# Drawing from Cognitive Studies of Mathematical Learning for Curriculum Design

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Abstract: *One of the problems that has engaged the attention of mathematics educators is how to facilitate learners in making sense of symbolic mathematics. Meaning for symbols, and warrant for reasoning about representations in mathematics, is drawn from different sources of control, which I broadly classify as semantic (real world referents), syntactic (rules and procedures) and structural (translations between representations). Researchers have explored in detail how to enrich semantic sources of control for reasoning about symbolism. Recent research on mathematical representations however has pointed to the importance of structure as a source of control. A long range view of different topics in elementary mathematics - whole numbers, fractions and beginning algebra - shows the importance of structural understanding. I provide an analysis of students' understanding of the domain of whole numbers drawing from available literature. Following this, I indicate briefly how by drawing on students' understanding of whole numbers, structural control on symbolic mathematics can be provided for while designing the curriculum in the topic areas of fractions and beginning algebra.*

## Introduction

Cognitive studies of students' learning of mathematics have steadily accumulated findings for some decades now. By cognitive studies, I mean studies that probe students' understanding and the change in this understanding as learning takes place. The category includes studies of student errors and misconceptions, theories that explain student conceptions, studies that examine students' response to instruction and micro-genetic studies of changes in strategies used to solve specific types of problems. In this article I want to examine the implications such studies have for curriculum and instructional design of specific areas of school mathematics education. In doing so, I will not be describing new trends in mathematics education research, as much as revisiting some older trends. I believe that the studies that I shall refer to, some of them from two decades ago, hold important implications for instruction that are yet to be elaborated. Indeed, some recent teaching studies have drawn insights from the earlier body of work. Such studies are difficult to implement and do not always have clear generalizable results. However, they push the understanding of the domain further, and a clearer and more refined understanding of topic domains, especially when shared with the community of practitioners, contributes to improving teaching and learning.

A recent review article attempts to comprehensively examine the implications of cognitive studies, as described above, for the domain of whole number knowledge (De Corte and Verschaffel, 2006). The article develops a four-part framework to draw implications from the findings of research studies for designing instruction. The four components discussed are (i) Competence (ii) Learning (iii) Instructional environments and (iv) Assessment. Some of the implications discussed by De Corte and Verschaffel generalize beyond the domain of whole number arithmetic, especially those concerning the learning process or the instructional setting. For example, the principle that learning is situated and collaborative is valid for learning any topic in mathematics or the learning of other school subjects. In this article, I shall not address such issues in any detail. Rather I shall take a more domain-centric view, and examine how studies of learning help to illuminate the structure of the domain from the viewpoint of teaching and learning. Hence of the four components listed above, my focus will be on the first, namely, competence. Here I shall seek to go beyond whole number knowledge, but still restrict myself to elementary mathematics.

A part of the aim in this article will be to study multiple domains to extract common features and also to become aware of important differences. The domains I will focus on are whole number arithmetic, rational numbers and beginning algebra. While each of these domains have been studied in great depth and detail over the last few decades, attempts to systematically compare the findings across these domains have been few. New insights are likely to arise from such a comparison; in this article I shall take some initial steps in this direction. As may be expected, one will need to go beyond empirical findings attached to specific tasks or concepts in order to obtain a more general perspective. At the same time, it is necessary not to restrict oneself to theories that are too general in scope, since instructional design has to concern itself with the details of a particular domain. Hence middle level theories will recieve more attention. In particular, I will try to interconnect conceptual and semantic theories of the domains of whole number arithmetic, rational numbers and algebra.

Researchers in mathematics education are paying increasing attention to the role of representations in mathematics learning. Mathematics is distinctive in that the objects of discourse are not accessible to direct perception or through instrumentation. The objects of mathematics have been described as virtual and as being constituted by the representations and the way representations are used in mathematical discourse (Sfard, 2000). Researchers have attempted to systematically outline a theory of the way representations function in mathematics and in the learning of mathematics (Goldin and Kaput, 1996; Duval, 2006). A careful attention to representations is certainly an important part of curriculum design. Hence some ideas and concepts introduced by these theoretical studies will inform the discussion below.

Besides natural language, humans use a wide range of representations (or ‘signs’ to use the terminology of semiotics) for communication and as an aid to remembering and thinking. A majority of non-linguistic signs in human cultures, which include icons, diagrams, certain kinds of inscriptions, gestures and tokens, have a visual or spatial aspect. These signs have a variety of functions. Some of them point to or call attention to something. Some stand for or represent other objects, and some add emphasis or tone to communication. Each function may be performed in a variety of ways. For example, representing can be depictive, iconic or metaphorical (Goldin-Meadow, 2006).

The use of one kind of representation, namely written symbols is salient to children in their experience of school mathematics. This may be a consequence of the fact that current pedagogical practices lay undue emphasis on symbol manipulation. However, learning to interpret and use symbols is an inescapable part of modern mathematics. The development of powerful mathematics became possible as increasingly efficient symbolism was invented. This is because in mathematics inscribed symbols have, apart from the functions of representations discussed above, an important additional function, namely, computation. When a problem is mathematized and represented with symbols, the solution proceeds usually by computing with symbols. Computing with symbols involves transforming a visual array of (written) symbols in specified ways to yield other interpretable symbols. The transformations are usually multiple transformations chained into a sequence. The computational and representational functions of mathematical symbols are often in tension. Preoccupation with one function may obscure the other. By virtue of the fact that computational processes in mathematics typically act directly on symbols (usually) inscribed on paper, symbols acquire an object like character. In the course of computation, symbols direct attention away from their possible referents and seemingly empty themselves of meaning (Arzarello et al., 2001). This leads to one of the central problems of mathematics pedagogy that has engaged researchers and curriculum planners – to restore meaning making as an integral part of learning mathematics.

Making sense or meaning is related to the assurance with which inferences are made. Sources of meaning are also at the same time sources of warrant for the inferences that one draws, or in the words of some mathematics educators, sources of control (Balacheff, 2001). There are multiple sources of meaning that can and must be drawn upon to make sense of mathematics. To take an example discussed by Sfard (2000), how does one know that $\frac{2}{3}$ and $\frac{12}{18}$ are the same (equal)? One can produce a diagram that shows how 2 parts out of 3 is the same as 12 parts out of 18. One can explore the situations where 2 cakes are shared equally among 3 children, and where 12 cakes are shared equally among 18 children, and discover that the children in both the situations recieve the same share. One may apply a rule, of multiplying the numerator and denominator in $\frac{2}{3}$ by the same number 6, and obtain $\frac{12}{18}$. For an individual student, each of these sources of meaning lead to different strengths of belief or assurance. Moreover, the sense of 'being the same' or 'being equal' in each of these situations is different. There is a danger that the use of multiple sources for meaning making can result in understanding becoming fragmented rather than integrated. Instructional design must not only ensure that there are opportunities to students to use these multiple sources, but must also allow students to integrate these different sources by noticing and drawing parallels between the different situations. Hence analogical reasoning has an important place in learning mathematics.

One can distinguish three sources of meaning or control for symbolic mathematics, while remembering that such distinctions will at times prove inadequate in dealing with the complex processes of reasoning and understanding. One source is the real world, which includes both physical or material objects and cultural objects belonging to the sphere of economics. The most common control of this kind comes from interpreting mathematical symbols to stand for quantities and operations with quantities. We will call this source of meaning and control 'semantic'. Another source of meaning or control is the set of rules that govern the transformation of symbols, which we will describe as 'syntactic'. A third kind of control emerges from drawing correspondences and analogies between different mathematical representations, which we will term 'structural'.

As students progress with learning mathematics, the relative importance of the different sources of control will change. As more diverse representations become accessible, one would expect reasoning to take on a more structural character. A preference for syntactic control may be a result of instructional exposure, or of extent of practice. We have come across instances where students who were exposed to concrete representations of the base ten units after years of unsuccessful learning of multi-digit algorithms for whole numbers, still preferred to check their inferences using algorithms rather than concrete representations. For example, to be sure of the result of adding 3 hundreds to 4 hundreds, they would set up and implement the vertical column addition algorithm.

Briefly speaking, when reasoning in a source domain exercises control over a target domain, the possibility and the extent of control depend on how well understood the source domain is and the richness of the connection between the source and the target domains. As several analyses have pointed out translating between representational domains can strengthen the understanding of not only the target domain, but also of the source domain. One of the claims made in this article is that students' understanding of whole numbers is a robust and continuing source of control for later learning. The curriculum designer is often faced with the challenge of how to allow for the possibility of such control. Since numbers are not objects in the world, but mentally and discursively constructed entities, in the classification that I have presented above, it would be appropriate to call this form of control 'structural'.

## Understanding Whole Number Arithmetic

Small counting numbers are among the simplest and the most 'natural' of mathematical concepts acquired by children. (By 'counting numbers' or 'whole numbers', I mean the positive integers.) This is attested by the fact that many cultures have developed counting numbers. What accounts for the ease or 'naturalness' of the counting number concept? One explanation may be found in the accounts by developmental psychologists. Firstly, the perception of numerosity may have an innate basis that is shared with many mammals and other animals. Many animals can distinguish between collections of objects that differ in number, as long as the numbers are small, or the differences are large. Human infants also display this ability. Counting acts are readily imitated by young children and may be a part of the growing language capability. Children reflect an understanding of the principles governing the language game of counting surprisingly early (Gelman and Gallistel, 1978). However, whether this has a link with innate conceptual preparedness to perceive numerosities is not clear. Children interestingly fail to use their counting abilities, which may be well developed, to make inferences about quantity in comparison tasks. The Piagetian number conservation task also elicits failures from young children indicating that it takes time to align their ability at verbal counting with their understanding of cardinality (Sophian and Kailihiwa, 1998).

Another explanation for the ease of the counting number concept may be found in the nature of the mental representations that constitute the number concept. There are suggestive findings from neuroscience research that number representations in the brain may reflect spatial organization akin to a mental number line (Izard and Dehaene, 2007). There are also converging analyses of students' developing competence in the domain of small whole numbers, which take into account the detailed findings of empirical studies (Fuson, 1992; Steffe, 2002). In the act of counting, number words function like symbolic tokens that are put in one-to-one correspondence with objects. The fact that the sequence of number words has a natural correspondence with a set of objects attended to sequentially, facilitates the consolidation of the internal representation of number. Indeed, spoken words are not externalized representations that have an enduring presence like material objects or inscriptions. Initially the number words are used in the presence of objects, then in the presence of symbolic objects like the fingers or counters, and finally the number words themselves serve as objects to be counted (Steffe, 2002, p.269). Reflection on the experience of counting and of adding and subtracting small numbers leads to the emergence of structure in the mental representation for number words. One fundamental change is the development of the cardinal structure on top of the ordinal structure of the number words, signified at first by the count-to-cardinal and the cardinal-to-count transitions and by the gradual development of a nested, breakable, countable sequence of numbers (Fuson, 1992). The development of such a representation of number is postulated to underlie the growing competence of the child in counting, and in addition and subtraction tasks. For sophisticated counting strategies such as counting up from a given number or counting up by a given number to emerge, the mental representation of the number sequence must be available as an object on which actions can be performed. These actions include the partitioning of the sequence into fragments or joining fragments of the sequence to form new sequences. Such actions form the basis for children's developing strategies to deal with unitary, as opposed to multidigit addition and subtraction operations (Fuson, 1992).

Analyses such as the above indicate that counting, unitary addition and subtraction are carried out by actions on internal mental representations that are analogous to actions on objects. Moreover, the analogies between actions on objects and on elements of the number sequence are direct and simple. Such internal representations function both as symbols, and as objects. Since they can be produced at will, they have an immediacy that is not a feature of external representations. This provides students with a strong source of meaning and a ground for assurance, atleast in dealing with small numbers.

Extending the secure understanding of the number concept beyond the first few numbers depends on understanding the decadal structure of the counting numbers. Internalizing this structure for increasingly large numbers takes place in steps, which constitute major cognitive achievements. Beyond the first ten or twenty numbers, the sequence must be continued on the basis of the decadal patterns implicit in the spoken number words. The complexity of this pattern varies across languages. Beyond hundred, a fairly regular decimal pattern takes over counting. Initial experiences of imitating or using the decadal and decimal patterns may be in the spirit of the language game of counting. Soon, cultural necessity drives and reinforces the identification of the decimal units in spoken number words with the concrete embodiments in the form of monetary currency. The syntax of the spoken number words in the language of instruction and the extent to which the children's lives are integrated into the monetary economy are both likely to influence their learning of base ten numbers in the early years of schooling.

The generalization of the number concept to larger numbers requires different acts of integration. Firstly, individual counting acts only cover fragments of the number sequence; integration of these acts takes place by inserting the fragments in their right positions. Such integration is guided by the decadal and decimal structure of numbers. Secondly, the decimal structure also provides a compression of the counting acts and facilitates a generalization by analogy. One can thus count in jumps of ten by counting the tens as though they were units. Both forms of integration facilitate the development of the representation of increasingly large number sequences. Sequential structures presumably develop in stages for clusters of the powers of ten, and is generalized beyond them again by analogy. The number names therefore show a periodicity in addition to the base structure. The thousands for example are counted in tens and hundreds, as though they were a new unit. One must note that the counting of higher units signifies an advanced understanding of multiplication, although it is still implicit. Over time, a generalized linear representation of the number sequence that incorporates the decimal structure develops and remains into adulthood. This is described by researchers as the sequence or 'positioning' understanding of numbers (Fuson, 1992; Treffers, 2001).

The sequence understanding is different from the structural understanding in which the number is composed through addition, subtraction or multiplication operations. Again this is guided by the base ten structure which facilitates the decomposition and recomposition of a number on the basis of the numbered and named decimal units. The structural understanding of multiunit numbers is an operational rather than a counting concept. It is consequent upon experiences with the operations of addition, subtraction, quantitative comparison and estimation, which call for composing and decomposing a number. Further, dealing with multiunits extends the implicit understanding of the multiplication operation. The number words use linguistic cues to code the additive and multiplicative composition of a number. In the number 'two thousand and twenty two', we have 'two [$\times$] thousand and [+] twenty [$2 \times 10$] [+] two'. We notice that 'and' is used between 'thousand' and 'twenty' to indicate addition, while addition is implicitly indicated by juxtaposing 'twenty' and 'two'. On the other hand juxtaposition of two and thousand implicitly indicates multiplication. Despite these inconsistencies, children learn to distinguish multiplicative and additive composition, presumably by using a combination of cues such as, linguistic modifiers, sequence, stress or rhythm in vocalisation. The pattern of cues changes and becomes more consistent as the units become larger.

For example, in Tamil, one of the major languages of Southern India, the additive and multiplicative composition is distinguished by linguistic markers. The word for 'twenty' in Tamil is *iruvathu* which is a modified form of *iru pathu*, which literally means 'two tens'. The word for 'twenty two' in Tamil is *Iruvathi erendu*. The multiplicative composition embedded in 'twenty' (two tens) is indicated by prefixing to the word for 'ten' the adjectival form of the word for 'two' – *erendu* changed to *iru*. The additive composition implied by juxtaposing 'two' after ‘twenty’ is indicated in Tamil by the vowel suffix *ee*. The suffix *ee* for additive composition is consistently followed even when bigger units are introduced: *Erendayirathi irunoothi iruvathi erendu* for ‘two thousand two hundred and twenty two’. Notice that the linguistic marker for multiplicative composition remains for the hundreds, but disappears for the thousands. In English, while the tens are modified to indicate multiplicative composition, for the hundreds and beyond, multiplicative composition is indicated by juxtaposition, as is additive composition. It is not clear if such language differences have instructional implications although it cannot be ruled out. After all these linguistic markers may have arisen in order to sensitize native speakers to the compositions embedded in number names. In many Indian languages, these modifiers appear in the way multiplication tables are recited, including in the multiplication tables for fractions[1].

We have chosen an example from an Indian langague that shows a relativel regular number word pattern, which is moreover transparent with regard to decimal structure. North Indian languages are less regular in this respect. The point of this linguistic exegesis however is that implicit understanding of additive and multiplicative composition is already in place when children begin to understand larger numbers. This implicit understanding is not often exploited in generalizing the understanding of the composition of number through different units. Most instructional approaches may focus on familiarising students with number names and consolidating their sequential understanding.

The complex understanding of the decimal composition of numbers receives support from the culture in the form of the base ten multiunits for monetary currency and other decimal measures. Monetary value is of course an abstract measure not always appropriate for young children. Additional pedagogical support for the multiunits through various concrete embodiments help children grasp the structure of numbers. Indeed studies of students’ learning of decimal numbers support the effectiveness of supporting constructs both for sequential and for structural understanding of multiunit numbers (Fuson, 1992; Gravemeijer and Stephan, 2002).

The positional encoding present in the base ten numeration system imposes a different order of difficulty. Here not only must children know the multiunit composition of numbers, but must decode the positional cues that indicate how many of each unit is present. Although this fact is well recognized, the distinction between the grouping and the positional principle is often obscured in designing curricular sequences, with both principles being conflated into a ‘place value’ concept. Many historical numeration systems such as the Egyptian, exhibit the grouping but not the positional principle. Children need to master both these principles. The decoding of the positional cues may happen in two ways, the first by translating the numeral into the number word, which explicit names the different units. However, children also need to be able to translate the numeral into other representations of the units familiar to them. This is especially important in making sense of the procedures that take advantage of the positional numerals.

With the positional numerals, children also first encounter the use of the written symbols in computation. Positional numerals have evolved so that simplified operational procedures

[1] In Tamil Nadu, traditional schools taught multiplication tables of fractions, a practice which was prevalent till perhaps the early twentieth Century (Babu, 2004)

can be constructed which consist of visually guided routines operating on inscriptions. In this respect, the positional numerals are similar to algebra and embody syntactic compressions akin to those present in algebra. This simplification of procedures comes at the cost of obscuring the operational composition of the number. However the numerals are different from algebraic notation in the compositional form being fixed and in the fact that language and culture support the unpacking of the compositional structure. For most educated adults this is internalized so well that the symbol '2736' is paradoxically seen as being more transparent than the full compositional form $2 \times 1000 + 7 \times 100 + 3 \times 10 + 6$. For some students of course, the compressed positional notation is a persistent source of difficulty.

To summarize, several factors combine to make the whole numbers more accessible to children in comparison to other mathematical concepts. From the viewpoint of curriculum design it is important to understand these factors in detail also as a preparation for the challenges that lie in the teaching and learning of later, more advanced concepts. As discussed, whole number learning already embodies complex cognitive accomplishments, which can function as valuable resources to be drawn upon by the curriculum designer. In studies that we have carried out, we have attempted to exploit the understanding of whole numbers to build pathways to the learning of rational numbers and of beginning algebra, which I shall briefly indicate in the respective concluding parts of the next two sections.

## Working with Fractions

Unlike in the case of whole numbers, most children fail to form a well developed and consistent representation of fractions. Several causes may underlie this failure. One important factor is that while whole numbers are used extensively in modern cultures, fractions with arbitrary denominators are hardly ever used. Measurement contexts bring forth only the decimal fractions, which extend the base ten system used for whole numbers. Even historically, measurement needs have been met by using only a subset of the fractions if at all. (The binary fractions used in the British system are an example). Many cultures avoided dealing with fractions by introducing new sub-units. In the present day world of commerce, a part of a whole (tax, interest, discount) is almost always described in percentage. Hence these dominant everyday contexts do not require fractions with arbitrary denominators as they are learnt in school.

What is the rationale then for teaching and learning fractions in school? The most important rationale is that fractions give notational and conceptual access to dealing with proportionality. Linear functions are ubiquitous and understanding such functions is an important goal of school mathematics. In solving problems dealing with proportional relationships expressed by the linear function $y = kx$, the need arises for inverting the relation to obtain $x = \frac{y}{k}$ or to obtain the rate $k = \frac{y}{x}$, which may lead to fractions. Moreover, in such situations one may obtain the measures for $x$ or $y$ as fractions. Students need to deal with these different possibilities with understanding. Problems involving the comparison of ratios also implicity use the notion of proportionality and require similar operations.

Researchers who seek to explain the difficulty of the fraction concept usually adopt one of two broad perspectives. The first views children's familiarity with whole numbers as hampering the learning of fractions. Procedures taken over from the domain of whole numbers are applied wrongly to operations with fractions. Even the fraction symbol is interpreted as being composed of two whole numbers. A more recent variation of this approach is the application of the conceptual change perspective borrowed from science learning (Vosniadou et al., 2007). In the conceptual change framework, difficulty in learning a new concept arises from the fact that it is in conflict with a robust conceptual structure or theory that is already in place. Children frequently respond to this conflict by accomodating the new concept, or new data, within the framework of the old concept, leading in many cases to 'synthetic' or 'hybrid' conceptions. Indeed for children who have been learning whole numbers over a period of a few years, fractions present new rules and relationships, which conflict with the whole number framework. Stafylidou and Vosniadou (2004) present a list of important elaments of this conflict which include differences in symbolization, ordering, the nature of the unit and the procedures for operating with fractions.

The conceptual change approach to students' understanding of fractions is useful in focusing attention on the precise differences between the whole number and fraction conceptual frameworks, differences that students fail to internalize even after years of instruction. This approach has so far framed its research in terms of an integrated, mature concept of rational number that is close to the formal mathematical concept. Rational numbers provide the new conceptual framework, which many students fail to absorb to varying degrees. However this approach sidesteps the findings that have accumulated on how students learn to make sense of fractions.[2] Curriculum framers and teachers seek to introduce fractions in ways that are meaningful even in the initial encounters. It is through a series of subtle changes in perspective, interpretation and with the learning of new notation and new ways of using a notation that the student begins to get a hold on the new concept.

The second approach to understanding the difficulty students have with fractions is more sensitive to the interpetative changes that students need to make as they begin to work with fractions. This approach seeks to explore the different ways in which students encounter and make sense of fractions. One widely accepted theory is the subconstruct theory formulated by Kieren (1980), which attributes the difficulty children have with learning fractions to the fact that the fraction concept consists of several subconstructs that are cognitively distinct. The widely favoured view is that five subconstructs of fractions are important, of which four are measure, ratio, quotient and operator. The fifth subconstruct, the one that students usually encounter first, is the part-whole subconstruct, which is linked to all the other subconstructs (Behr and Fuson, 1992; Charalambous and Pitta-Pantazi, 2007). Hence although the fraction symbol denotes one mathematical object, namely, a rational number, the same symbol may be interpreted in these different ways when we apply the notion to different situations. Kieren postulated that subconstructs are mental entities that are more integrative than schemes, and are hence further up in the theoretical hierarchy in comparison to schemes. One may think of them as a set of linked external and internal representations along with transformations on the

[2] See the criticism by Behr et al. (1993) that, from the teaching-learning point of view, the notion of a rational number as an element of an infinite quotient field is overly simplistic.

representations. Fundamental to the idea of a subconstruct is the integration of representation with the situations in which the representations are invoked. Empirical studies have found that the subconstructs may be acquired relatively independently of one another (Kieren, 1993) and indicate that the extent to which various subconstructs are understood may be an outcome of the instruction students are exposed to (Charalambous and Pitta-Pantazi, 2007).

If subconstructs do play an important role in learning, then several questions arise both from the viewpoint of the adequacy of the theory and from the viewpoint of curriculum design. First, are subconstructs peculiar to fractions? Can one sensibly claim for instance that the number concept also consists of different subconstructs? Possible candidates for such subconstructs are the cardinal, ordinal and labelling uses of number, which lead to distinct structures and properties. One could also point to the various interpretations of the addition and subtraction operations in the combine, change and compare situations, which children appear to think of in distinct ways. Analogous examples may be found in other topics as well – arithmetic expressions may signify both instructions to carry out a process and the result of that process. One might argue of course, that in comparison to these examples fractions exhibit a more rampant polysemy. Alternatively the fact that different fraction subconstructs seem difficult to reconcile may indicate that current conceptualizations of fraction instruction are inadequate precisely because they fail to integrate the different subconstructs at the conceptual level, unlike in the case of other topic areas, where the different meanings are better integrated.

From the viewpoint of the curriculum designer, the subconstruct theory raises another set of questions. Are all subconstructs important, or can children do with only some? This issue has only recently begun to be investigated systematically (Moseley, 2005). What is the sequence in which children best learn the different subconstructs? How do the different subconstructs become integrated into one unified concept of fraction or rational number? What role do different representations play in the construction of these concepts?

Traditional curricula begin with applying the part-whole subconstruct in introductory fraction activity through the use of the area model. Even though the use of the fraction notation to indicate ratios and the division operation is introduced in the traditional approach, it is the part-whole interpretation that remains dominant throughout the treatment of fractions. The limitations of this approach have been pointed out by several researchers, including the criticism that it does not induce students to move out of whole number conceptions (See, for example, Kieren, 1993). Fractions become important not in the context of counting, but in the context of measuring, while activities centred around the part-whole subconstruct restrict themselves to counting and comparing the number of parts. The importance of the fraction symbol lies in the fact that it expresses the flexible construction of units from a given unit. It uses a common notation for the two fundamental processes operating in the activity of measurement: the creation of sub-units by equal partitioning (described as the splitting scheme by Confrey, 1994), and the iteration of a unit. In other words, it provides an integrated notation for multiplication and for division. Vergnaud (cited in Behr et al. 1992) has pointed out the fruitfulness of understanding the fraction symbol as denoting the concatenation of the multiplication and the division operations. This idea is embodied in the operator notion of fractions, which has been analysed in detail by Behr et al. (1993). In our work with middle grade students, we have used the operator interpretation of fraction as an integrative construct, that brings together and integrates students' experiences with other fraction constructs and lays the foundation for the powerful use of fractions in understanding proportionality.

It is productive to compare the fraction notation with the notation for whole numbers. Like the latter, the fraction notation also embodies the composition of the number denoted, but unlike the notation for whole numbers, it embodies a fundamental ambiguity. The fraction notation can be read as embodying the multiplication operation – a whole number times a unit fraction. The unit fraction is the new unit, constructed by dividing the base unit: $\frac{m}{n} = m \times \frac{1}{n}$. The fraction can also be read as embodying a division operation: $\frac{m}{n} = m \div n$ or $\frac{m}{n} = \frac{1}{n} \times m$, that is, taking a $n^{th}$ part of $m$. This ambiguity arises because of the assymetrical interpretation of the factors in multiplication: $\frac{1}{n}$ as unit or as operator. (See Vergnaud (1988) for more details on this ambiguity and ensuing difficulties.) An important step in the understanding of fractions or rational numbers is the integration of these two interpretations. In a teaching approach that we have adopted students explicitly study the equivalence of the two interpretations of fraction by comparing the interpretation of fractions as a measure and as a quotient. An example of such a task is shown in Figure 1. In the figure, the student compares the quotient and measure representations for the fraction $\frac{3}{8}$. The caption above the left column says 'each child's share' and the one above the right column says '(by) unit fraction'. The student has mistakenly drawn only 7 stick figures instead of 8. (For more details see Naik and Subramaniam, forthcoming.) In this example, the students reason by translating between the representations. The control however, is largely semantic, and arises from their understanding of the equal sharing situation.

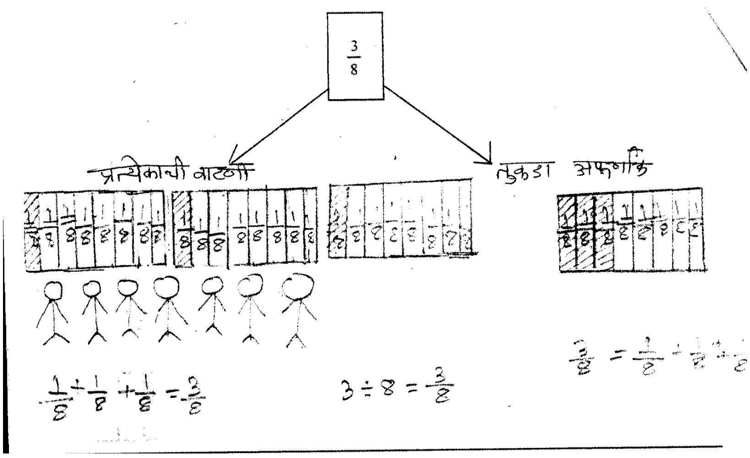


Figure 1: Combining the measure and quotient interpretation of fraction

In our approach the integration of the operator and ratio subconstructs is achieved by interpreting ratio in terms of the operator. Developing the operator construct relies on the fact that students have an intuitive understanding of the multiplicative relation between whole numbers. This intuition can be realized with the resources they already have only when one number is a (small) integral multiple of the other. Moreover, students are unaware of how to designate the inverse multiplication relation (that is, when the target number is a factor of the other). The fraction notation makes this possible. Figure 2 shows how students make sense of the forward and inverse multiplicative relationships. This paves the way for the realization that fractions allow one to represent the multiplicative relation between any two numbers. Students' progress in understanding the multiplier/operator notion is guided by their prior knowledge of whole numbers. Hence the form of control that is exercised here is primarily structural.

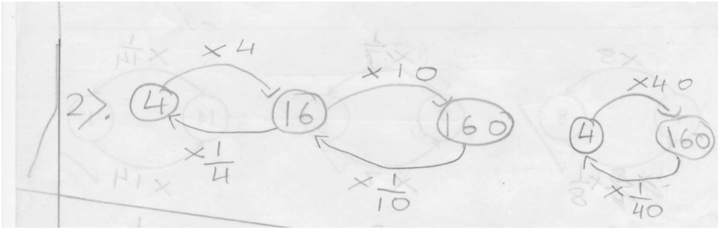


Figure 2: Fraction as operator: multiplicative relation between numbers

We postulate that an implicit grasp of the multiplicative relation underlies the understanding of ratio. In simple situations students are usually forthcoming in expressing a ratio in terms of the multiplicative relation. Thus the ratio of 3 to 6 is seen to be the same as ratio of 4 to 8 because in each pair, the second number is two times ('double') the first. Generalizing the multiplicative relation between any two whole numbers using the fraction notation extends students' resources by allowing them to represent the ratio (the multiplicative relation) between two arbitrary whole numbers. The application of the multiplier concept (fraction as operator) in understanding ratio and proportion problems is facilitated in our approach by the use of the double number line (see Figure 3). The double number line is a graphical representation of the two measure spaces that are linearly related in a situation where proportionality obtains. It indicates visually the 'within measure space' and the 'across measure space' ratios expressed through the operator construct.

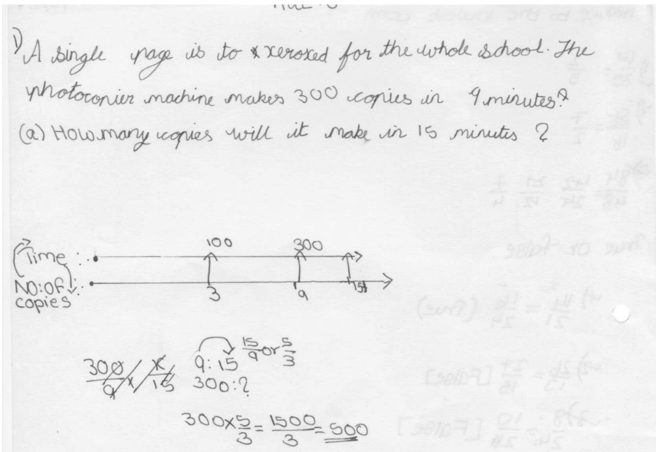


Figure 3: Understanding proportion using the double number line and the operator construct

I have indicated here briefly how opportunities to integrate the different interpretations of fractions can be provided to students. The semantic control of students reasoning as they learn fractions by linking representations of fractions with concrete embodiments and situation is has been well explored by earlier researchers. Notably, Streefland has shown how much progress can be made by keeping equal sharing situations at the heart of fraction learning ( Streefland, 1993). However we need a better grasp of how structural control of the students' thinking and learning processes can be facilitated. In this, the intuitive understanding of the multiplicative relation between whole numbers plays a part. This allows the operator subconstruct of fractions to develop as an integrating interpretation, paving the way for a structural understanding of the rational number concept.

## The Transition from Arithmetic to Algebra

The difference between arithmetic and algebra has occupied a central place in the research on algebra learning and teaching. The conceptual change perspective, at least in this case, has been anticipated and well explored. As researchers came to recognize that the competence that students have gained in arithmetic in primary school can actually hamper the learning of algebra, a variety of responses have evolved to address the problem. Some researchers have sought alternative conceptualizations of algebra. Others have advocated an early introduction to algebra by 'algebrafying' the elementary arithmetic curriculum, that is, by introducing algebraic thinking, and sometimes introducing rudimentary algebraic notation in the primary years (see Lins and Kaput, 2004).

Traditionally the route to algebra has been through arithmetic. Algebra is thought of as encoding the general rules and properties of arithmetic operations such as the commutative, associative and distributive properties, and exploiting such encodings to obtain transformation rules and equivalences of different symbolic expressions. Prior to the transition to algebra, students' knowledge of arithmetic is enriched in the traditional curriculum by having them work with arithmetic expressions. Primary school arithmetic familiarizes children with the binary operations done singly, but arithmetic expressions encode a sequence of binary operations rather than a single binary operation. This is new to many students and marks the first point of transition to algebra. Hence working with arithmetic expressions in the traditional curriculum helps familiarize students with the order of operation conventions that ensure that each arithmetic expression, even when written without brackets, has a unique value. This is expected to prepare the ground for work with algebraic expressions, which yield arithmetic expressions when the variables are substituted and hence take on unique values when the conventions for order precedence are followed. Thus algebraic expressions become representations of functions. Since the conventions governing algebraic expressions reflect the conventions for precedence of operations, it is expected that students who understand these conventions will also understand the structure of an algebraic expression.

However, students do not make the expected smooth transition to algebra and encounter many hurdles. For example, students are habituated through arithmetic to obtain a 'closed' answer or a single number as the result, which leads them to misunderstand notations like $3 + x$ and $3x$ as being equivalent. The arithmetic bias operates in this case by producing a 'closure' in the form of the expression, similar to obtaining a number as the result in arithmetic. A deeper account of the hurdles that students encounter is provided by researchers who draw on the work of Piaget and others on the development of abstract concepts. They argue that expressions such as $3 + x$ have multiple meanings in algebra and it is necessary to treat them in a flexible manner as both processes and products (Sfard, 1991; Tall et al., 2000). For example, $3 + x$ can both be understood as a process of adding any number to $3$ or as standing for the result of that process, which is the number obtained as the sum. The process-product duality is also found in the quotient interpretation of fractions: $\frac{m}{n}$ can denote either the process of dividing $m$ objects into $n$ equal shares or the result of the process, namely, each share. Students who fail to adopt the flexible dual view of fractions may have trouble seeing $\frac{2}{3}$ and $\frac{12}{18}$ as 'being the same'.

Another difficulty caused by arithmetic expressions in relation to algebra learning is that students fail to perceive the structure of arithmetic expressions. Indeed, they may interpret the structure variously without even being aware of the requirement that each numerical expression can have only one value. As one might imagine, this seriously hampers understanding symbolic algebra. Part of the reason lies in the fact that students do not appreciate the necessity of representing a sequence of binary operations. Problems in arithmetic can be solved simply by actually carrying out the binary operations one by one. Contexts where such representations are necessary, whether in problem solving or in expressing functions (formulae) or in justifying and proving (see Bell, 1995) already presume a facility with symbolic expressions that students may still need to attain.

In an approach to algebra that we are currently developing for the middle grades, the focus is on working with symbolic arithmetic to draw out students' intuitions in arithmetic and to build on them. As I argued in the discussion on whole numbers, the operational composition is already embodied in the complex place value notation for numbers. In this approach to algebra, the understanding students already have about the operational composition of a number is strengthened and enhanced. Students can generalize and extend the idea of composition and arrive at different representations for a number. Each representation carries compositional or relational information about the number. Thus the two expressions 5+7 and 9+3 may denote the same number, but they express different relational information. Students can appreciate this and soon develop an interest in generating expressions that denote the same number.

A key strategy here is to deflect students away from the goal of computing an expression. Arithmetic expressions are not to be interpreted as instructions to compute but as reflecting the operational composition of a number. For this, students need to clearly distinguish the units in an expression and how each unit contributes to the value of the expression. This is the key idea that underlies what we have described as the 'terms approach' to evaluating arithmetic expressions (Subramaniam and Banerjee, 2004). In the terms approach, students parse the given expression into terms and flexibly combine terms rather than add and subtract numbers in a sequence dictated by precedence rules. Each term contributes to the value of an expression: simple positive terms increase the value, while simple negative terms decrease the value of the expression, acting in a compensating manner. The rule for precedence of multiplication is absorbed into the visual concept of a product term, that is distinguished from the simple term. Product terms can be combined with simple terms only after they are themselves reduced to simple terms. Exceptionally they may be combined with other product terms which have the same factor. This approach was developed using a teaching experiment methodology with iterative trials using different groups of students. More details about this approach and the empirical results obtained may be found in Banerjee (submitted).

The approach also represents an operationalizing of the notion of reification or process-product duality that characterizes the growth of mathematical knowledge. This is entailed by the change in perspective from computing an expression by 'operating' on the numbers designated to evaluating the expression by 'combining' terms. From the proceptual point of view (Gray and Tall, 1994), the expression as a whole denotes a sequence of processes and also the result obtained after applying the processes, which is the computed value of expression. More importantly, here I wish to point to the proceptual nature of the elements of the expression: each term denotes the result of an operation, and designates an operator which modifies the value of the expression. This 'immanently' proceptual view frees students from rigid computational rules and allows them to see the elements of an expression in relational terms. This prepares them for the representation of a function by a symbolic expression, and also opens the door to understanding transformations of expressions and how transformations affect the value of an expression. This understanding is fundamental to the simplification procedures and the manipulation of forms that build facility with symbolic algebra.

To summarize, one of central problems that mathematical pedagogy has to deal with arises from the fact that symbolic mathematics, while being an important and essential part of mathematics, presents enormous hurdles to students, who fail to engage meaningfully with symbols. Researchers have sought to overcome this problem by exploring approaches that enable students to exercise control over symbolic mathematics in deeper ways. Much of this research explores largely semantic forms of control. Developing forms of structural control can be complementary to this process. Indeed, as researchers have pointed out (Sfard, 2000) this gains increasing importance as students grow in their mathematical understanding. In this article, I have attempted to take a longer ranging perspective of elementary mathematics that compares findings from different topic areas spread across the school years. From such a perspective, forms of structural control emerge as important. I have also attempted to indicate briefly how forms of structural control on symbolic mathematics can be provided for while designing the curriculum in the areas of fractions and beginning algebra by drawing on students' understanding of whole numbers.

# References

Arzarello, F., L. Bazzini, and G. Chiappini (2001). A model for analyzing algebraic processes of thinking. See Sutherland et al. (2001), pp. 61–81.

Babu, S. (2004). Tamil mathematical manuscripts. In *Proceedings of epiSTEME-1 conference*. Mumbai: Homi Bhabha Centre for Science Education.

Balacheff, N. (2001). Symbolic arithmetic vs algebra the core of a didactical dilemma. postscript. See Sutherland et al. (2001), pp. 249–260.

Banerjee, R. (Submitted). *Developing a Learning Sequence for Transiting from Arithmetic to Elementary Algebra*. Ph. D. thesis, Homi Bhabha Centre for Science Education.

Behr, M. and K. C. Fuson (1992). Research on whole number addition and subtraction. In D. E. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, pp. 243 – 275. New York: Macmillan & Co.

Behr, M. J., G. Harel, T. Post, and R. Lesh (1993). Rational numbers: Toward a semantic analysis – emphasis on the operator construct. See Carpenter et al. (1993), pp. 13–36.

Bell, A. (1995). Purpose in School Algebra. *Journal of Mathematical Behavior 14*(1), 41 – 73.

Carpenter, T. P., E. Fennema, and T. A. Romberg (Eds.) (1993). *Rational Numbers: An Integration of Research*. Erlbaum, Hillsdale, NJ,.

Charalambous, C. Y. and D. Pitta-Pantazi (2007). Drawing on a Theoretical Model to Study Students Understandings of Fractions Journal Educational Studies in Mathematics. *Journal Educational Studies in Mathematics 64*(3), 293 – 316.

Confrey, J. (1994). Splitting, Similarity, and Rate of Change: A New Approach to Multiplication and Exponential Functions. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics*. State University of New York Press.

De Corte, E. and L. Verschaffel (2006). Mathematical thinking and learning. In W. Damon, R. Lerner, I. Sigel, and A. Renninger (Eds.), *Handbook of child psychology (6th edition). Volume 4: Child psychology and practice*, pp. 103 – 152. John Wiley & Sons.

Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in the Learning of Mathematics. *Educational Studies in Mathematics 61*(1-2), 103 – 131.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. E. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, pp. 243 – 275. New York: Macmillan & Co.

Gelman, R. and C. R. Gallistel (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.

Goldin, G. A. and J. J. Kaput (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe et al. (Ed.), *Theories of Mathematical Learning*. Mahwah, NJ: Lawrence Erlbaum.

Goldin-Meadow, S. (2006). Nonverbal communication: The hands role in talking and thinking. In W. Damon, R. Lerner, D. Kuhn, and R. Siegler (Eds.), *Handbook of Child Psychology, Sixth Edition, Volume Two: Cognitive Perception and Language*. N.Y.: John Wiley & Sons.

Gravemeijer, K. P. E. and M. L. Stephan (2002). Emergent models as an instructional design heuristic. In K. Gravemeijer, R. Lehrer, B. van Oers, and L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education*, pp. 145 – 169. Dordrecht: Kluwer Academic Publishers.

Gray, E. and D. Tall (1994). Duality, ambiguity and flexibility: a proceptual view of simple arithmetic. *Journal for Research in Mathematics Education, 26*, 115 – 141.

Izard, V. and S. Dehaene (2007). Calibrating the mental number line. *Cognition*, doi – 10 – 1016 – j – cognition – 2007 – 06 – 004.

Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. Kieren (Ed.), *Recent Research on Number Learning*, pp. 32–55. ERIC/SMEAC, Columbus, OH.

Kieren, T. E. (1993). Rational and fractional number: From quotient fields to recursive understanding. See Carpenter et al. (1993), pp. 261 – 288.

Lins, R. and J. Kaput (2004). The early development of algebraic reasoning: The current state of the field. In K. Stacey, H. Chick, and M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra. The 12 th ICMI Study*, pp. 73 – 96. Boston: Kluwer.

Moseley, B. (2005). Students early mathematical representation knowledge: The effects of emphasizing single or multiple perspectives of the rational number domain in problem solving. *Educational Studies in Mathematics 60*, 37 – 69.

Naik, S. and K. Subramaniam (forthcoming). Integrating the Measure and Quotient Interpretation of Fractions. In *Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education*. Mexico.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics 22*, 1 – 36.

Sfard, A. (2000). Symbolizing mathematical reality into being – or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, and K.McClain (Eds.), *Symbolizing and communicating in mathematics classrooms*, pp. 37–98. Mahwah, NJ: Lawrence Erlbaum.

Sophian, C. and C. Kailihiwa (1998). Units of counting: Developmental changes. *Cognitive Development 13*, 561 – 585.

Stafylidou, S. and S. Vosniadou (2004). The development of students understanding of the numerical value of fractions. *Learning and Instruction 14*, 503 – 518.

Steffe, L. P. (2002). A new hypothesis concerning childrens fractional knowledge. *Journal of Mathematical Behavior 20*, 267 – 307.

Streefland, L. (1993). Fractions: A realistic approach. See Carpenter et al. (1993).

Subramaniam, K. and R. Banerjee (2004). Teaching arithmetic and algebraic expressions. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Volume 3, pp. 121 – 128. Bergen, Norway.

Sutherland, R., T. Rojano, A. Bell, and R. Lins (Eds.) (2001). *Perspectives on school algebra*. Dordecht, Netherlands: Kluwer.

Tall, D., M. Thomas, E. Davis, Gray, and A. Simpson (2000). What is the object of encapsulation of a process? *Journal of Mathematical Behavior 18*, 223–241.

Treffers, A. (2001). Numbers and number relationships. See van den Heuvel-Panhuizen, M. et al. (2001).

van den Heuvel-Panhuizen, M. et al. (2001). *Children learn mathematics*. The Netherlands, Freudenthal Institute.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert and M. Behr (Ed.), *Number Concepts and Operations in the Middle Grades*. NCTM. Reston, Virginia.

Vosniadou, S., A. Baltas, and X. Vamvakoussi (2007). *Re-Framing the Conceptual Change Approach in Learning and Instruction*. Emerald Group Publishing.