTS umbrella. It is aimed at providing a national platform for teachers at the Undergraduate level to improve teaching methods and share pedagogical insights.

learning and life outside school. It stresses that the knowledge that students bring to the classroom from their life outside, and the diversity of ability and ways of thinking within the classroom are resources for teaching and learning and not hindrances. Specifically with regard to mathematics, it gives precedence to the goal of mathematical thinking or mathematisation, rather than "knowing mathematics" as a set of rules and facts. Clarity of thought, pursuing assumptions to logical conclusions, the ability to handle abstractions, problem solving are what are considered central to mathematics and worthwhile aims of mathematics teaching and learning (NCERT, 2006a).

Although ideas such as child-centred learning are not new, NCF 2005 has been effective in changing the discourse on education in a system-wide manner. Teachers are now more open to the idea that their teaching approach needs to undergo fundamental change. However, there is very little clarity about what this change really amounts to in terms of classroom teaching and learning, and schools and teachers look for help as they try to interpret the message of the new curriculum framework. In terms of implementation of NCF 2005, besides a significant change in the textbooks, administrators of major school systems have tried to implement reform measures through directives and circulars. However, it is widely acknowledged that in order to support change in classroom teaching there has to be system-wide preparation involving not only teachers, but also other stakeholders like administrators, principals, education officers as well as parents. This situation creates a potential for change as well as a challenge by way of designing in service teacher professional development that addresses teachers' needs to comprehend the vision of teaching and learning as articulated in the NCF.

Policy perspectives relevant to mathematics Teacher education

Vision of mathematics and its teaching

While the importance of mathematics as a subject for the elementary school has been felt for a long time in Indian education, the issues of why mathematics and of what mathematics to teach have been contentious. The Nai Talim, Gandhiji's influential perspective document on education, emphasised language and mathematics as the basic core of the school program (Sykes, 1988). In independent India, the report of the landmark Education Commission of 1964-66 (also known as the Kothari Commission) emphasised mathematics as essential for national development since education in science and engineering was dependent on mathematics. The commission thus made mathematics compulsory up to Grade 10. It recognised that teaching through lectures was prevalent in most science and mathematics classrooms and recommended emphasis on developing understanding of basic principles rather than "mechanical teaching of mathematical computations" (Government of India [GOI], 1966, Ch. 8, Sec. 8.66). Recognising the importance of subject knowledge the commission recommended “20%” of teacher training time to be devoted to developing adequate knowledge of subject matter and for relating it to methods and materials of teaching (GOI, 1966, Ch. 4, Sec. 4.14). For secondary teachers this was to be done in collaboration with university departments.

The National policy on Education (GOI, 1986), which was the major landmark document after the Kothari commission, and its subsequent revisions also emphasised mathematics but the focus in these was to develop the capability of using mathematics in daily life and in applications in other areas. The understanding of mathematics teaching for improving its everyday application and the capability to handle mathematical aspects in other subjects of study were the core concerns in National Curriculum Framework for School education (NCERT, 2000), which also emphasised the need to develop capability of doing mathematical calculations.

The NCF 2005 made a break from this and emphasised developing the capability to abstract, use and understand logical forms, grasp ideas and discover, create as well as appreciate patterns. The idea of mathematisation and giving learners the space to discover the way mathematics functions was an important change in the NCF 2005 formulation. It also urged focus on developing concepts and learners’ own ways of solving problems and building new algorithms rather than remembering short cuts and efficient ways to calculate.

Recommendations about teacher education in various documents

Recognising the importance of teachers in improving the quality of education, the Kothari commission (1964-66) recommended “securing a sufficient supply of high quality recruits to the teaching profession” by increasing the status of teachers, “providing them with the best possible professional preparation” and “creating satisfactory conditions of work” (GOI, 1966, Ch 3, Sec 3.01). To improve teacher education in the country the commission recommended professionalisation of teacher education and urged that isolation of teacher education institutes from university life, from schools and from one another be removed. It recommended reorganisation of teacher education programmes at all levels, including the reorientation of subject knowledge and improvement in methods of teaching and evaluation. It recognised problems in teacher preparation programmes like set pattern and rigid techniques for practice teaching done for a few isolated lessons, which was unsupervised or ill supervised. Therefore it recommended that student teachers should be oriented in the first phase to teaching and working of schools through observations and teaching individuals and groups before teaching the whole class. In the second phase, student teachers should be involved in “block teaching” (teaching continuously) for a period of 2 to 6 weeks. It recommended increasing the duration of teacher education programmes at primary and secondary levels from 1 to 2 years to allow deep study of fundamental concepts in the subject matter.

The Kothari Commission recognised the need for the continuing professional education of teachers and called for "the organisation of a large scale, systematic and coordinated programme of in service education, so that every teacher would be able to receive at least two or three months of in service education in every five years of service" (GOI, 1966, Ch 4, para 4.56). It recommended that continuing in service education be based on research inputs.

The National Commission on Teachers (GOI, 1983- 85) advocated a 4-year integrated course after 12th grade that combined a university degree in a subject with a teacher qualification, having at least 4 weeks of internship in the fourth year. For sharing human and material resources for in service professional development of teachers, it recommended establishing school complexes, which would include schools within the radius of 5-10 miles having 1-2 higher secondary schools, 6-7 middle schools and 30-35 primary schools. The commission advocated that desirable competencies of the teacher for recruitment should be on the basis of practical research. While acknowledging the woeful inadequacy of in service education, the commission recommended that classroom and practical needs of teachers should be identified by surveys and studies. The programs should be announced well in advance and feedback from schools and teachers should be taken after in service courses. Resource persons for teacher professional development were recommended to be from diverse backgrounds – university professors, people from industry and agriculture and practising teachers and supervisors. The in service course should be in the workshop mode where materials are developed which teachers take with them for use in classrooms. The commission noted that what teachers need most "is a change in the climate of schools, an atmosphere conducive to educational research and enquiry".

The New Education Policy of 1986 recommended a rapid expansion of the infrastructure for education of teachers at the elementary level through the setting up of institutions at the district and block levels, which would deal with both pre service and in service teacher education (GOI, 1986a). NPE 1986 attempted to break the separation between pre and in service teacher education by considering both as phases of a continuous process thus acknowledging the need for career long professional development of teachers. It recommended that mathematics teaching should be focused on analysis and reasoning and enable use of technological devices for analysing cause effect through interplay of variables (GOI, 1986a, para 8.17)

The Acharya Ramamurthy committee in 1990 (GOI, 1990) emphasised the role of actual field experience during internship to foster professional growth of teachers. The Committee explicitly stated that "in service and refresher courses should be related to the specific needs of the teachers. in service education should take due care of the future needs of teacher growth; evaluation and follow up should be part of the scheme" (as cited in NCERT, 2006c, pp. 4) It recommended adoption of "internship model" by having brief theoretical orientation followed by 3-5 year supervised teaching under mentors.

The Yashpal committee report titled "Learning without burden" (GOI, 1993), which had a major impact on the revision of the school curriculum, recommended restructuring of the course content of teacher education to serve the changing needs of school education and making teacher education more practice oriented. The National curriculum framework 2005, which attempted to implement the recommendations of the "Learning without burden" report in a systemic manner, acknowledged the problems in teacher preparation as teachers are prepared for disseminating information rather than fostering reasoning in mathematics. While the teacher education infrastructure has indeed expanded vastly, issues of poor quality and low relevance of teacher preparation remain. Further, teacher education institutions have tended to focus more on pre service education leading to the neglect of in service education.

The National Focus Group on teacher education (NCERT, 2006c) remarked on the inadequacy of teacher education and how despite various recommendations of commissions they have remain unchanged in terms of their "substance, experience offered and modalities adopted" (p.3). It recommended "recognising the active 'agency' in institutionalising the process of school curriculum renewal" by creating "reflective practitioners" (p.25). The position paper by National focus group on teaching of Mathematics (NCERT, 2006b) recognised the problem of inadequate teacher preparation leading to primary teachers reproducing techniques experienced in their schooling, the pedagogy adopted rarely "resonating with findings of child psychology" and inability to link formal mathematics with experiential learning. On the other hand due to curriculum revision secondary teachers are faced with content in which they are not confident and thus unable to make connection within and across mathematics while relying on notes/guides available in the market. The Focus Group recommends that professional development have a specific focus on mathematics as opposed to 'generic' teacher training. Recommendation is also done for generation of large number of freely available resources and networking among teachers as well as with college teachers and research mathematicians to enhance their pedagogic competence.

A renewed attempt to address the problems of pre service and in service teacher education is made by the new National Curriculum Framework for Teacher Education (abbreviated hence forth as 'NCFTE') (NCTE, 2009). While reiterating and elaborating on earlier recommendations in pre service teacher education, the NCFTE also puts forth several principles that need to govern the design of in service teacher education programs. These include,

- designing programs with clarity about aims and strategies for achieving these aims

- allowing teachers to relate the content of the program to their experiences and also to find opportunities to reflect on their experiences
- need to respect the professional identity and knowledge of a teacher and to work with and from it (NCTE, pp. 66-67)

Most pre and in service programmes view teachers as mere agents of the state, and as implementers of curricular and reform directives. Hence they do not directly address the teacher's own conceptions of teaching, learning and mathematics gained from her own experience. Thus revisions in pre service teacher education curricula and in service modules tend, over the years to acquire "add-ons" while not aiming to address teachers' beliefs and attitudes at a fundamental level.

Pre service teacher education

Nearly all schools in India require students to study mathematics as a compulsory subject upto Class 10. The primary school mathematics teacher in India would typically have completed 12 years of school, while a secondary mathematics teacher may be a graduate or post-graduate of mathematics or science. Although the National commission of Teachers (GOI, 1983-1985) recommended twelve years of initial schooling as basic qualification for entry into elementary teacher education programs, its large scale acceptance was achieved only into the late 1990's (Rajput & Walia, 2001). The most recent figures compiled for all types of schools (private, government and government aided) indicate that roughly 19% of regular primary teachers have completed only 10 years of school (Mehta, 2011). However, nearly 46% of primary teachers (Grades 1 to 5) have a university Bachelor's or Master's degree, while for elementary teachers as a whole (Grades 1 to 8), the figure is nearly 57%.

The preparation to become a certified primary teacher requires a two year Diploma in Education (D.Ed.) programme following 12 years of school, and for a secondary teacher, undergoing a one year Bachelor of Education (B.Ed.) programme following a University degree. However, for the country taken as a whole, roughly 22% of primary teachers have a B.Ed. qualification, which is higher than the requirement of a diploma, while 20% of primary and elementary teachers have no teacher qualification (Mehta, 2011). It is only recently that a sharper distinction has been introduced between elementary and secondary teaching qualification, with the RTE Act stipulating that only D.Ed. and not B.Ed. is a qualification to become a primary teacher. NCFTE (2009) has recognised that elementary education and early childhood education have been neglected as "distinct areas of knowledge with their own distinct concerns, concepts and methodological perspectives (NCTE, 2009, p. 10).

The eligibility for getting admission in a regular course of B.Ed is 50% marks in the university Bachelor's degree, while it is 55% for doing it through Correspondence. The duration of the regular course is one year while it is 2 to 4 years when done through correspondence. The Master of education programmes are for one year after B.Ed. and serve as preparation for becoming a teacher educator and researcher in the field of education. Bachelor of education and Master of education are conducted in either colleges affiliated to the university or by departments of education of the concerned university.

D.Ed. Programmes typically require entrants to have completed 12 years of school, but only a very small proportion of students take up mathematics as a subject in the senior secondary school (Grades 11 and 12). The two year D.Ed. programme has besides the component of teaching methodology, a subject component including mathematics. Although most student teachers who join the D.Ed. programme have done mathematics upto Class 10, they have no confidence in their own ability to learn mathematics or to solve problems in mathematics on their own. The mathematics component in the D.Ed. Programmes, like in school, emphasises remembering known solutions to problems, and does not encourage a genuine engagement with the content. While recognising this NCFTE (NCTE, 2009) has recommended enhancement of entry qualification and duration of training making it equivalent to degree programme and bringing these isolated institutions under universities for their management. It must be noted that teacher and the teaching profession in India has a low social status and becoming a teacher is the last choice for most entrants into the population.

Among the graduates and post-graduates who complete the B.Ed. programme, the capability of even those who have studied mathematics at the University level is limited, since most University mathematics programmes do not give the learner any confidence in the subject, fostering a view of mathematics as a set of limited problems that have been already solved. The tasks that students learn to complete is not one of formulating and solving problems that cannot be solved by using known principles but of solving problems that can only be solved with a known trick. It is possible that this attitude to mathematics and learning, and their lack of confidence in mathematics leads them, as school or college teachers, to shun dialogue in the classrooms.

Teacher education institutions

Institutions in India that prepare teachers at all levels (pre-primary, primary and secondary) are run by the Government as well as by private bodies, with both types of institutions offering the same degree. Some programmes like the integrated 4 year B.Sc.Ed. (which combines a university degree with a secondary teacher qualification) are run only in a few Government institutions like the Regional Institutes of Education. An innovative 4-year integrated programme in Elementary Education (B.El.Ed.) is offered by the Delhi University through a few of its affiliated colleges. The integrated programmes however have not spread beyond a few institutions (Walia, 2004).

For pre service training, the National council of teacher education (NCTE), a statutory body of the central government, is responsible for planned and coordinated development of teacher education in the country. The NCTE lays down norms and standards for various teacher education courses, minimum qualifications for teacher educators, course content and duration and minimum qualification for entry of student teachers for various courses. It also grants recognition to institutions (government, government-aided and private) interested in undertaking such courses and has in built mechanisms to regulate and monitor their standards and quality. Financial support is provided by both state government as well as central government to different institutions.

In service training is provided by a large network of government owned teacher training institutions at various levels of hierarchy. The National Council of Educational Research and Training (NCERT) along with its six Regional Institutes of Education undertake design and implementation of in service programmes for both teachers and teacher educators. Along with advising and assisting the government of India in academic matters related to school education, the NCERT serves the function of supporting educational research and training in educational research methodology, developing school curricula, textbooks and other learning material, materials for teacher education, training of teachers, teacher educators and officers, publication and dissemination of research through journals, and programmes with different countries for exchange of educational materials and faculty members.

At the state level, the state councils of educational research and training (SCERT) prepare modules for and conduct teacher training for teachers and teacher educators. The colleges of teacher education and Institutes for advanced learning in Education (IASE) provide pre service (B.Ed) and in service training to secondary teachers and teacher educators, develop materials for teachers and conduct surveys and Research. The District Institutes of Education and Training (DIETs) provide in service and pre service education for elementary teachers.

Stage	Government	intake	private	intake
Pre-primary	16	746	219	14102
Elementary	757	49089	4831	298278
B. Ed.	224	20031	5731	609496
B.Ed. open university	24	13800	2700	16500
B. El. Ed.	-	-	13	545

Table 1: Teacher education institutions and their intake by category (Source: Rajan, 2012)

Teacher education curriculum and its revision

In the post independence era from the 1950's to the 1970's, pre service teacher education mainly emphasised theoretical aspects like discussing aims of mathematics education, inductive and deductive method, analytic and synthetic method, focus on Herbartian steps of preparation, and presentation and application for planning lessons (Chel, 2011). The mathematics method paper had a weightage of 10% of the total marks. The student teachers were expected to make charts and other teaching aids but there was no emphasis on relating mathematics to out of school experience or to other subjects.

The comprehensive curriculum framework for teacher education was released in 1978 which adopted a task oriented approach to teacher education viewing teaching as a series of concrete and hierarchically graded tasks. It had practical aspects of teaching as its focus as it suggested that student teachers should be put through a series of simulating, micro teaching situations before being pushed into actual classrooms. The weightage of the mathematical component was raised to 22.5%. The assessment of content was made by asking student teachers to solve problems from different content areas of school mathematics upto class 12. In the earlier syllabus there was no separate evaluation of mathematics content. However in 1990's there was criticism of this move as teachers and teacher educators felt that testing of content separately without integration with methods is redundant since teachers have already been tested for it in their undergraduate degree programme (Chel, 2011).

Following a major revision of the school curriculum, the "National Curriculum for Elementary and Secondary Education" (NCERT,1988) recommended integration of theoretical understanding with practical application and recommended more weightage to practical application, leading to a revision of the teacher education curriculum. A major watershed development in teacher education was the establishment of National council for Teacher Education (NCTE) as a statutory body in 1993. The NCTE brought out a "Curriculum Framework for Quality Teacher Education" in 1998, which was the first to provide stage specific guidelines for teacher education. It defined several areas of commitment, competence and performance to serve as guiding principles for teacher education programmes (NCTE, 1998). The competencies for teachers were established with a view to supporting the achievement of the Minimum levels of Learning for students in classrooms as laid down in a document on the "Minimum levels of learning" (GOI, 1990). It expected teachers to express learning outcomes in the form of constituent competencies and behaviours that indicated mastery learning. It was assumed that minimum levels of learning are to be achieved uniformly across students. There was focus on developing diagnostic tests and therefore construction of "Achievement test" was assigned additional weightage in the B.Ed. course. The questions in the achievement test were categorised as knowledge, skills, understanding and application. Remedial teaching was recommended after diagnosis of mistakes through the test, but it was not clarified as to how remediation is to be done in order to help students learn.

In the 1998 teacher education curriculum, integration of content with methodology was introduced in the form of "pedagogical analysis of concepts" having weightage both in theory and practical papers. The purpose of pedagogical analysis was to make a student teacher "conversant with the objectives of teaching a unit, the entry behaviour of the pupils, the classroom management and evaluation strategies" and thus make him/her more "effective and confident in his/her interventions in the classroom" (NCTE, 1998). The total weightage of mathematics in the B.Ed. Curriculum was raised to 28.5% of the total marks.

The National Curriculum Framework for Teacher Education (NCTE, 2009) is the most recent attempt at a thorough overhaul of the teacher education curriculum. It contains many new proposals, but is yet to be implemented across universities in the country. It advocates teacher education to be open and flexible, emphasising dialogical exploration rather than didactic communication, diversity of social contexts and learning spaces as sources of inspiration, and teacher education based on reflective practice rather than on a fixed knowledge base (NCTE, 2009). Major revisions in curricular areas are recommended and attempts have been made to draw upon theoretical and empirical knowledge as well as student teachers' experiential knowledge. The attempt is to focus on the learner, develop teachers' understanding of self as well as the social context, critically examine disciplinary knowledge and develop professional skills and pedagogic approaches to address needs of learners. Each curricular area has a theory and related "field based units of study" (practicum) in which the student teacher is expected to undertake projects, field work, clinical interviews, observation and analysis and interpretation of qualitative data to generate knowledge and continually seek clarity of ideas. The teaching of the subject is now conceived as "pedagogic studies" under which linkages among learner, context, subject discipline and the pedagogical approach has to be established. The shift in view of what is considered as knowledge is evident through inclusion of a course like "knowledge as construction through experiences" as compared to the earlier focus on disciplinary content in textbooks as knowledge. Another important aspect is the emphasis on research related to student learning in different areas, studies on addressing learners' misconceptions and engagement with epistemological questions. These indicate an important shift in recognising centrality of the student and her learning in teacher education. The practicum course work includes "hands- on experience at developing curriculum and learning materials, designing appropriate activities.. and formulating questions to facilitate learning" (NCTE, 2009, p. 38).

The duration the internship has been increased to minimum of 6-10 weeks for the two year programme and 15-20 weeks for the 4 year programme to allow sustained engagement with learners. The school internship is expected to provide opportunities for reflection on one's own beliefs and practices while trying out unconventional pedagogies.

A look at some B. Ed. syllabi across country

A look at various B.Ed. syllabi in Universities across the country (Pune, Gujarat, Rohtak, Mumbai, Indraprastha-Delhi, Tamil Nadu) indicates that there are differences in terms of emphasis on content. The B.Ed. syllabus typically consists of theory courses dealing with philosophical and sociological aspects of education, with psychology in relation to education, school administration and management. In some syllabi courses on educational technology or educational innovation are included. The student is typically expected to choose one elective course from among courses such as environmental education, guidance and counselling and mental health. The student has to choose two subject specific method courses. All these courses include a theory portion for which most universities allot 50% marks and some having as much as 70 % marks (Gujarat, Rohtak). The theoretical component of the mathematics methods course would comprise from 5% to 14% of the total marks alloted for B.Ed. Within the mathematics method course the content of mathematics is discussed only in the context of analyzing textbooks of various grades. The practical component of B. Ed course dealing with mathematics comprises of around 30 to 48% of the the total marks alloted. The major portion of practical marks are allotted to practice teaching including micro and macro lesson teaching. Other component of practical includes lesson plans, practical records and construction of achievement test for students. Thus the total mathematical component of the B. Ed courses ranged from 31 to 55% of the total marks. Besides practical work related to subject, other practical work like school based and community based activities which include case studies of students, psychological experiments, etc., are included in almost all universities.

The mathematics method course has broadly three foci. The first comprises the nature of mathematics, its aims, its connections to other subjects and contributions of great mathematicians. The second focus is on specific methods and maxims of teaching like "inductive–deductive method", which in a few universities include "models of teaching" like advanced organiser model, concept attainment model, etc. Out of school activities for mathematics as well as development of math clubs, math laboratory design have also been included in some syllabi. The third focus is on content enrichment for which some universities prescribe study from school textbooks, while others expect students to formulate specific methodologies for teaching a particular topic and in some rare cases ask student teachers critically analyse school textbooks. To assess content some universities (for e.g., Mumbai university) have a "content enrichment" component where in students are expected to do self study of the subject they have chosen for the special methods course. Tests are conducted internally by colleges based on the syllabus of state board for Grades 9 to 12. This reflects a concern for building proficiency in mathematics at that level. But focus on school textbooks for developing understanding of content might make it difficult for student teachers to go beyond textbooks while teaching. As a teacher one needs to have proficiency of developing problems to enable student learning which does not have its place in the teacher education curriculum. Also the kind of mathematics that is needed for teaching of mathematics is different from what is typically learnt in school as identified by several researchers (for example, Ball, Thames and Phelps, 2011).

Pedagogical content analysis, which includes identification of concepts, listing behavioural outcomes, listing activities and experiences and listing evaluation techniques, is included in 3 of the 6 syllabi. However it is not clear how it leads to construction of knowledge that is useful in classroom teaching since there is no indication of students teaching a topic after doing pedagogical content analysis and getting some insight about student learning.

What has not changed over the years (since perhaps the 1950's) in the B.Ed. syllabus is discussion of aims and objectives of mathematics education, maxims of teaching, methods of teaching like "inductive, deductive, analysis, synthesis" methods, techniques of teaching like "oral work, drill work, brain storming, self study" and preparation of teaching aids like charts, models and lately "power point presentations". Most of these topics adopt a view of teaching without considering students thinking thereby preparing teachers for transmitting information in different ways (NCERT, 2006b). Clearly the teaching of methods is unlikely to effect a change in the way mathematics is taught in classrooms and developing students' understanding and reasoning in mathematics as envisioned in NCF 2005, even though though there is a substantial component of practice teaching.

Most B.Ed. syllabi devote about 20 hrs for Micro teaching, 10 hours for integrated lesson, 15 hours for preparing 2 simulated lessons and around 150 hours for preparing 10 practice lessons for each of the 2 semesters in the B. Ed course).

What is lacking in the syllabi is a perspective of teaching that makes the child the centre, and views her conceptions and sense making process as an important part of the teaching and thus the teacher preparation process. In contrast, teaching is fragmented into its components which are dealt separately with a hope that this will impact teaching in classrooms. Its not clear if the B.Ed. programme allows opportunity for students to think critically about their own mathematics learning, teaching practices prevalent in schools, curriculum and textbooks.

The NCFTE 2009 in its radical departure from earlier teacher education curricula has recognised the importance of developing an understanding of the learner, and classroom based teaching and research work as important tools to such understanding. The proposed syllabus for B.Ed. based on NCFTE 2009 has incorporated many interesting features. The pedagogy for teacher education has been proposed to include "focused reading and reflection, observation-documentation-analysis, seminars, case studies and school based practicals and workshops" besides lecture-demonstration. The assessment of student teachers has been recommended to include reflective journals, products like lesson plans and observation of student teachers in various contexts of teacher education. The school based experience has been aimed at preparing teachers for "understanding and developing meaningful learning sequences appropriate to the specificity of different levels of learning and also mobilize appropriate learning resources for them". Pedagogical analysis of content now includes content analysis, identification of various content categories and skills, task analysis with reference to learning objectives, student capabilities and learning approaches, learning resources, possible assessment modes, visualising learning situations, organising learning sequences and contextualising learning. The integration between theoretical and practical aspects of teaching has been proposed through designing learning situations which allow teachers to scaffold learning, clarify fallacies and misconceptions, and reconstruct meaning that teacher has to facilitate in classroom. Comparative textbook analysis has also been proposed.

Practice teaching in the teacher education curriculum

Over the years people have realised that pre service teacher education is too theoretically oriented and efforts have been made in several teacher education curricula to make it more practically oriented by increasing the weightage assigned to practical aspects of teaching (NCTE 1998; NCFTE, 2009). In the 1998 curriculum practice teaching was advocated for about 40 lessons, i.e., 20 lessons each for the two subjects chosen for specialisation in B. Ed program. However NCFTE 2009 listed the following as major drawbacks of the current model of practice teaching.

- Treatment of school curriculum and textbooks as 'given'
- Fastidious planning of lessons in standardized formats with a view to fulfil ritual of delivering required number of lessons
- Repeated practice of isolated lessons being considered as sufficient for professional development
- No opportunity for student teachers to "examine their own biases and beliefs and reflect on their own experiences as part of classroom discourse and enquiry"
- Theory courses having no clear connections established to practical work and ground realities
- The evaluation protocol is too theoretical and excessively quantitative.

In light of the above problems in pre service education NCFTE 2009 has recommended "School internship" through 'partnership model' where trainees develop new materials that function as resource for regular teachers. The duration is to be a continuous period of 4 days a week for 12-20 weeks after observing classroom for one week. Sustained engagement with schools is visualised through teachers participating in all school activities, conducting classroom research and developing learning resources. Recognising the importance of practice teaching the framework views it as both an "evaluation tool for effective teacher education as well as its critical quality indicator" (NCTE, 2009, p. 41).

Innovations in pre service teacher education

The four year integrated course of Bachelor of Elementary Education (B.El.Ed.) is an innovative programme introduced over a decade ago in the Delhi University. It is aimed at preparing teachers for the elementary level of school in contrast to B.Ed. Programmes, which typically focus on the middle and secondary school level. However, it includes more relevant and useful courses for preparing teachers for teaching mathematics as compared to the B.Ed. curriculum. The course outline on "Core Mathematics" in the B.El.Ed. curriculum indicates that "various concepts and operations will be reconstructed through activities and problems, using concrete materials as often from the kitchen as from mathematical kits, to arrive at solutions or conduct investigations. This would be followed by reflective discussions on the concepts, solutions, results and the methods used" (Maulana Azad Centre for Elementary and Social Education [MACESE], 2001). The course includes study of concepts like number and measurement, space and shape, algebra and number patterns.

Another course in "Logico-mathematics education" includes the following: understanding the nature of children's logico-mathematics thinking through exposure to theories by Piaget, Vygotsky and Dienes; language and mathematics; critical study of pedagogical constructs like zone of proximal development, drill, memorization and algorithmization; research on children's learning in specific areas and content specific pedagogy for numbers, fractions using ready made kits. The course on "pedagogy of mathematics" deals with "helping children develop a mathematical view of the world; initiating student's investigations and independent activity and problem solving strategies" (MACESE, 2001).

Here we see a more wholistic view of teaching as compared to the B.Ed. syllabi discussed earlier where different aspects of teaching were dealt with separately and then student teachers were expected to incorporate them in their classroom teaching. The 'aggregation' view of teaching in the B.Ed. syllabi assumes that any method, teaching aid can be useful in teaching any concept. There is no scope of exploring how a particular teaching aid helps in concept formation. Understanding this might contribute more towards building knowledge for teaching of mathematics rather than knowing how to make different teaching aids without consideration of content in the teaching learning process. Unlike some B.Ed. syllabi which include "drill work" in techniques of teaching, the B.El.Ed. syllabus is progressive in giving an opportunity to engage student teachers in critical study of practices like drill but also concepts that have propensity for being used as buzz words like "zone of proximal development". Further, the course attempts to connect teacher education with research in education making efforts to bridge the gap between research and practice. As compared to the B.Ed. syllabus this course keeps the child at the centre of the teaching-learning process and assumes a view of teaching which encourages construction of knowledge through investigations and using students' ideas and strategies in teaching.

Realising the need for focus on content knowledge in teacher education

The typical educational experiences of a teacher in school or university do not prepare her or him to engage with mathematics, to struggle to find a solution to a problem, to examine a concept from different points of view, to make connections, to reason and provide justifications, all of which are stressed by the new curriculum framework (NCERT, 2006b). In a typical B.Ed. programme, the focus is almost entirely on pedagogical technique, and content is assumed to have been mastered earlier. The fact that such education leads to a grossly inadequate preparation of the mathematics teacher with regard to her understanding of mathematics is well recognized (Ravindra, 2011).

The Teacher Eligibility Test (TET), now made an essential qualification by the Right to Education act to secure a teaching job in any school, acknowledges the importance of content knowledge. There are different tests for primary and middle school level aspiring teachers having 150 multiple choice questions. Out of the 150 questions in the primary level test, 30 questions are devoted to Mathematics, of which 15 questions are based on the content in the school textbooks and the remaining 15 on pedagogical issues like error analysis and related aspects of teaching and learning, and "understanding children's reasoning and thinking patterns and strategies for making meaning and learning". For the elementary level mathematics teachers (Grades 6 to 8), again 30 questions are devoted to mathematics, of which 20 are devoted to content and 10 to pedagogical issues. (A similar pattern is followed for science.) Teachers need to get 60% correct answers to pass the test. The recent results show an extremely low pass percentage of 5.5% for primary teachers and 6.5% for middle school teachers. The Human resource development minister ascribed it to the mushrooming of private teacher education institutions (12689 private institutions as compared to 1178 government teacher training institutes), whose quality of teacher preparation may be poor. (Teacher tests results...Kapil Sibal, 2012).

These results are consistent with the findings of other studies such as Banerji & Kingdon (2010), Ravindra (2007), Dewan (2009), which have revealed the unsatisfactory status of knowledge of mathematics of regular school teachers. This state is a reflection of teachers' own education which valued only rote memorization of procedures on the one hand and lacked opportunities to re-learn mathematics in a meaningful way during professional education and during the course of their career on the other hand.

With the change in school curriculum following the National Curriculum Framework 2005, the demand for better understanding of the content and alternative pedagogy has increased. Teachers in elementary and middle grades not only have to make their students fluent in computational mathematics but also address *process* goals in the learning of mathematics, such as reasoning, using multiple ways to solve problems, justifying their solution, making generalizations and conjectures, analyzing the mathematical work of others, etc. (NCERT, 2006b). However there have been few teacher education programs in India, which have focused on the skills and knowledge required to facilitate this kind of teaching. Research studies of teachers' knowledge in other countries have identified pedagogical content knowledge (PCK) as a specialized form of knowledge required for teaching of mathematics and subject matter knowledge (SMK) as a coherent, connected and deep understanding of mathematics (Shulman, 1986; Ma, 1999). Although PCK and SMK are widely acknowledged now as essential components of teachers' knowledge, the preparation of content, and pedagogy revolving around content, is rarely the central focus of any phase of teacher education in India. Teacher education needs to provide opportunities for deepening teachers' knowledge of mathematics and of pedagogy revolving around mathematical practices.

While the teacher education policy documents and the curricula of some innovative programmes acknowledge the importance of content knowledge, actual policy measures suggest the opposite. With the passing of the Right to Education Act, and the consequent pressure to universalize elementary education, most states are faced with a shortage of teachers. This situation has led to multiple cadres of teachers and the appointment of *para-teachers* without the requisite teacher qualification (Govinda and Josephine, 2004). This policy measure reiterates the assumption that a primary teacher does not need to know mathematics beyond the level that he/she is going to teach. Thus there are very low expectations by policy makers regarding the level of content knowledge required of a primary teacher.

In service professional development of mathematics teachers

As emphasised in the policy documents, the central and state governments in India have made efforts to include in service Teacher Professional Development (TPD) as an integral part of the school education system. According to a recent report (Mehta, 2011), 35% of all elementary school teachers in India received in service training in the year 2007-08. However, in service programmes do not follow a well thought out structure and there is no regulatory mechanism that ensures the relevance, quality and suitability of the training provided. In India, workshops are an important component of TPD programs on which maximum time, effort and resources of the state are spent. TPD workshops are often organized in an ad hoc manner on the basis of expediency, sometimes driven by the need to utilize funds (MHRD, 2009, p. 2. Also pp. 15-16). There is no clear consensus about what needs to be done in these workshops and how it is to be done. The vision underlying most of these programs restrict teachers' agency to implementing a new textbook, a pre-designed pedagogy or a prescribed assessment technique. TPD programs however need to have a broader vision of what the needs of a teacher as a developing professional are, and must address issues of knowledge, beliefs, attitudes and practices in a comprehensive manner, rather than in the narrow context of a particular reform.

NCFTE 2009 has now identified several principles related to content and pedagogic approach in in- service programs. It is recommended that spaces are provided to teachers for sharing their experiences in terms of content and pedagogy while providing autonomy in planning and teaching practices thus recognising professional identity of the teacher and building on it. The design of the program should thus be based on clear aims and vision of how they will be achieved while incorporating post programme support or extensive interactions over time with the same resource group for continuing professional development. Further, use of distance media, sabbatical for study and research, attending meeting and conference and development of professional foras, resource room and materials have been recommended.

In service TPD Initiatives

In Service teacher education at the level of the district is organized and provided largely by the District Institute of Education and Training (DIETs), with an overall co-ordination at the state level by the State Council of Educational Research and Training (SCERTs). Other institutes established by government like Institute of Advanced Study in Sducation (IASE), university department of educations are also involved in the effort. At the national level, the NCERT is involved in development programmes and resource material for TPD. For in service professional development of people working in colleges and universities several "Academic staff colleges" have been established (66) in several universtities which provide orientation as well as subject oriented refresher programmes.

There have been major initiatives in the in service training of teachers over the last few decades, in roughly two phases. The first phase began with two programmes initiated in the 1980s at the national level called Programme of mass orientation of School teachers (PMOST) and Special orientation of primary teachers (SOPT). The emphasis in these programmes was on methodology and how to teach in the classrooms, rather than on the content of mathematics. The SOPT also saw beginning of idea of Minimum Levels of Learning (MLL) in education, which was further reinforced by the report on MLL published by the Ministry of Human Resource Development in 1991 (GOI, 1986b). The document viewed learning as occurring in separate small chunks, each of which could be mastered separately by repeated practice. In the SOPT and subsequent MLL based programmes, teacher training was seen merely as a forum where teachers would be given activities and materials that they could use in the classroom.

In a typical SOPT programme of 7 days, 3 sessions would be on mathematics. The training modules included detailed descriptions of what kind of activities could be done with children. The modules assumed that children have similar views and follow similar ways of learning and therefore suggested how an activity could proceed with a group of children. The emphasis was on activity and use of materials. The key words were hard spots, MLLs, competencies, assessment, diagnostic testing and remedy as well as activities, modules and demonstrations.

These efforts were followed in the second phase by the capacity building programmes under the District Primary Education Programme (DPEP) and similar projects supported by many multi-lateral partnerships. In service training in these programmes centered around "joyful learning" and presentation of activities to teachers. The orientations were marked by an attempt to introduce games and other interesting devices into classrooms without necessarily looking at the nature of the concepts to be transacted or the nature of mathematics. The activities that were developed involved a lot of movement, play, singing and use of materials but there was little thought about how this could be related to conceptual development in mathematics. The time spent in mathematical thinking on these tasks was much smaller than the total time required for the activity and most of the effort was spent on ensuring that children had fun. The pattern of training in the DPEP continues to influence newer initiatives such as the Sarva Shiksha Abhiyaan (Education for all mission) and the Rashtriya Madhyamik Shiksha Abhiyaan (National secondary school mission). Thus a continuing influence of these programmes has been the emphasis on technique or activity and a reduced emphasis on mathematical understanding or thinking.

Some in service initiatives have introduced important elements that have a significance for teacher professional development. The Shiksha Karmi (Education worker) initiative of the 1990s in the state of Rajasthan emphasised the autonomy of the teacher in its in service programmes, and developed a critique of the top-down "transmissionist" model of in service training (Sharma & Ramachandran, 2010). The main features of the programme involved selecting local youths to act as teachers in dysfunctional schools while they get continuous and intensive training through out the year and are supported by village education commitees. The "Shikshak Samaksha" (Teachers' empowerment) project in Madhya pradesh involved teachers meeting once a month in the resource centres to discuss their problems, experiences and suggestions to make their teaching interesting. The teachers were provided regular academic support (Mohanty, 1994). Andhra Pradesh Primary Education project (APPEP) included the use of demonstration lessons given to group of children to illustrate new pedagogic techniques and making the classroom interesting by displaying and organising children's work. Teachers planned and generated activities for teaching at the teacher centres established at sub-district level in 23 districts (Mohanty, 1994). So the major departure in these innovations is providing regular academic support and discussion of teachers' experiences in the classroom. The influence of these and similar initiatives have led the new National Curriculum Framework for Teacher Education to stress the need to respect the professional identity and knowledge of a teacher and to work with and from it (NCFTE, 2009, pp. 66-67).

The Project in Science and Mathematics (PRISM) initiated in year 2000 involved collaboration of Homi Bhabha Centre for science Education with the Bombay Municipal Corporation. The objective was to strengthen teachers' understanding of fundamental principles, creating an environment in the classroom for students to ask questions and helping teachers and students to go beyond the textbook. HBCSE members worked directly with 50 resource teachers for a year to develop their capacities to train the larger group of teachers working in about 250 schools. There was focus on developing conceptual understanding through discussing usefulness of teaching aids (for e.g., bundles of matchsticks of 10 or 100 for place value concepts and operations). Activities were done with teachers to challenge the belief that all mathematics problems have only one correct answer by asking teachers to formulate open-ended questions. The approach adopted for the resource teachers included planning of lessons followed by one teacher teaching students while other colleagues observed the lesson, in a manner similar to Japanese "lesson study". The lesson would be followed by intensive discussion focused on the teaching as well as student responses and thinking, followed by planning for subsequent lessons."Model lessons" by HBCSE team members, problem solving, observing simulated teaching and teaching in schools of participant teachers was part of the program (Burte, 2005).

The "Prashika" experiment in primary education was an innovative programme launched by Eklavya, a leading voluntary organization working in the area of elementary education for many decades. This programme included a teacher education component, for which the description "orientation programme" was used instead of "teacher training". The word "orientation" reflected the Prashika standpoint that teacher education cannot be completed in a 20 day contact period programme, which serves only as an initiation into engaging with teaching, trying out things and "learning from experiences" (Agnihotri, Khanna & Shukla, 1994, p.127). Thus the teacher development was conceived as "gradual, ongoing, interactive and collaborative process of change" (Agnihotri et al., 1994, p.122). The major objectives of the programme were to

- Create an awareness of the learning process and bring about attitudinal changes.
- Cultivate skill and confidence
- Help teachers acquire knowledge

- Develop those operational skills that are needed to put curriculum in practice
- Help teachers in a sense to become their own informal researchers (Agnihotri et al., p. 126).

The Prashika approach focused on building teachers' understanding of the child, curricular understanding for creating appropriate activities and enhancing creativity of the teacher by overcoming inhibitions and enagaging in activities like drawing, singing and role play. The expectation was that the teacher will function as a "partial source of information and knowledge" while being able to "plan a multiplicity of activities, observe carefully their implementation and analyse the feedback to modify and change the activities" (Agnihotri et al., 1994, p. 120). The pedagogy adopted during teacher orientation emphasized establishing equality among resource persons and teachers by realizing that much can be learnt from teachers, flexible plans for the programme which could be modified based on the needs of the group and getting feedback from teachers, resource persons and observers for revising materials for classrooms and deciding teacher orientation agenda.

Recognising the limitations of teachers' knowledge of mathematics, Prashika placed emphasis on enhancing conceptual knowledge of teachers. "A large number of them know rules and formulas, but they are often incapable of handling questions like why and how a particular algorithm works" (Agnihotri et al., 1994, p. 135). One of the principles behind teacher orientation activities was to let teachers enjoy mathematics to ensure that at least some of it is taken up in the classroom. The vision of teaching mathematics involved using concrete materials at early stages and then moving to abstract concepts, opportunities for children to articulate their understanding, opportunities to make hypothesis and make their own problems, allowing expression and exploration of alternative procedures and attempt to understand *why* children make mistakes. Over the course of the engagement, teachers made important realisations like "reciting numbers upto 100 is not counting", students appear to understand and solve sums correctly in classroom when the topic is being done but not later and problems in developing functional understanding of concepts like place value even when student are able to understand their abstract nature (Agnihotri et al., ibid, p. 131).

Another well known voluntary educational organisation working with teachers, Digantar, offers a "Certificate course in foundations of education". The mathematics component of the course emphasises that teachers must be involved in "*doing* mathematics" to understand the nature of mathematics through emerging patterns and rules. In the contact sessions, teachers engage in problem solving followed by discussion on how general rules can be derived by comparing the approaches used by participants. Teachers are also involved in discussing theoretical aspects of mathematics teaching through discussing readings and papers (for e.g., absolutist and conceptual change view of mathematics discussed in the writings of Paul Ernest). Teachers are also encouraged to speak about areas of mathematics where their understanding is weak. Other colleagues are urged to help their peers in overcoming these weaknesses. Group work and presentations by groups is central to the pedagogy adopted for teacher orientation (Digantar, 2008).

Recent initiatives by NCERT have focused on developing a range of resources useful for teacher training including the development of an "in service teacher professional development programme" having 5 day workshops every year for in service teachers and heads of schools. The Training package of the programme for mathematics includes mathematics kits, source book for assessment and ICT Kits (Pattanayak, 2009). NCERT has been promoting Mathematics laboratories for a number of years. The need for maths lab has been mentioned in the school curriculum frameworks (NCERT, 2000; NCERT, 2005). As a result, the Central Board of Secondary Education has introduced Maths lab as a part of the curriculum for secondary school. Maths Lab Manuals containing suggestions for various activities for different concepts and instructions on how to do them have been developed by NCERT. Some educationists have cautioned against the excessive promotion of the idea of a maths lab since it may foster an incorrect epistemology of mathematics (accepting verification in a few cases as a substitute for proof), and may encourage drawing a sharp distinction between classroom teaching and "activities" done in the lab (Dhankar, n.d.).

The Department of Education in Science and mathematics in the NCERT organises orientation programmes for teachers and master trainers (who teach teachers) to strengthen the teaching of Science and mathematics e.g. orientation on "activity based teaching" in mathematics. The draft of a textbook on pedagogy of mathematics has been prepared recently for use in teacher preparation in line with NCF 2005 recommendations for moving from content to process and "transformation of procedural level understanding to conceptual level understanding" (NCERT, 2011). It includes experimentation and activity with low cost materials and teaching of mathematics through games, puzzles and visuals along with curriculum construction in mathematics at various stages with examples. Enrichment material has been prepared in collaboration with practicing teachers at the higher secondary stage on themes like conceptual understanding, applications and misconceptions. A teacher training manual for class 1 and 2 teachers has also been developed by "Group arithmetic" cell established in NCERT for strengthening early mathematics development programmes (NCERT, ibid).

For the promotion of mathematics several programmes have been started at state levels like Metric Melas, Math festivals, Math forum, Math clubs and even Maths Marathon. At "Ganit Melas" (Math Fairs) alternative teaching learning materials, activities and methods of assessment are presented to participants, i.e., teachers and students. The development of self learning and interactive learning material by teachers have also been undertaken by various states. (Pattanayak, 2009)

Teacher education through distance education

Looking at the high demand for trained teachers and the inadequate infrastructure to train teachers, distance education plays an important role in providing avenues for pre service teacher education as well as in service professional development. Many practicing teachers achieve certification after going through distance education programmes.

The Indira Gandhi National Open University, the leading open university in the country, has developed materials specific to mathematics for a certificate program on "Teaching of primary school mathematics". The course is a broad based course meant to encourage learning of mathematics and the appreciation that it is not merely abstract and unrelated to our experience. The course is taken by teachers, parents, Bachelor degree program students, and persons working in mathematics education in various capacities.

The course aims at making the learner of the course appreciate the difference between understanding and doing mathematics on the one hand and merely using algorithms on the other. It includes discussion on the. It engages with the understanding about the nature of mathematics and the purpose of learning it common among the general public. Key ideas about how children learn and applying these ideas to build engaging classrooms, materials and assessment systems is an important part of the discussion. The certificate course comprises of two parts, a basic and an advanced course.

The print material for the course has been developed by a team of authors from a range of institutions who have contributed to innovations in mathematics education (Indira Gandhi national Open university [IGNOU], 1996). The course material focuses on concepts considered difficult while giving detailed illustrations of various teaching strategies in the areas of numbers, fractions and measurement. It is unique in dealing with topics like statistics and probability. Other interesting topics include development of spatial abilities of children, multiplication and division by a fraction, importance of estimation in fractions, understanding of simple algorithms, mathematical logic and language of mathematics and engaging in constructing proofs and ways of doing it with children. The teacher is encouraged to make the "model of learning" in her mind explicit and engage in inquiry about how children learn and how classroom processes influence learning. The student teacher is urged to actually try the activities given as illustrations with children, develop a sensitivity towards how students learn, try out variety of interactive learner oreinted methods of communicating mathematics, critically evaluate one's method of teaching mathematics and alter it to suit the situation of the learner and thus develop arguments to support one's experience and understanding.

Informal feedback from the course participants suggests that they enjoy studying it and being confronted with a host of new ideas (Parvin Sinclair, personal communication,).

The many innovative elements in the course make its success critically dependent on the availability of good counsellors at the study centres. Many counsellors and evaluators for the courses also find much to learn from the programme. However, the efforts put by learners on the activities and project work do not generally meet expectations, especially where counsellors are few or are unavailable. The overall percentage of students passing the exams is about 25%. While most students pass the assignments, very few pass the project component, which usually takes longer and multiple attempts.

Research in Teacher education

The Kothari commission mentioned the lack of adequate research on "problems under Indian conditions" (GOI, 1966) and absence of high quality original books on pedagogy and educational science in modern Indian languages as two major weaknesses that constrained the professionalisation of education as a discipline.The quality of research in teacher education leaves much to be desired. The research undertaken is largely questionnaire or survey based within a quantitative paradigm. The experimental research undertaken is usually of the form where comparison of conventional teaching with innovative method is done and the innovative method is found to be significant in improving the achievement of students. The use of case studies and ethnographic studies are rare. Most research studies do not take care to operationalise terms used and interpretations of terms vary from study to study. The tools used in research are mostly adapted from research done in other countries and the background or rationale for tool use is not made explicit. There is also not much infrastructure support for carrying research in teacher education institutes. Walia (1999) found that out of 150 elementary teacher educators in her study sample, only 11 undertook research studies. Out of 77 secondary teacher educators only 22 % undertook sponsored research.

NCFTE 2009 observed that there is very little research on effectiveness of training programmes and research does not provide thorough understanding about the interventions reported. The research reported has been anecdotal and impressionistic and there has been reporting of even contradictory findings depending on who is doing the research.

The need for Professional development of teacher educators

As emphasised by NCFTE 2009, it is imperative to develop programs for professional development of teacher educators. In most teacher education colleges, the majority of teacher educators are not graduates in mathematics thus have limited content knowledge. Additionally, most teacher educators are recruited from among teachers of secondary schools and thus don't have experience in teaching of mathematics at the primary level (GOI, 1966) even though they may be educating primary teachers, as for example in the DIETs. NCFTE 2009 notes that the lack of professional preparation of teacher educators is the weakest aspect of teacher education in the country.

At present the qualification requirement for teacher educators for the elementary stage is B.Ed. and for the secondary stage is M.Ed. although PhD and M.Phil. carry a weightage. The M.Ed. program is taught as an extension to B.Ed with little preparation for taking on the role of teacher educator. There is no "practicum" requirement for M.Ed., that is, M.Ed. students are not required to teach teachers. However, teacher educators need to have knowledge about supporting learning of children as well as of adult teachers. Secondly, there is need for teacher educators at the elementary stage to be proficient in areas of science, social science, mathematics and languages along with understanding of young child. The B.Ed course may not address this need as it is focused on preparing teachers for the secondary level. The Kothari commission had felt the need to raise the required qualification for teacher educators for secondary teachers. It recommended a double masters degree along with study of teacher education as a special subject and recommended that a fair proportion (10%) of teacher educators should hold a PhD degree. NCFTE 2009 recommends that M.Ed. be developed as a program for preparation of teacher educators where stage specific specialisation can be done like early childhood, elementary or secondary teacher education. Further specialisation in fields like mathematics education can be offered in M.Ed.

The pedagogy adopted by teacher educators in most teacher education institutions is mostly lecture method (Walia, 2004; NCTE, 2009; Ravindra, 2007). After NCF 2005 there has been shift in thinking about pedagogies adopted in teacher education with NCFTE 2009 recommending teacher educators engaging teachers with learners in real contexts while reflecting on the larger socio- political context in which the learner as well as teacher herself is situated. This can be done by bringing experiences of teachers centre stage and allowing for reflection on, for e.g., their own position in society in terms of gender, caste, etc.

Teacher educators hardly get opportunities for their own professional development. With significant shifts in thinking about teaching and learning as espoused in NCF 2005, there is now urgent need to engage teacher educators in discussions around this new vision of teaching. There are also no institutes designated for professional development or preparation of teacher educators in India. In order to reform the teacher education system in India it is important to first undertake professional development of teacher educators themselves. For teacher educators there are demands to understand both how children learn and how teachers learn to be able to support development of reflective practitioners.

A new M.A. programme in elementary education was launched in 2006 at the Tata Institute of Social Sciences in Mumbai, in collaboration with four other leading institutions that have made innovative contributions in elementary education. This is one of the very few Master's level university programme in India focusing wholly on elementary education.

The course prepares students for key roles in educational innovation, including the role of teacher educators. A pedagogy of mathematics course is offered as an elective course in the programme. The course discusses contemporary pedagogical and learning issues in connection with the content of elementary mathematics and exposes students to key research contributions in the field of mathematics education. A strong emphasis on equity issues with readings drawn from across the world is a notable aspect of the course. Several students who completed their course also completed a field research component in mathematics education; a few of these studies have been presented at conferences on mathematics education in the country.

Conclusion

Revisiting the goals of teacher education

It is pertinent at this point to ask what the goals of pre or in service mathematics teacher education must be. Studies across several countries have emphasised the role of specialized knowledge for teaching, teachers' beliefs and attitudes in shaping classroom teaching, and the need for teacher development programmes to address these. Studies of teacher development and teacher change have emphasised the creation of communities of inquiry and building the professional identity of a teacher. These insights are reflected in the needs identified by teacher educators through long experience of working with teachers in India.

Several components of knowledge that is needed to teach mathematics remain so far inadequately addressed in the educational trajectory of teachers. An important need is strengthening teachers' knowledge of mathematics, which includes not only an understanding of the concepts involved but also an appreciation of the nature of the discipline and its specific nuances. A second aspect that teachers need to feel assured about is the need for children to learn mathematics – why should children learn mathematics and what mathematics should they learn. A third aspect that the teacher needs to know involves the learners: what strengths and experiences do they bring to the classroom and how do these shape their capability to learn? A fourth aspect is understanding how mathematics needs to be addressed and engaged with in the classroom keeping in mind the above.

Teachers' beliefs and attitudes are also crucial. They are related to the components of knowledge described above, but are also independently directed. These attitudes, which may arise from prejudices, include their notions about the nature of mathematics, about children, their background and learning capability, about classroom processes and about what the purpose of education including of mathematics education can be and should be. It is quite common for educators and administrators to believe that children from disadvantaged socio-economic backgrounds are incapable of learning mathematics, either because of an inherent lack of ability or because they do not have the cultural preparation and attitude to learning. The teacher also needs to have confidence in her own ability to do mathematics in order to encourage students and to give space for their thinking. Thus both pre and in service teacher education needs to address these gaps that have been created by a poor education system.

Challenges

The first and foremost challenge is of the scarcity of institutions and qualified people, to address the needs of a huge population of teachers. While in many states, the situation is one of a shortage of qualified teachers, and of teacher education institutions, in some states there is an excess of qualified teachers. This is a consequence of the massive thrust in expanding teacher education following the Kothari Commission report in the late 1960s. However the expansion has been at the cost of creating a deep structural limitation that affects both pre and in service teacher education. Teacher education has been hived off as a professional stream outside mainstream university courses and disconnected from other knowledge intensive professional courses, resulting in a commercialisation of teacher education, which has been the main engine for expansion. This has led to an absurd view of 'teaching' as an activity divorced from what is being taught.

Further, teacher education, which was designed to draw on disciplines like psychology, sociology, history and philosophy, has become de-linked from the developments in these disciplines, as also from their dynamic interplay with the Indian socio-political cultural milieu. The separation of pedagogy from content on the one hand, and from the social sciences on the other, has had far-reaching consequences. It has resulted in the near irrelevance of teacher education to the practice of teaching, and to a diminished status of the teacher in the academic community. Other short term measures taken without hindsight or a long term vision, have resulted in a weak infrastructure for teacher education even where it exists.

Another challenge is the paucity of resources and materials available to teachers for their own growth. The diversity of languages in India is an issue to be tackled since what materials exist are mainly in the English language, and are inaccessible to the vast majority of teachers.

The way forward

The challenge of the divorce of pedagogy from mathematical thinking and content is one of the deep structural problems that needs to be addressed. As discussed earlier, some efforts in this direction have been made by integrated programmes that offer a University degree together with a teacher qualification such as the four year B.Sc.Ed. Programme launched by the NCERT. Another programme which has had an impact in Delhi is the four year B.El.Ed. programme, which emphasises a better integration of the disciplinary foundations of education with pedagogy and intensive practical work in schools, but not so much the integration of pedagogy with subject matter. These intitiaves represent a trend of forging stronger links between University based disciplines and teacher education. A development with a far-reaching consequence would be if regular university degree programmes linked the learning of concepts to learning to teach it as well, demanding attention both from students and faculty to how concepts could be taught and learned. There is in general a need to build strong links between universities and knowledge creating institutions and the work of teaching at all levels. Similarly, in many parts of the country there exist active mathematics teachers' associations focusing on talent search and nurture, promoting problem solving and popularising mathematics. They have weak linkages to teacher education institutions, and to nation wide in service TPD initiatives. A platform for forging these links could be provided by teacher conferences on challenges in mathematics education and efforts to address them.

There is a need to reformulate pre service teacher education programs to address the issues discussed above including in particular, understanding of the cultural and social background of children, the social processes that they face in school, how their language and culture could be a resource for their learning, understanding the purpose of education for society and be aware of the expectations of students, the capabilities of all children and the strengths specific to the group, how children learn mathematics and what conceptual understanding of mathematics means. An integration of the communities involved in pre and in service teacher education would bring it closer to the practice of teaching and also take advantage of situations where pre service student teachers and in service teachers are enrolled in the same programme. A mode of teacher education that combines face to face contact with distance and school based teacher development also needs to be explored for its potential.

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9. Innovations and initiatives in mathematics education in India

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Introduction

The past four decades (beginning 1970s) have seen enormous changes in the field of education in India and numerous organizations (governmental and non-governmental) have taken steps in response to or in reaction against the policies adopted by the government vis-à-vis education and curriculum documents released from time to time. Universalization of education and education for democracy have become the new agendas for the country. Mathematics has been a subject with a large number of student failures, a reason for students dropping out of school and a cause of fear and anxiety among students. Mathematics and science, which have played the role of gatekeepers for accessing higher education, have for many years aroused interest among several intellectuals to make an effort on the ground and make a difference in children's attitudes to and understanding of these subjects. Various commissions set up by the government have recommended mathematics as compulsory for all students due to its importance for the growth of biological and physical sciences and technology. Thus, many efforts in the form of interventions at various levels have been made in the country. However, these have not translated into detailed documentation or systematic research studies to gather evidence of their actual impact on students' thinking and learning, teacher development or systemic changes. The recent change in the public discourse on general education to a more progressive one, imbibing the constructivist philosophy and keeping the welfare of the child in mind led to further efforts in the field. This article aims to first describe some of these efforts made in the area of mathematics teaching and learning and then raise certain issues and challenges which mathematics education and research in the area faces in this country.

Interventions and initiatives in the teaching and learning of mathematics

The interventions which have been made in the way of influencing the teaching and learning of mathematics have been in varied directions: curriculum development, interventions in schools and classrooms, in-service teacher training, nurture programmes to train students to build capacities to think mathematically and solve problems of various complexities, and popularization of mathematics, largely to attract students to take up higher mathematics.

Curriculum and material development

Some of the earliest attempts to improve teaching and learning of mathematics were in developing alternative curricular materials. The Khushi-Khushi series of books for teaching mathematics at the primary level developed by Eklavya, a non-governmental organization (henceforth NGO), based in the state of Madhya Pradesh and the material designed by another NGO, Digantar based in Rajasthan are two such examples. They associated closely with local schools and teachers and the communities they intended to work with. These attempts were premised on children as active learners and faith in their ability to think independently and create knowledge. They took into consideration children's background (socio-economic, language, local traditions, culture, environment, etc.) in the designing of learning material, understanding that these factors influence learning of mathematics in specific ways and considered these as ways to make mathematics meaningful. These initial attempts were based on some understanding in the areas of child development, language learning and mathematical skills and abilities of children. They relied heavily on Piagetian stages of cognitive development and promoted constructivist and discovery based learning, used games and activities and helped students learn from concrete experiences or structured materials before moving to abstract concepts. However, teachers were given adequate space to change, modify and add to the illustrative material given in the text as per the needs of their classes and children. They paved the way for many subsequent interventions.

Much of the curriculum development activity in the country has been guided by these assumptions and philosophy of teaching and learning but has varied in the way these translated into the design of textbooks and materials. We can see three distinct trends in the designing of these interventions. One set of interventions used plenty of games and activities to introduce concepts and ideas as well as to strengthen procedures. The thrust of these interventions is the use of games and activities and less importance is given to sequencing of concepts across grade levels or ideas within a concept. They also systematically included student generated strategies, giving scope to generate problems given some numerical sentences, and recognition of patterns in numbers and operations.

The Eklavya group made attempts to combine the teaching and learning of different subject areas like language, mathematics and science (Khushi-Khushi, 1998) but this approach worked well only for the initial Grades 1-3. As the students went up the grades, the teaching of different subject areas had to be separated. The Grade 4 Khushi-Khushi has three books, half of one of these books is devoted to mathematics and in Grade 5, one full book is on mathematics. Eklavya's effort, in particular, was a broad based effort in collaboration with the state education department, involved many mathematicians and scientists and aimed to impact the system and society directly. Teachers were integral to the design and development of activities and ideas in such interventions and thus a large number of classrooms were available for trialing them. In the process, these initiatives conceptualized and developed mechanisms for supporting teacher preparation (pre and in-service). The School Mathematics Project started by the Centre for Science Education and Communication, University of Delhi (1992) was another systematic attempt to develop an alternative primary mathematics curriculum in schools. It involved teachers, educationists and scientists, and built on the experiences of Eklavya. This process inspired other groups in the country to take up similar exercises later on, and some state boards (like Kerala and Tamil Nadu) used extensive participatory processes by which large numbers of teachers were involved in textbook preparation and trialling.

Another set of interventions has been based on the use of very structured materials in a systematic way, to gradually build new ideas and concepts. Suvidya, a Bengaluru based NGO, developed materials for concepts which are more amenable to such treatment. Navnirmiti, a Mumbai based NGO, designed a range of materials to cover almost all concepts and ideas in primary mathematics, embodying the philosophy of learning by doing. Jodo Gyan, a Delhi based NGO, made further extensions in the same line to incorporate the philosophy of Realistic Mathematics Education and designed material and teaching learning trajectories accordingly. Centre for learning resources, based in Pune is another organization, which has worked extensively with primary level mathematics, developing simple low cost teaching material and teacher training packages for using these materials.

A third set of interventions is more conceptually driven, introducing the concepts gradually and in a logical sequence (does not strictly follow the Piagetian stages of child development), deepening students' understanding through various activities aimed at clarifying the concept and related ideas and procedures (Homi Bhabha Centre for Science Education, a research institution in Mumbai).

Mathematics educators like P. K. Srinivasan and Shailesh Shirali have contributed by writing books and primers with the aim to clarify the content and with numerous examples of interesting problems (e.g. Srinivasan, 2004; Shirali, 2000).

The National Curriculum Framework 2005

Experiences with interventions like the ones mentioned above together with experiences gained through the Adult Literacy Programme (Rampal, Ramanujam and Saraswati, 1998) contributed to the discussions towards the country's new National Curriculum Framework (NCF-2005) (NCERT, 2005) and subsequent revision of the textbooks. The perparation of NCF-2005 was a huge exercise and led to significant changes in the way one thought about teaching and learning of mathematics and the way textbooks for children were written. The framework clearly explicated a philosophy and an approach to teaching and learning and systematically tried to address social justice questions. It was guided by the Constitutional values of India as a "secular, egalitarian and pluralistic society, founded on the values of social justice and equality" (NCERT, 2005). It proposed five guiding principles for curriculum development:

> (i) connecting knowledge to life outside the school; (ii) ensuring that learning shifts away from rote methods; (iii) enriching the curriculum so that it goes beyond textbooks; (iv) making examinations more flexible and integrating them with classroom life; and (v) nurturing an overriding identity informed by caring concerns within the democratic polity of the country.

This effort too held the same assumptions about children, knowledge and teaching-learning as the earlier interventions and emphasised teaching as a professional activity. In line with the earlier experiences of the interventions, the textbooks which were written after the NCF-2005 deliberations, include various voices and backgrounds of children and adults who surround them. The tone of the books is reader friendly and the books have many visuals, games, activities and open-ended tasks. This nation-wide exercise had tremendous influence on state level development of curricular document and textbooks. Some such examples will be given later.

In-service teacher training and teacher education programmes

Teacher education programmes across the country have not changed as a result of the nationwide deliberations during the formulation of the NCF-2005. They continue to be the weakest link in our education system. They do not systematically provide any content specific inputs to help the trainee teacher to get prepared for her job. The Indira Gandhi National Open University made the first and almost the only attempt to run a teacher education programme for primary mathematics teaching with special emphasis on content and conceptual clarity. The School Mathematics Project, which was started by the Centre for Science Education and Communication, University of Delhi, was another attempt to work with primary mathematics teachers in schools. Sandhan, an NGO in Rajasthan, has been associated with two prominent programmes in the region, Shiksha Karmi and Lok Jumbish. They have been involved with training locally recruited educational workers as teachers, creating teaching-learning material for children in mathematics and other subjects and setting up support structures for them in collaboration with other organizations. A more detailed account of teacher preparation programmes and their professional development can be found in Kumar, Dewan and Subramaniam (this volume).

Mathematics popularization activities and nurture programmes

Leading mathematicians spread across the country in many national level institutions (Indian Institutes of Technology, Indian Institute of Science, Institute of Mathematical Sciences, Tata Institute of Fundamental Research, Indian Statistical Institutes, etc.), many individuals, and the National Council of Educational Research and Training contribute towards popularizing mathematics among students, teachers and the community at large. *Ganit Mela* and *Metric Mela* (mathematics fairs held in villages where adults are involved in answering questions raised by children based on estimation) organized in different parts of the country are some attempts to take mathematics to the community.

In this country of great diversity, we also have a very promising group of students spread across the country and across grade levels. Initiatives have been taken to retain their interest in mathematics and motivate them to pursue a career in mathematics. Some such programmes are the Mathematics Training and Talent Search (MTTS), Rural Mathematics Talent Search (RMTS), Mathematical Sciences Foundation (MSF) and the Mathematics Olympiad. These aim to promote independent thinking among students, make challenging mathematics accessible to them, show applications of mathematics in various walks of life and interact with experts in the field. The MTTS programme is meant for students pursuing undergraduate and postgraduate degrees. The RMTS aims at identifying and nurturing talent in the rural areas of Orissa. It holds a mathematics competition at the grade 6 level, designed on the lines of the Olympiad competition. The selected candidates are trained for the next three years, meeting them twice every year. MSF is involved in arranging innovative programmes related to the teaching, understanding, learning and application of mathematics at the school, college and post-graduate levels. The mathematics Olympiad activity is undertaken by the National Board for Higher Mathematics and aims to support mathematical talent among high school students in the country. It culminates in the selection and training of the Indian team for the International Mathematical Olympiad every year. An important part of these nurture programmes is providing scholarships for students even at the undergraduate level for pursuing mathematical study, and enabling their systematic interactions with research mathematicians. This site provides interesting research problems and another possible solution to get important mathematics accessible to a large number of students.

Institutional and state level initiatives

The Homi Bhabha Centre for Science Education, Mumbai is the only institute working in a focused manner in the area of mathematics education. It contributes to the field of mathematics education through its many research projects, a doctoral programme and the epiSTEME series of conferences which are held every two years and provide a platform for sharing research ideas with participants within and outside the country. These then feed into the many consultative activities which the institute is engaged in, both at the level of teacher development and student learning. In the recent years members from this institute and a few individuals from other organizations have made efforts to publish their ongoing work as short and long research papers and chapters in books. Bose and Subramaniam, 2011; Naik and Subramaniam, 2008; Banerjee, Subramaniam and Naik, 2008; Subramanian, Subramaniam, Naik and Verma, 2008; Menon, 2009 are a few examples.

Initiatives at several other levels by individuals and teacher organizations spread through the country have shaped the way mathematics teaching and learning is viewed in the country today. Groups like the Tamil Nadu Science Forum (TNSF), Kerala Sastra Sahitya Parishad (KSSP) and teacher associations like the Kerala Mathematics Teacher Association (KMTA), Association of Mathematics Teachers in India (AMTI) have played major roles in the past two decades in mobilizing teachers to form networks, conducting teacher trainings in order to enrich their content knowledge, connecting their mathematics knowledge to the world around them, expanding their knowledge by giving them tools to look at the same concepts and ideas in multiple ways. The participation of teachers in these activities is entirely voluntary and the interest of some mathematicians in this endeavour makes the activity exciting. The experience and contributions of TNSF and KSSP have fed into the development of state level new curriculum framework, textbook writing and material development. During the nationwide Sarva Shikha Abhiyan (SSA, Education for all) movement in the 1990s, TNSF actively contributed to the development of activity based learning material. Similarly, teachers, teacher educators and mathematicians have made efforts to bring technology into the secondary and higher secondary mathematics classrooms to make that learning space lively and enjoyable – engaging students in problem solving, challenging projects and providing support to understand the abstract concepts dealt at this level. The TIME (Technology and Innovations in Mathematics Education) series of yearly conferences deals with this theme.

Concluding remarks

Mathematics education is not an established discipline in this country and few systematic studies exist in this area. Due to the efforts of many, a substantial amount of work has happened by way of improving teaching and learning of mathematics. But a lot more needs to be done. Impressions of researchers or teachers involved in developing alternative curriculum and carrying out the classroom interventions indicated significant improvements in children's attitudes towards mathematics. They also indicated better understanding of the content but systematic studies are required to assess their actual impact on students' learning. In the absence of strong empirical evidence and sound theoretical background, policy formulation becomes a difficult task. This holds true for the NCF-2005 as well where studies are required to critically examine the translation of guidelines given in the document to the textbooks and in the classroom. A few small scale studies, carried out in the primary and middle school grade levels, do indicate that a lot needs to be still achieved to fulfil the visions of the document. This may also indicate the need to critically examine the underlying assumptions in the design of the framework and the textbooks and the organization of content across grades. One needs to address the question of children's learning of mathematics as a discipline (with certain concepts, ideas, language, symbols, ways of reasoning and arguing, dealing with abstractions and generalizations) till the middle school level, which may serve as the terminal point of education for many children in this country. One also needs to ponder whether changing the framework and revising the textbooks would automatically lead to the desired overall change.

Teacher preparation continues to be the weakest link in our education system. The departments and colleges have not been able to come up with a good model of training teachers at both the pre-service and in-service levels. Simultaneously, efforts have to be made to develop capacities among teacher educators and administrators in the system.

There are relatively few individuals who are contributing to innovation in mathematics education in this vast country. As can be seen from the above description of initiatives, much of this comes from a few individuals or from a few non-government organizations. There is no systemic structure to support and strengthen such work. Although the list of contributors is not an exhaustive list and there are many others, including private support in the form of corporate social responsibility today, one needs to worry about quality of the various efforts made and critically look at the underlying philosophy. There is also no forum where different groups and individuals showcase their work and discuss issues relevant to mathematics education in the country.

Similarly, assessment is another area which has not radically changed since the NCF-2005 came into being. This is also one area which needs serious rethinking and research.

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10. Mathematics education research in India: Issues and challenges

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Background

Mathematics education research, the world over, has tried to study a variety of issues. These have included understanding construction of mathematical knowledge among students, teachers, various groups and communities; understanding how they acquire this mathematical knowledge and use it to think and organize their experiences or organize their teaching; understanding ways in which social, political, economic and ideological factors influence the curriculum, content, teaching and learning of mathematics and thus access to mathematics for all children/ students. Many studies have also been conducted to understand the relation between mathematics (as a discipline, its epistemology, history) and mathematics education. In this context, we see specific studies having been conducted with regard to issues related to the content of mathematics at different levels of education, processes involved in learning mathematics, and how different aspects of the classroom culture/process contribute to the acquisition of mathematical knowledge (Sierpinska and Kilpatrick, 1998). Mathematics education research has evolved over decades of research, starting from Thorndike and the many critiques of his theory which focused on meaningful and purposeful mathematics to complex research designs and multiple disciplinary frameworks in order to explore the issues impinging on teaching and learning of mathematics. Many theories have been adapted and modified from other disciplines and theories of teaching and learning have been formulated in mathematics education. However, this development has not reached or is not shared uniformly across the globe.

Mathematics education and research in mathematics education

The place of mathematics as a subject in the school curriculum in India has always been valued and the need for improvement in its content and pedagogy has been emphasized by various commissions on education constituted by the Government of India. The progressive tone that is seen elsewhere in the world is also visible in this country. Time and again a need has been expressed in these documents of taking mathematics beyond mechanical computations and focus on understanding basic principles. Kapur (1997) in the Fifth Survey of Education Research (1988-92) had pointed out that "the main object of mathematics education research is to be of help in improvement of classroom learning and teaching. It is therefore natural that a large number of studies should be concerned with different aspects of this problem". He went on to suggest various dimensions of this research, including study of effective teachers, instruction based on the use of computer aided and other technology, error analysis, styles of learning among different groups of students, remedial teaching methods, study of attitudes, socio-economic and other personal factors influencing learning. More than two decades later, many of these issues, in their true sense (in the way we understand the domain and purposes of mathematics education and research in mathematics education), are yet to be researched.

There have been many initiatives and interventions in the area of mathematics teaching and learning but they have not been documented and analyzed; nor do we have enough illustration of the understanding, thinking and reasoning of students participating in these initiatives (See Chapter 9, this volume). What we know about their success (or not so successful attempts) is more anecdotal than based on systematic investigation. However, their contribution to the mathematics education scene in the country is immense. They explicated a philosophy of teaching and learning based on an understanding of the child and his/ her capacities to learn and think. These earlier attempts and deliberations made possible new ways of thinking about teaching and learning of mathematics that are seen today and have become part of the National Curriculum Framework (NCF-2005) (National Council for Educational Research and Training [NCERT], 2005) and the new textbooks developed after this.

Typically, the departments of education in various state universities, some private universities and the NCERT along with the five Regional Institutes of Education (RIE) set up by the NCERT (to cater to specific regions and states of the country) are involved in carrying out research in the area of education, including subject specific research, like mathematics. In general, the mathematics departments (or any other discipline based department) of these universities are not involved in research in the area of mathematics *education* at any level. The education departments have traditionally taken on the task of in-service and pre-service teacher education and in this endeavour are isolated from other departments of the university, both structurally and intellectually. Each department may have one or just a few faculty members with mathematics as their area of specialization. These departments offer undergraduate (mandatory pre-service teacher education programme for teaching at the secondary school level), postgraduate degrees in education and pre-doctoral and doctoral level programmes, with an emphasis on educational studies. Largely, students graduating from the postgraduate programme take up a teaching or an administrative position in colleges, universities or schools. (There are many more avenues open today, with more and more state as well as commercial companies putting in large money in education.) Very few of these postgraduate students get trained in the departments for research in the area of education, and even fewer in mathematics education. The strict entry qualifications as well as mandates of the departments do not allow easy movement of students from departments of education to departments of mathematics and vice-versa at postgraduate and research levels.

Given this scenario, it is not difficult to understand that research in the area of mathematics education has not been a high point in the country. A large amount of research that has been carried out has focused on very traditional psychometric models. There is some evolution which research in mathematics education has gone through in the last decade and there is an attempt to study and explore some issues in the teaching and learning of mathematics. A synopsis of studies of both kinds will be presented in the next section.

Traditional research in mathematics education in India

A review of some of the educational abstracts and surveys of education (Buch, Joshi and National Council of Educational Research and Training, 1991; Kapur, 1997; NCERT, 1999) reveal the following trend in research in mathematics education.

A large number of research studies in India have focused on designing diagnostic tests in order to identify learning difficulties in some content area or the other and standardizing them. These studies in general are followed by remedial teaching; there are other studies which only design a remedial teaching approach after administering a suitable standardized test. However, the thesis, reports or abstracts available describing these attempts do not give very specific insights into the test items, the approach adopted for remedial teaching or illustration of students' understanding of any concept or idea. Largely, the tests led to the identification of difficulties students face in learning various ideas in mathematics, and the remedial teaching produced significant difference in students' performances.

- Exploratory studies tried to understand the factors responsible for poor achievement or failure in mathematics. The design of these studies generally included the use of standardized scales/ tests followed by statistical analysis to find possible correlates/ factors. The following factors were found:
    - Lack of pre-requisites
    - Difficulties with understanding the language
    - Difficulties due to certain kinds of teaching
    - Mathematization of verbal problems and the interpretation of solutions
    - Socio-economic factors

  - Intelligence, attitude, study habits, reasoning power, spatial visualization
- Studies also tried to list the kinds of errors which students commit while working on problems in certain areas.
- Studies explored reasons for mathematics anxiety and fear among students, language issues in mathematics teaching and learning, nature of mathematics and its pedagogical implication.
- Comparison of different methods of teaching, development of activities and kits for activity centred learning, functionality/ need of some policy (like the prescription of minimum levels of learning for each grade level) or some feature like the mathematics laboratory have also been studied.

This trend of research has been highly governed by psychometric designs and models of data analysis and has dominated the Indian mathematics education research scene. Some of the studies in this tradition are no doubt significant, but they are much limited by their methodologies and outlook. Thus, we have not been able to develop a deeper understanding of the issues in a way that is useful for designing policies on teaching and learning of mathematics or for designing the curriculum. Statistical significances achieved with standardized tests and scales are not always sufficient to grasp the complexity of the matter of study. We need to develop multiple theoretical frameworks and sufficient empirical basis to study and intervene in the problems of mathematics education. For example, we need to understand how say, socio-economic factors or language or reasoning influence teaching and learning of mathematics. What kinds of teacher actions, curriculum materials or classroom environment are responsible for discrimination against certain sections of society and leads to students' losing interest in the subject or dropping out of school? What is needed is increasing the depth, quality and variety of the studies conducted rather than large sample, survey kind of data on which certain statistical techniques are employed to derive conclusions.

New trends in mathematics education research in India

Over the years, however, we do see some change in this paradigm of research, especially in some studies after the year 2000. These studies/approaches are more analytical in nature, make efforts to understand the teaching or the learning process through designs of study other than experimental methods. Incidentally, many conferences focusing on science, technology and mathematics education have been held around this time in the country (e.g. CASTME-UNESCO-HBCSE International conference, February 2001, epiSTEME conferences held every two years by the Homi Bhabha Centre for Science Education, Mumbai), some of which have had an international character and gave an opportunity for researchers and practitioners to share ideas with a wider audience. Two conferences (First and Second conference on mathematical education in South Asia) held in the year 1956 and 1960 at the Tata Institute of Fundamental Research, Mumbai were precursors to this modern trend of research. These early conferences were attended by a large number of eminent mathematicians from across the world, including Hans Freudenthal. They were geared towards tertiary level and research in mathematics and the discussions revolved around the teaching of various branches of mathematics in the light of modern developments and hoped to provide an impetus to teachers of mathematics in South Asia. In the first conference, Freudenthal spoke about Realistic Mathematics Education. Having been hosted in the country's premier research institute in fundamental sciences, the deliberations and ideas generated in these conferences did not reach the wider society and universities in the country. It is only now that we are trying to catch up with the developments in mathematics education research worldwide. Some efforts have been made by individuals in the recent years to submit and present research papers in international conferences like the Psychology of Mathematics Education and the International Congress for Mathematical Education and submit research based articles in journals of repute. I will try to illustrate the nature and details of these studies in the following paragraphs.

Teaching and learning of mathematics in the elementary grades

In a large country as ours and the few research studies that have been conducted so far, elementary mathematics has been the most researched area. Groups or individuals have engaged with the crucial areas of elementary mathematics, both in designing intervention studies as well as conducting exploratory studies in order to understand different aspects of teaching and learning of mathematics.

Teaching interventions

Intervention studies have been conducted at the primary and the middle school level in order to increase students' understanding of particular areas of mathematics like number concepts and operations, geometry, fractions, algebra. Different groups of individuals have made the effort of designing alternative routes of learning these areas and simultaneously placed them in the context of trends in international research. The highlight of these studies is the detailed discussion of students' understanding of the concepts within the teaching-learning situations. Menon (2004, 2007, 2009) has been trying to explore possible alternatives to a variety of topics in primary mathematics curriculum like angles in geometry, place value and understanding of numbers and word problems. This series of studies is highly influenced by the Realistic Mathematics Education approach developed by the Freudenthal Institute (van den Heuvel-Panhuizen, 2001) with which Menon combines a Vygotskian perspective (Menon, 2009). In her efforts to develop an alternative geometry trajectory, she reported on grade 1 and 2 children's understanding of angles, which was found to be very much within their zone of proximal development. The author thus made a case for introducing such ideas much earlier in the formal curriculum than is done currently. In another study, she looked at the place and ways in which word problems are introduced in textbooks and argued for alternative approaches. She highlighted the use of word problems for mathematization of reality so that situation specific models can be developed as tools for mathematical reasoning. She has also argued for of moving away from an analysis on the place value understanding of numbers and developing number sense using the empty number line.

Fraction has been considered to be a difficult concept and arguments have been put for dropping the topic from the primary grades; its utility being very limited in daily life and largely taught for introducing rational numbers. This has been a motivation for a few groups to explore this area more conceptually, its relevance and its different interpretation so as to make a more meaningful teaching intervention possible. Subramanian and Verma (2009), Naik and Subramaniam (2008), Subramaniam and Naik (2007) give some glimpse of such studies. These studies use one or more interpretations of fraction (measure, share/ quotient, ratio, operator) and use these as models for building children's understanding in the primary school and illustrate their reasoning in various problem situations. These interpretations give meaning to the concept of fractions and the tasks are so designed that they make it relevant to children's lives. The studies show the relative ease with which children use these interpretations/ models to think about fractions and their effectiveness vis-à-vis the part-whole (a more static) interpretation of fractions. Sankaran, Sampath and Sivaswamy (2009) tried to build a computer aided tool using the part-whole interpretation for teaching fractions and based it on the cultural aspects of the learners (building a bead necklace of different colours, tiling an area). They concluded that such an approach which uses a familiar idea to teach difficult concepts is useful and helps enhance children's conceptual understanding.

Banerjee, Subramaniam and Naik (2008) describe the evolution of a teaching approach for beginning algebra and highlight the use of arithmetic in specific ways to help students make the transition. The study used concepts like equality, terms (additive components of the expression), expressions, and value of the expression in order to make the transition from arithmetic to beginning symbolic algebra. These concepts and ideas not only helped in understanding symbolic transformations in algebra but also enabled the students to reason about numerical and algebraic expressions, with respect to their value. Moreover, these ideas also allowed them a way to think about tasks which dealt with proving and generalizing. It was somewhat natural for the students to see that in order to complete these tasks they needed to make a representation (arithmetic/ and or algebraic) and that transformations on them would lead to a value or an equivalent expression, which bore relevance to the solution of the task.

Exploratory studies

A few exploratory studies on children's understanding of numbers and basic operations on them have been conducted. In one study, Khan (2004) explored three different groups of children's understanding of these, namely *paan* (betel leaf) sellers, newspaper vendors and children going to municipal schools in Delhi. The *paan* sellers, who sell a greater variety of articles at different prices than newspaper vendors, were the better of the three groups and there was not much difference in the performance of the other two groups. However, the interesting difference was in the manner the *paan* sellers and the newspaper vendors approached the problems and solved them in comparison to the school going children. These two former groups of children were able to understand the meaning and the import of the problems, had devised many alternative ways to work with numbers during their transactions but were limited by their inability to work with all kinds of numbers; the school going group was more concerned about correct answers, than methods, processes, reasoning. Another of her studies (Khan, 2008) explored children's acquisition of number concepts, with focus on their representation in a formal language. One of the important lessons to be learnt from this study is the complexity in learning this very basic concept in mathematics and how both language (in this case Hindi) and teachers' failure to appreciate the challenges involved make things difficult for children very early in their school life.

A study to increase cognitive capacities of children through what the author called "thinking mathematics" lessons has also been conducted (Chilakammari, 2001). It used the CAME model (cognitive acceleration in and through mathematics education) and the experimental study showed that children's learning was accelerated as a result of mathematics lessons based on thinking and sharing of ideas in the classroom.

Work in progress

Similar work around other concepts and areas of primary mathematics is visible now by individuals (the regional conferences and the national conference held under the auspices of National Initiative in Mathematics Education – NIME-2011-12 – give a glimpse of such work). Not all of these have taken the shape of fully fledged research but are initial attempts in the field of mathematics education and grappling with problems in the field. Many of these studies deal with exploring students' understanding of concepts in areas like numbers, fractions, geometry, algebra, measurement and probability and difficulties students face in learning them. Investigators are trying to design alternative ways/ trajectories which help students make sense of key concepts and procedures in various content areas (numbers and basic operations, arithmetic, geometry, algebra, measurement, etc.), explore classroom dynamics, relation between curriculum, textbooks, pedagogical practices and students' learning, and teachers' knowledge through students' mathematics. Attempts are being made to use particular theories (e.g. Lesh's model of multimodal representation) in designing teaching and learning tasks. Researchers are beginning to look at a crucial aspect of mathematics education, that of assessment, and its role in giving insights into students' learning and its possible use by teachers, designing of assessment tasks and its recording and communication. Studies on mathematics among communities and in the culture are also found. Government level initiatives have led to collaboration between some Indian mathematics educators and Swedish counterparts in the area of use of technology for mathematics teaching. These do give a ray of hope that mathematics education would find a place as a discipline and as a research domain in the country. However, these are a small number of people placed in select organizations and institutions (elite and better off institutions and organizations) and a large number of research studies in the education departments of mainstream universities still follow the old paradigm of research.

Higher secondary and higher education

Studies in the higher secondary level are even fewer. Parameswaran (2007a, 2007b, 2010) has been studying factors which influence high school students' understanding of abstract definitions like graphs or concepts like limits and infinitesimal quantities when they encounter these for the first time. One of the major factors which do not allow them to fully understand these complex ideas is students' prior learning of similar but more concrete ideas. Her study with expert mathematicians revealed the cognitive tool they employ to develop deep understanding of mathematical definitions (Parameswaran, 2010).

Efforts have also been made to systematically use e-learning platforms, graphic calculators, Computer Algebra Systems, and dynamic geometry softwares in the higher secondary classrooms to make mathematics interesting and provide challenging tasks to students and make their learning of abstract mathematics meaningful. Several small classroom experiments have been conducted with high school students in the designing of interesting projects (Ghosh, 2001, 2011; Asija, 2011, Kathuria, 2011).

Higher education has largely been a neglected area as far as mathematics education is concerned. The Indira Gandhi National Open University made some efforts to collect data on a bridge programme (the Bachelors Preparatory Programme, BPP) that they had launched to allow access to the undergraduate degree programme for people who had no formal high school leaving certificate. Feedback collected over a 3-year period suggested the need for curricular revision so as to meet the needs of the students (such as simplification of content, basic mathematics with emphasis on mathematical thinking and comfort with dealing with mathematical language) (Sinclair and Varma, 2001).

Teacher education

Teacher education is another area where there is a lot of scope for research. In India, few such studies exist, although many teacher interventions have been conceptualized and implemented. Rawool (2001, 2007) has been making efforts to document and analyze the effect of simulation of teaching practice on trainee teachers to learn to teach. The programme itself and the study tried to identify elements that are critical for helping trainee teachers to develop a theoretical framework about teaching and learning, finding motivations for student to engage in learning, designing and organizing non-traditional learning environments. This is one of the few reports that give some detailed account of thinking, reasoning and questioning that was happening among the group of trainee teachers. It tried to document the difficulties which these trainee teachers were facing, with respect to language, comprehension, their own comfort with content, etc., while participating in this pre-service programme.

Summing up trends in research

It is evident from the above description, that certain kinds of research studies have been undertaken so far. Groups or individuals have largely addressed content related issues, in the elementary grades. They have focused on exploring students' understanding of content areas covered in these grades or have designed intervention studies which help overcome the difficulties students face in learning them. Thus, both the psychometric and other designs of studies share a similarity with respect to the question(s) that have been addressed. As can be expected, the more recent studies attempt to report on students' reasoning and thinking within the situation of learning and illustrate the ways in which a particular teaching-learning situation impacts their learning, highlighting both the positive aspects and the challenges. This provides a rich set of data using which one can hope to design a curriculum, which have some theoretical and empirical basis and based on an understanding of the challenges likely to be faced in their implementation.

Further, there are hardly any research studies conducted at the secondary and senior secondary levels dealing with the issue of content and students' understanding of them. The few people who are involved at this level, have either tried to explore difficulties which students face while learning a new concept/ idea or have designed computer intensive environments in order to give students an opportunity to make sense of the mathematics they are learning. A deeper analysis of student work and their overall cognitive and affective gains in the later kind of studies would be useful for making a case for creating space for such components in the curricula.

Uncovering the gaps in research

In a country of the size and diversity of India where many socio-economic-political dynamics operate, the number of research studies in mathematics education is far too few to have any significant impact on either policy or on our own understanding of children's levels and their capacities. This is evident from the complete absence of reflection of such work in the new textbooks which were written post NCF-2005.

The above is coupled with the fact that our engagement with theories is minimal and is restricted to using one or the other theory for purposes of a study, leading to more descriptive reports than analytic ones. These theories come from various disciplinary frameworks of the social sciences as well as there are theories generated within mathematics education. The separation of education from social sciences, and making education almost like a collection of skills, increases the chances of theories being mechanically used rather than engaged with critically. The object of study being very complex, research studies in the "progressive" trend would gain by a deeper engagement with theory and bringing in some rigour in the methodology used. It is possibly this reason, that these studies do not find themselves readily used for making policy or curricular decisions.

There is also a need to engage with deeper issues of teaching and learning of mathematics at all levels and not only work in the broad areas of content of mathematics, as has been the case till now. These include issue related to representations and symbols in mathematics, meaning making, language issues, reasoning, argumentation and proving, use of technology, understanding classroom cultures, teacher education, socio-political-economic questions and its impact on mathematics education, affect and mathematics teaching and learning, and assessment. It is these issues which impinge directly on the framework we choose for designing studies or conception of a curriculum. The research studies have largely been silent on these issues or at least have not addressed them directly.

Very little attention has been paid to research in teacher education, which is one of the key areas of research in mathematics education. Much of curriculum reform or any change in teaching and learning of mathematics would be difficult to actualize as we do not clearly understand what aspects of the teachers' personality and their academic achievements are responsible for being able to teach effectively mathematics that is considered important for students to learn. Documentation of efforts made for professional development of teachers is essential to envisage a new design of curriculum and teacher preparation programme as well as in-service teacher development. Research must be able to explore characteristics of "good" teachers, its relation with their prior qualification, teachers' pedagogical content knowledge, their beliefs, attitudes towards teaching, learning and mathematics. Moreover, it is through research that we would be able to get a glimpse of teachers' lives – their daily practice, the experiences that shape their practices (both academic and non-academic decisions in the classroom), their own professional development with increasing experience as a teacher.

Issues and challenges and ways ahead

Systemic issues

One of the major challenges that India faces in developing mathematics education as a research area is the lack of systemic support. The university departments and colleges of education have not been able to provide the space and support for establishing traditions of content specific and subject specific research with sound theoretical frameworks and well designed empirical studies. This is also due to the fact that education departments are isolated from departments of subject disciplines, like mathematics, which can provide inputs on the content aspect in mathematics education research. Thus, even when lot of research studies have been conducted in the area of education (using the frameworks and theories in education), few have been conducted in mathematics education. The studies conducted in the departments still follow the traditional style of research with focus on psychometric designs, studies on lists of errors in different areas, studies in the area of fear and anxiety in mathematics, nature of mathematics and its pedagogical implication. In the recent years, one can see some change in these departments as well with some titles moving beyond the ones mentioned earlier (like, making sense of the classroom environment, teacher practice). However, the issues being very complex, they often lack theoretical and methodological rigour. Also, students who undertake the masters dissertation, which has a narrow scope, do not often get enough opportunities, despite their motivation and interest, to continue their work and contribute to the field. What one needs is to make these places rich in both print and human resources and students should be exposed to career opportunities in research in the area early on.

The studies mentioned in the discussion earlier have come from a few individuals, institutions or non-government organizations. There is an acute shortage of experienced faculty/ researchers across the country, who can take up research issues in mathematics education as well as contribute effectively to teacher education programmes. There is no systemic structure to support and strengthen such work. This becomes a vicious cycle – the fact that there are few researchers in the area makes it difficult to establish departments specific to mathematics education which can conduct research studies in the area and the lack of such departments makes it impossible to produce researchers. It is largely personal interest and motivation which drives individuals to do research in the area.

Issues related to diversity and complexity of the subject matter

In the Indian context, questions of meaning, symbols, classroom environment, technology and access to quality education are very important, given the diversity that exist across the country, in terms of physical and human resources and culture and context of people.

The debate about meaningfulness of the mathematics learnt or the mathematical activity has taken several twists and turns. Not so long ago, learning mathematics was thought to be meaningful as it provided access to prestigious professional or academic career. All children went through a certain kind of mathematics for ten years[1], which prepared them successively for the next stage, keeping the end in mind (mathematics for professionals or mathematicians). In some time, it was found that most children do not succeed in this endeavour and are therefore not able to gain through their mathematics learning/ teaching, but develop anxiety, fear, lack of confidence and hopelessness. Thus, the need arose for rethinking the mathematics curriculum. The root cause of this failure among students was identified to be the meaninglessness of the mathematical activity. The first effort to infuse meaning, through large scale governmental and non-governmental initiatives, was to increase activities, games, concrete materials and word problems (signifying application of mathematics to real world) in the mathematics classroom. Although one found overall gains in attitude and confidence of children who participated in them, no systematic attempts were made to collect data of students' learning of mathematics in these situations. All the same, many felt that this mathematics too was not very useful and the applications in terms of word problems were rather contrived. Thus, another attempt to make mathematics meaningful came by embedding mathematics in real world contexts of children and thereby engaging with critical theories and pedagogies of education (seen in the new textbooks written by NCERT).

In the process of taking any of these policy decisions, we need a more nuanced understanding of what "meaning" is – what types of activities can be considered to be meaningful, what cannot be meaningful, positioning and sequencing of these activities, emphasis on different aspects of mathematics, etc. What role do concepts and symbols play in this process? How does understanding of symbols progress? What is the relation between concepts, procedures and symbols in mathematics? What kind of classroom cultures enable children to not only make sense of the mathematics that is a lived reality for them but also transcend it and move into the world of mathematics, deal with the abstractions? What is the role of communication, reasoning and argumentation in the classroom? What strategies can be evolved to take this forward and learn to communicate in the language of mathematics, for example, writing proofs? In what ways can technology help teaching in the classroom? What is the role of the teacher in such a classroom? What kind of knowledge and preparation should the teacher have? What challenges and issues arise while implementing any of the above in the classroom?

Thus, the process of arriving at a resolution for the idea of "meaning" in mathematics is a complicated one. It immediately gets intertwined with many other things, including questions of access and quality education. If relevance or context is given priority over

[1] Mathematics has been a mandatory subject for all school going children till grade ten across the country for several decades.

ideas to be learnt in a mathematics lesson, then we have to explore the extent to which an equitable access to mathematics learning can be provided, given the diversity in the country. How would this impact their later learning? What vision do we have of a mathematics learner who would exit school after grades 8, 10 or 12? Can this be different for different kinds of learners? At what grade level can we start differentiating between students? Is it possible to completely remove considerations of the long term ends or directions students may choose to take?

These are certain issues and questions which mathematics education research must illuminate. These cut across different levels of schooling and different content areas within mathematics, thus are broad and overarching. Debates and answers to these issues are important to be able to make the kind of decisions that are taken while formulating a new curriculum or other policy decisions. Research studies undertaken must be able to address one or more of these issues, directly or indirectly. It is only on the basis of theoretical and empirical research dealing with critical issues such as these that we will be able to make more informed policies and be able to better analyse our existing policies.

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