HBCSE Report

MathWiki

A Resource for Problem Solving

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Contents

Introduction

In India access to educational material remains an almost insurmountable hurdle for the vast majority of students. The new Information and Communication Technologies (ICT) promise to solve this problem to a great extent. The *Mathwiki* project aims to use ICT to build resources for mathematics education. *Mathwiki* is a web based resource of problems in school mathematics that can be used by students and teachers. It is based on the wiki philosophy – that users can also contribute to the material on the mathwiki pages, blurring the distinction between author and reader. It uses technology already developed and being used successfully in developing material based on the wiki idea, such as the free web based encyclopedia, *Wikipedia*. *Mathwiki* has been developed on the free Mediawiki platform, which is also the platform for *Wikipedia.*

The Wiki philosophy

A wiki is web-based material that can not only be read over the internet, but can also be edited and modified by readers. The underlying belief is that by allowing readers to contribute to and shape the material, it gets written faster and becomes continuously better. Wiki is like other collectively authored material, with the difference that there is very little restriction on editing and changing the material.

To users of Wikipedia, it often comes as a surprise that material of such high quality as one finds in the Wikipedia articles, is built through 'unvetted' contributions from readers. That this approach would work was the idea of Ward Cunningham, the inventor of the wiki name and concept and the creator of the original Wiki engine.

Modern web technologies allow many authors to read and edit a single database over the internet. This is made possible by the Wiki engine which manages the documents that comprise the wiki. The engine also provides a simplified interface that allows users to edit material with a minimal knowledge of syntax and formatting conventions.

Mediawiki, a popular Wiki engine, is the basis for *Wikipedia* and also for *Mathwiki.* Mediawiki allows features such as quick searches, classification of different kinds of pages, summary of contributions by authors, and most important, ready access to the entire history of a page listing every modification by every user, and to all past versions of the page. This is especially useful when we need to revert to an older version in case errors are inadvertently saved or in the case of an attack of vandalism.

The immediate objective of the Mathwiki project is to build a collection of interesting problems in high school mathematics with solutions. Although this is a small step, we are aware of the great possibilities that the technology enables. For example, building a learning resource on the web in a collective and shared manner, may shape our understanding of pedagogy and even content in new and deep ways. We would like to keep these possibilities open from the very beginning, by reflecting, articulating and inviting discussion on how best the content and modes of interaction can be organized.

Although information and communication technologies hold great promise for education, their real impact on education, especially in a developing country like India, have been limited. Since this technology is rapidly changing and evolving, it may take a while before stable forms of ICT-based educational practices begin to emerge. Added to this is the learning curve that must be crossed by educators in adjusting to the new technology. Mathwiki is conceived as a step in the direction of using ICT to build resources for education that are widely accessible.

Problem solving in mathematics

Mathematical problems have formed a part of cultures in many regions of the world. Solving mathematical problems has been pursued in a variety of contexts: practical application in commerce or architecture, astronomical calculations, recreational activities and also problem solving for its own sake. In old Indian mathematical texts, we often find problems presented in attractive language such as this problem from Bhaskara's *Lilavati*.

> A peacock perched on the top of a nine foot high pillar sees a snake, three times as distant from the pillar as the height of the pillar, sliding towards its hole at the bottom of the pillar. The peacock immediately flies to grab the snake. If the speed of the peacock's flight and snakes slide are equal, at what distance from the pillar will the peacock grab the snake.

While problems solving is common in the school mathematics curriculum, the widely prevalent attitude is that one must 'know' how to solve a problem. The stress is on learning methods of solution, and very little on 'discovering' a solution. The experience of struggling to solve a problem for long periods of time and then discovering a solution is almost wholly absent form the experience of a typical school student.
Polya captures the pedagogical value of such experiences in these words,

> If a teacher can satisfy the curiosity of the student by facilitating the solution of problems by his own means and he experiences the tension and enjoys the triumph of discovery. Such experience at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a life time. (Polya, 1945, p. v.)

So the most important element of problem solving is to solve problems that one has not solved before, to discover solutions, not to recall them. The most appropriate setting for such an activity would be one where students solve a problem collectively in the classroom or perhaps in a problem solving club or camp. A webpage could not be expected to generate sufficient motivation and interest to overcome the common 'inertia' that students experience in tackling problems. Often however, students and teachers do not have access to problems, especially ones that are designed to encourage problem solving, with illuminating comments about how one could discover a solution or about connections to other problems. There are many excellent books on problem solving, but these are often difficult to obtain in many cities. This is the rationale for developing a resource like *Mathwiki.*

The Organization of Mathwiki

Problems in the *Mathwiki* are grouped under broad topics: algebra, geometry, number theory, etc. At present only problems on geometry and algebra have been included. A list on the margin of every page allows ready access to the topics. Each problem in a topic is listed by giving it a name. This is made necessary by the software in its current form. We decided not to change it since giving a problem a name may have pedagogical value by structuring learning and recall.
Clicking on the name of the problem leads one to the problem page, which contains the problem, hints and comments about the problem. (We would like to hide the hints unless they are clicked, but this is yet to be implemented.) The solution is given on a separate page. Problems are usually, but not necessarily, sourced from the literature. Hints, solutions and comments are added by mathwiki writers. Writers add value by choosing hints carefully, formulating solutions concisely, clearly and transparently, and offering comments that make the problem interesting and facilitate remembering and recalling it.

We are looking for an appropriate way to store and present problem chains. Problems may be related to one another because of a similarity in the problem itself or in the solution technique. Problems could also lead to other problems through generalization or extension. Two problems could also be related because the same heuristics are useful in solving both, or because the same key diagram underlies both problems. In the present organization of mathwiki, these elements are included in the text for each problem, often under comments. However we intend to develop these as separately organized, searchable 'attributes' of problems.

We realize that problems can form a valuable resource for the teacher. So we plan to add a feature that allows teachers to select a set of problems and obtain a ready made worksheet, a separate page of hints and a solutions page. These can be printed out for use.

Regular users of Mathwiki are encouraged to obtain a 'log in' identity. Contributions to the Mathwiki can be traced through these identities and authors can be given credit for their work. The quality and value of *Mathwiki* wil depend on a community of people committed to maintaining it and contributing regularly. We hope that such a community will be formed over time.

The *Mathwiki* URL is http://web.gnowledge.org/wiki.

Acknowledgement

The installation and troubleshooting of the *Mathwiki* was done single handedly by Manoj Nair, who is also continuously maintaining the webpages. We are indebted to him not only for his contribution, but for the enthusiasm he has shown in the project. We would like to thank Sahil Sheth and Shweta Naik for their contributions to the problem solving camp.

We also thank colleagues at HBCSE for giving us support and help in many ways, Nagarjuna, who facilitated the setting up of the *Mathwiki* server, H.C. Pradhan, Sugra Chunwala and Chitra Natarajan.

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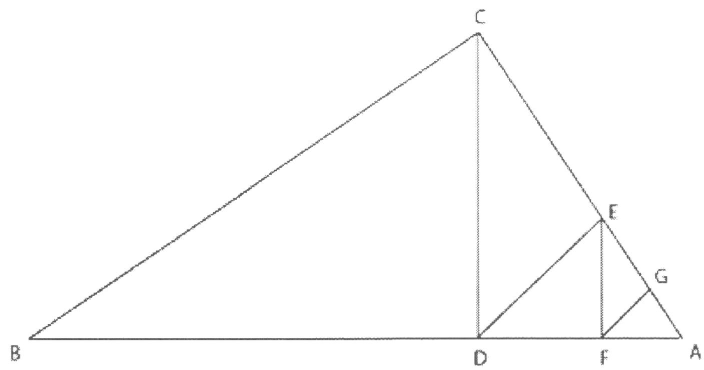
GEOMETRY

Recursive embedding of right triangles

ABC is a 30^0-60^0-90^0 right triangle. CD is the altitude from the right angle vertex C to the hypotenuse AB. Similarly DE is the altitude from D onto side AC in right triangle ACD, EF is the altitude from E onto AD and FG is the altitude from F onto the side AE.



a) Find the ratio of the lengths of the two sides AG and AB.

b) What is the ratio of the area of triangle AEF to triangle ABC?

Hint: Use the relation of the sides in a 30^0-60^0-90^0right triangle.

Solution to Recursive embedding of right triangles

Solution: (a)

In Δ ABC, $\angle C = 90^0$ and $\angle B = 30^0$

Therefore, AC = 1/2 (AB)

(Since the side opposite to 30^0 degrees angle in a right angled triangle is half of the length of the hypotenuse)

In Δ ADC, $\angle C = 30^0$ and $\angle D = 90^0$

Therefore, AD = 1/2 (AC) = 1/2 (1/2AB) = 1/4 (AB)

In ΔADE, $\angle D = 30^0$ and $\angle E = 90^0$

Therefore, AE = 1/2 (AD) = 1/2 (1/4 AB) = 1/8 (AB)

In ΔAEF, $\angle E = 30^0$ and $\angle F = 90^0$

Therefore, AF = 1/2 (AE) = 1/2(1/8AB) = 1/16 (AB)

In ΔAEF, ∟F = 30^0 and ∟G = 90^0

Therefore, AG = 1/2(AF) = 1/2(1/16AB) =1/32(AB)

AG = 1/32(AB).

Solution: (b)

Area of ΔAEF/ΔABC

$$\frac{1/2(AE).(FG)}{1/2(AB).(DC)}$$

$$\frac{1/8.(AB)(FG)}{(AB).(DC)}$$

$$= \frac{1/8(FG)}{(DC)}$$

$$= \frac{1/8(1/16(AB)}{1/2(AB)}$$

Since, FG = 1/16(AB) and DC = AC = 1/2 (AB)

Therefore the ratio of areas of the ΔAEF/ΔABC

= 1/64.

Comment

For many figures we can draw smaller figures inside the figure in a recursive manner. In each step the smaller figure bears a certain ratio to the larger figure. Here are some examples:

Example 1:

In any triangle join the mid-points of the sides. The smaller triangle that is formed is similar to the original triangle. What is the ratio of the sides of the smaller triangle to the corresponding sides of the larger triangle? What is the ratio of the area of the smaller triangle to the larger?

This step can be continued recursively.

Something surprising is found when we do this for any quadrilateral. Join the midpoints of the sides of a quadrilateral in order to obtain a smaller quadrilateral. What type of quadrilateral do you obtain?

Example 2:

If you take any rectangle and join one pair of midpoints of opposite sides, the rectangle is divided into two equal smaller rectangles. But these smaller rectangles are not in general similar to the original rectangle.

However for a special rectangle, this procedure gives two smaller rectangles that are similar to the original rectangle. This rectangle has the sides in the ratio $1:\sqrt{2}$.

A4 size paper, which is commonly used, is this type of rectangle. So if you fold A4 paper exactly in half, you get two smaller sheets whose sides also have the ratio $1:\sqrt{2}$

This size of paper is A5. (A3 size paper is double that of A4.)

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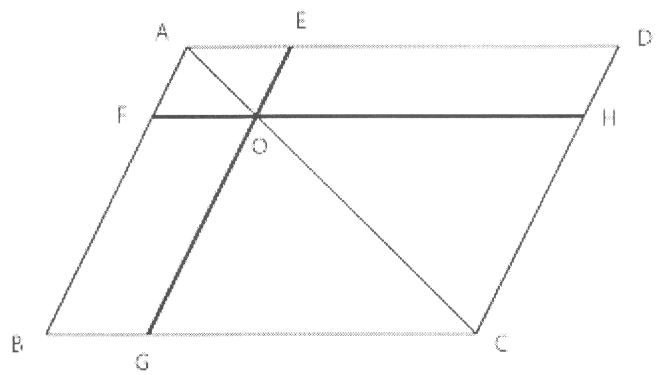
Complement Parallelograms

ABCD is a parallelogram. Take any point, say O on the diagonal AC, and through this point draw lines parallel to the sides of the parallelogram (EOG and FOH in the figure). Prove that the smaller parallelograms that are formed on either side of the diagonal (EOHD and FOGB) are equal in area.



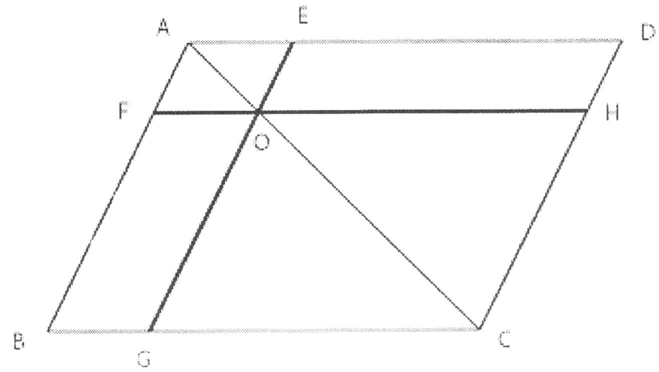
Hint 1: The diagonal of the parallelogram splits it into two equal halves.

Hint 2: Each half of the parallelogram in turn consists of three parts.

Fact: Such parallelograms are called "complement parallelograms" about the given diagonal.

Solutions to Complement Parallelograms

Read the problem well and see the diagram. The solution is given below.



We know that the diagonal bisects a parallelogram into two congruent triangles (proof?). Thus in parallelogram ABCD we have triangles ABC and CDA congruent. That would mean that among other properties that they both must have, their areas are the same. Again, as FH and EG and parallel to the opposite sides of the parallelogram, AFOE and OGCH would themselves be parallelograms as well. Thus for either of them, their diagonals (AO and OC respectively) would bisect them into triangles equal in areas, meaning triangle AEO would have the same area as AFO and OHC the same as OGC. From the above equalities we can say,

Area (Δ ABC) - Area (Δ AFO) - Area (Δ OGC) = Area (Δ ADC) - Area (Δ AEO) - Area (Δ OHC)

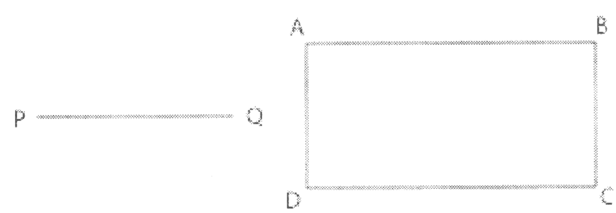
That leaves us with

Area (parallelogram FBGO) = Area (parallelogram EOHD)

This is what was asked.

Comment: It is interesting to discover that the complement parallelograms might look different but have equal area.

Related problem:

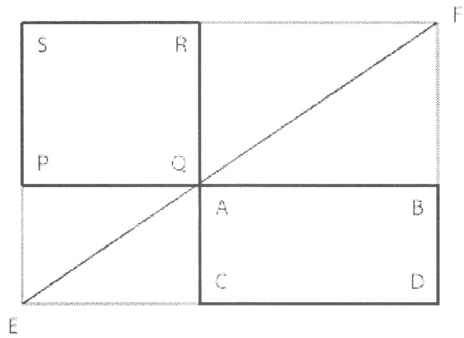


Given a rectangle ABCD and a length PQ, construct a rectangle PQRS equal in area to ABCD. (You are allowed to use only a compass and an unmarked ruler.)

Hint: Apply the property of the above problem to a rectangle.

Solution to Related Problem on "Complement Parallelograms":

Let us draw a rough diagram of what we are asked of by looking at the previous problem and comparing it to the one we are given. We can see that we are asked to construct something that would look like this:



The two rectangles are marked with their edges in bold. Of the two, we are given ABCD and side PQ of rectangle PQRS is given too. We find from the figure that A and Q are the same points.

Let us draw ABCD with PQ as shown above. From P we draw a line parallel to AC which meets DC extended at some point E. From E we use a straight edge and draw a line through A (or Q) that meets DB extended at F. We draw through F a line parallel to AB that meets CA extended at R and EP extended at S. We have the desired rectangle PQRS.

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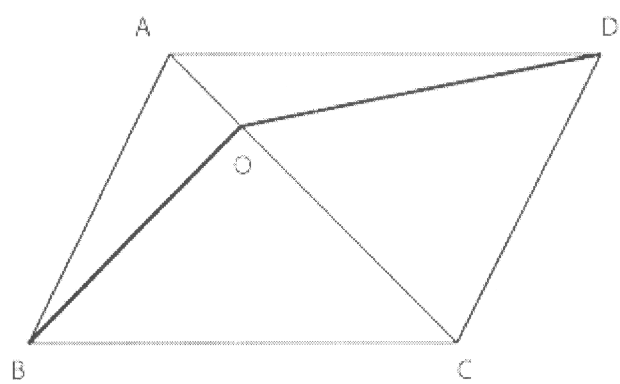
Complement Triangles

From a point O on the diagonal AC of a parallelogram ABCD two lines OB and OD are drawn. Show that the area of the two triangles AOB and AOD are the same.



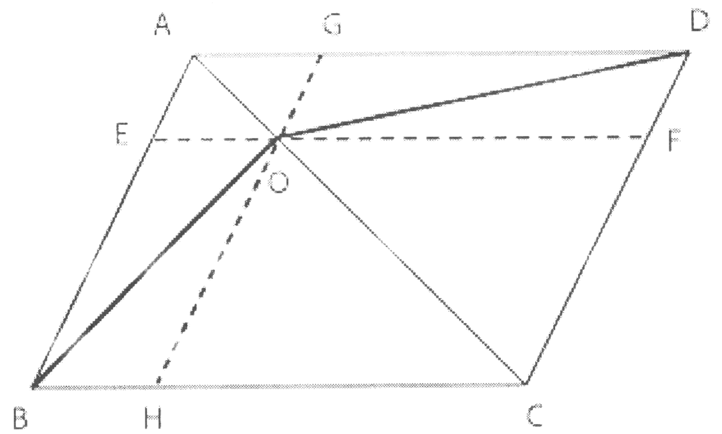
Hint 1: Draw lines through O parallel to AB and AD and use the same method of problem 1.

Hint 2: Apply the meaning of the area of a triangle for both the triangles remembering that A = ½ * (base) * (corresponding altitude). Choose AO as the base for either triangle.

Fact: Such triangles are called "complement triangles" about the given diagonal.

Solution to complement triangles

There are two ways to solve this problem as it is given in the hints, so let us try them.



1. We draw lines through point O parallel to the two adjacent sides of the parallelogram and make the figure appear similar to the first problem. Using the same arguments as before, we can say that triangles ABC and ADC have equal areas as indeed do triangles OFC and OHC as well triangles AEO and AGO. Subtracting the area of triangles AEO and OHC from triangle ABC leaves us with the area of the parallelogram EBHO. This area must be equal to that of parallelogram GDFO which comes when we subtract the areas of triangles AGO and OHC from the triangle ACD. Writing the remainder of the proof in notation form,

Area of || gm EBHO = Area of || gm GDFO

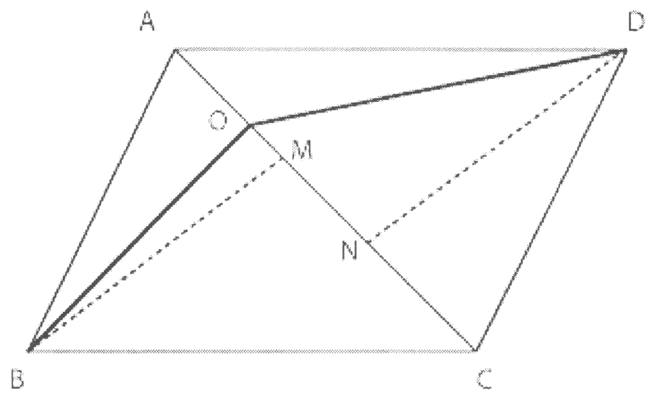
½ (Area of || gm EBHO) = ½ (Area of || gm GDFO)

Area of Δ EOB = Area of Δ GOD

We already have the areas triangles AEO and AGO as equal. Adding each to the left and right hand sides of the last equality completes the proof

Area of Δ AOB = Area of Δ AOD

2.



This time we drop perpendiculars from sides D and B which would enable us to have the areas of triangles ADC and ABC in terms of them. We see from the figure that triangles AOD and AOB have a common base AO. Remembering that the area of a triangle is ½ (base*perpendicular height), we can say that

Area of Δ AOB = ½ (AO * BM) while Area of Δ AOD = ½ (AO * DN).

Now BM = DN as they are the corresponding altitudes of congruent triangles ABC and CDA.

Thus Area of Δ AOB = Area of Δ AOD

Comment: This problem is interesting as it helps realize the essential property of the area of triangles, viz. one can take any side as the base and then find the corresponding altitude. It also helps in using the property that the diagonal of a parallelogram bisects it into two congruent triangles.

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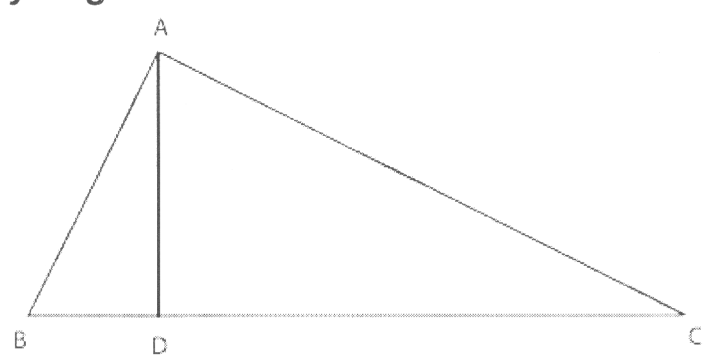
Reciprocal Pythagorean Theorem



In $\triangle ABC$, $\angle A = 90^\circ$ and $AD \perp BC$.

Show that $\frac{1}{AD^2} = \frac{1}{AB^2} + \frac{1}{AC^2}$

Hints:

i) Use the similarity property of triangles or use the concept of area of triangles.

ii) Use the Pythagorean Theorem.

Comment: We already know from the Pythagorean theorem that the sum of the squares of the sides add up to the square of the side of the hypotenuse. But from this problem we see that if we take the sum of the squares of the reciprocals of the same sides, then they add up to the square of the reciprocal of the altitude of the triangle to the hypotenuse.

Notice that the sides AB and AC are also altitudes. So we could see the relation above as the relation between the three altitudes of a right triangle.

Related problem:

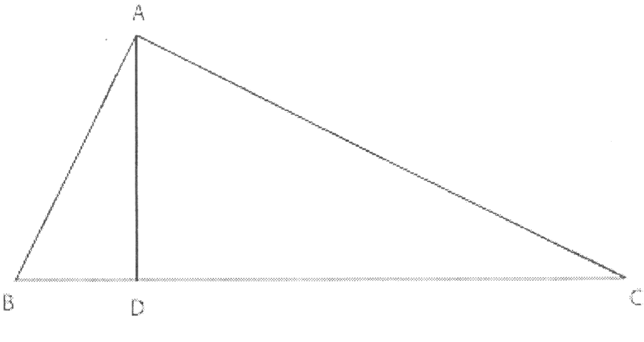
The Pythagorean theorem may be generalized for any triangle with sides a, b, c opposite vertices A, B, C respectively as

$a^2 + b^2 + 2ab(cosC) = c^2$

Show that the relation between the altitudes h_a, h_b and h_c drawn respectively from the vertices A, B, C is given by

$$\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{2cosC}{h_a h_b} = \frac{1}{h_c^2}$$

Solution to Reciprocal Pythagorean Theorem



We start with the area of the same triangle

$$area(\triangle\ ABC) = area(\triangle\ ABC)$$

That implies, from the formula of the area of triangles,

1/2 AD.BC = 1/2 AB.AC

$\Rightarrow$ AD. BC = AB. AC

$$\Rightarrow \quad AD = \frac{AB.AC}{BC}$$

$$\Rightarrow \quad \frac{1}{AD} = \frac{BC}{AB.AC}$$

$$\Rightarrow \quad \frac{1}{AD^2} = \frac{BC^2}{AB^2.AC^2}$$

$$\Rightarrow \quad \frac{1}{AD^2} = \frac{AB^2 + AC^2}{AB^2.AC^2}$$

but, $BC^2 = AB^2 + AC^2$ (by Pythagoras' theorem)

$$\Rightarrow \quad \frac{1}{AD^2} = \frac{1}{AB^2} + \frac{1}{AC^2}$$

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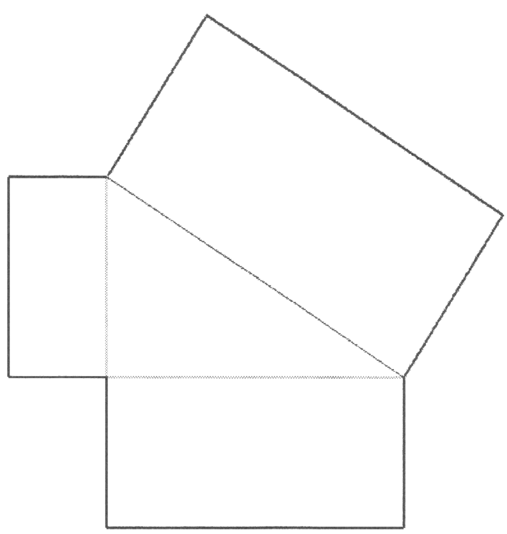
Rectangle Pythagorean Theorem

Similar rectangles are constructed on the sides of a right angled triangle as shown in the figure to the right. (Similar rectangles are those that have their corresponding sides proportional in lengths).



Establish a relation among the areas of the three rectangles.

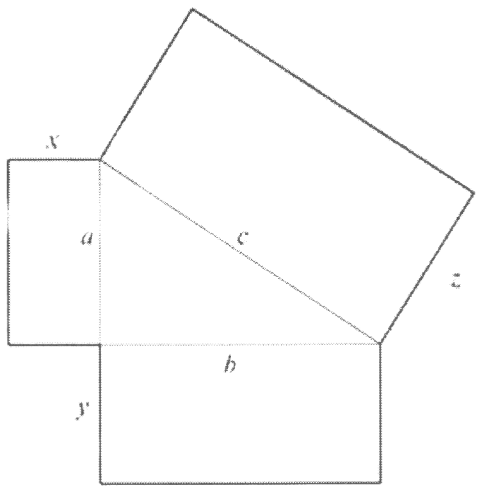
Imagine regular polygons of the same number of sides are constructed on the sides of the triangle now. Will the above relation hold?

Hint: Use the Pythagorean Theorem.

Comment: This is a generalization of the Pythagorean Theorem. In the case of the Pythagorean Theorem, we had the special case of squares. However, this can be further generalized to include any regular polygon drawn on the sides of the triangle.

Solution to Rectangular Pythagorean Theorem

Let the sides of the given right triangle be a, b and c.



Let the lengths of sides the given rectangles be (a,x), (b,y) and(c,z), such that

$\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ (By similarity statement)

$\Rightarrow$ a = x.k (or) x = a/k; b = y.k (or) y = b/k

and c = z.k (or) z = c/k

The area of the rectangle with sides a and x = a.x square units is

$$\frac{a.a}{k}$$

Using the above relations for x, y and z, we get the areas of the above rectangles as

$$\frac{a^2}{k} - - - (1)$$

$$\frac{b.b}{k} = \frac{b^2}{k} - - - (2)$$

and

$$\frac{c.c}{k} = \frac{c^2}{k} - - - (3)$$

But, we know from the Pythagoras theorem, $a^2 + b^2 = c^2$

$$\Rightarrow \frac{a^2 + b^2}{k} = \frac{c^2}{k}$$

$$\Rightarrow \frac{a^2}{k} + \frac{b^2}{k} = \frac{c^2}{k}$$

Thus the area of rectangle of sides a and x units + the area of the rectangle of sides b and y units = the area of the rectangle of sides c and z units.

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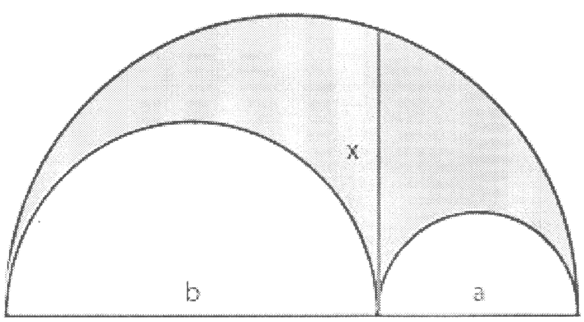
Embedding of Semicircles

In the figure given alongside, we have two small semicircles of diameters a and b units drawn into a larger semicircle. Let x be the length of the tangent drawn from a point on the circumference of the larger semicircle to the common point of contact of the two smaller semicircles.



Show the area of the shaded region (in grey) is equal to the area of the circle whose diameter is x units.

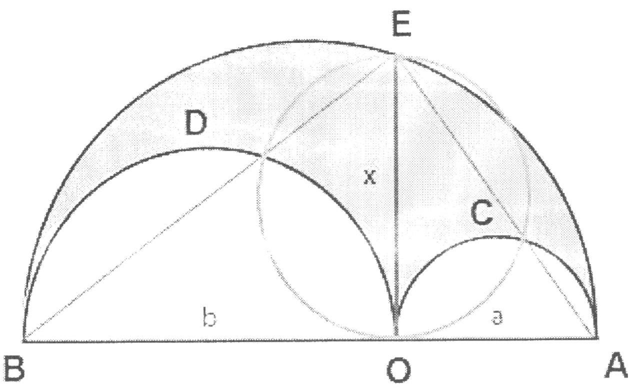
Hint: Establish a relation between x, a and b using the property of similar triangles.

Comment: This problem is also referred to as the '**Shoemaker's Knife Problem**'. The Greek word for Shoemaker's knife is 'Arbelos'.The Arbelos is the shaded area inside the larger semi-circle and outside of the 2 smaller semi-circles. It was so-named because the shape resembles the blade of a knife used by ancient leather workers.

The Shoemaker's knife has attracted the attention of famous mathematicians such as Archimedes, Descartes, Fermat, and Newton. The esthetic appeal of the figure makes the arbelos an interesting source of many problems in Geometry.

For more information, go here: [1] (*http://www.cut-the-knot.org/proofs/arbelos.shtml*)

Solution to Embedding of Semicircles



The area of the shaded region is clearly equal to the difference of areas of the big semi circle and sum of those of the two smaller ones as we can see from the figure alongside.

Thus area of shaded region = area of semicircle AEB - (areas of semicircles ACO and ODB)

$$= \frac{\pi}{2}\left[\frac{(a+b)}{2}^2\right] - \frac{\pi}{2}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right]$$

$$= \frac{\pi}{8}\left[(a+b)^2 - a^2 - b^2\right]$$

$$= \frac{\pi}{4}ab \quad \text{---- (1)}$$

From similar triangles AOE and EOB, we have the identity $x^2 = ab$ ---- (2) (show it!)

Now the area of the circle drawn in green of diameter x is $\frac{\pi x^2}{4}$

From equations (1) & (2), it follows that the two areas are the same.

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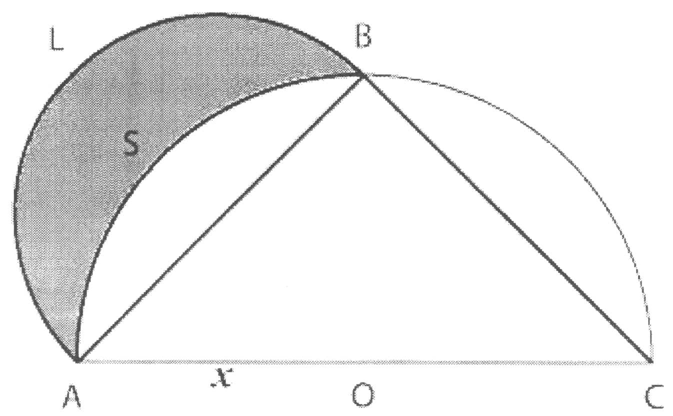
Area of Lune

The figure shown alongside consists of a semicircle of radius x units with an isosceles triangle ABC drawn inside it. On one of the equal sides of the triangle (say AB) another semicircle ALB is drawn.



Compute the area of the shaded region.

How is the area of the shaded region related to the area of the triangle AOB?

Do you see something interesting and maybe strange about the area of the shaded region, which is made of arcs and no straight lines as compared to the area of most circular regions?

Comments: The shaded region in the figure is called a lune. A lune is the area cut off by one circle from the interior of a smaller one.

Though the lune is bounded by two circular arcs, its area is independent of π .

Solution to Area of Lune

The area of the LUNE (shaded region) =

Area of the semicircle ALB - the area of the segment ASB on chord AB -- (1)

Area of the semicircle ALB = $\dfrac{\pi\ \sqrt{2}^2 x^2}{2.4} = \dfrac{\pi\ x^2}{4}$

Area of the quarter circle ASB = $\dfrac{\pi\ x^2}{4}$

Therefore the area of the semicircle ALB is equal to the area of the quarter circle ASB.

So substituting in (1) we have

Area of the LUNE = Area of the quarter circle ASB - the area of the segment ASB on chord AB = Area of triangle AOB = $\frac{x^2}{2}$.

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Product of Diagonals of Cyclic Quadrilateral

In a cyclic quadrilateral prove that the product of its two diagonals equals the sum of the products of opposite sides. In other words, if ABCD is the cyclic quadrilateral, show that

AD·BC + AB·CD = AC·BD

Comment: This theorem is known as **Ptolemy's Theorem**.

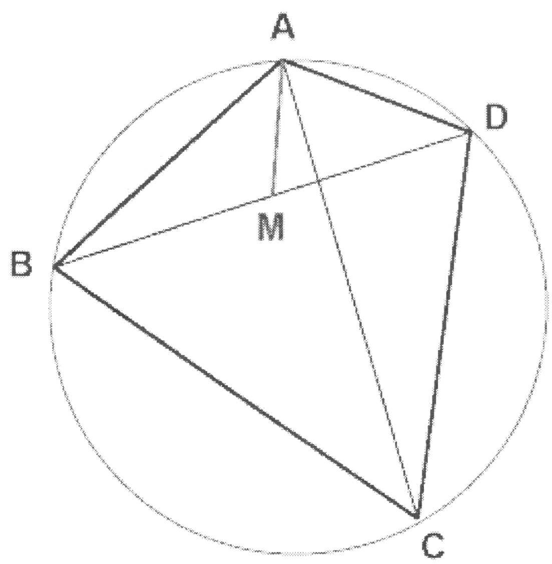
Hint: Take a point M on BD such that $\angle ACB = \angle MCD$. Now use the property of similar triangles to proceed.

Related Problem:

ABCD is a cyclic quadrilateral and AC is the diameter. If AB = 6 units, AD = 7 units and DC = 9 units, find the length of BD.

Hint: Use the above result and the Pythagorean Theorem.

Solution to Product of Diagonals of Cyclic Quadrilateral



On the diagonal BD locate a point M such that angles MAB and DAC be equal. Now compare triangles ABM and ACD. The angles MAB and CAD are equal by construction. Angles ABM and ACD are equal since they are subtended by the same arc.

Therefore, triangles ABM and ACD are similar.

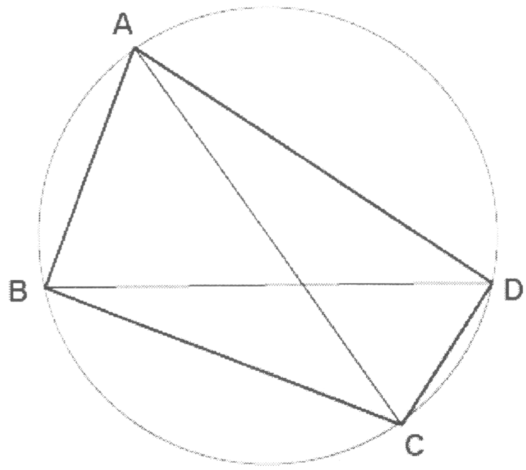
Thus we get AB/MB = AC/CD, or AB·CD = AC·MB.

Now, angles MAD and BAC are also equal. Angles BCA and MDA are equal since they are subtended by the same arc AB. So triangles MAD and BAC are similar which leads to AD/DM = AC/BC, or AD.BC =AC.DM.

Summing up the two identities we obtain

AB·CD + BC·AD = AC·MD + AC·BM = AC·BD

Solution to Related Problem:



In $\triangle ADC, \angle D = 90°$ (angle in a semicircle)

Thus, $AC^2 = AD^2 + DC^2$

$AC^2 = 7^2 + 9^2 = 130$

In $\triangle ABC, \angle B = 90°$ (angle in a semicircle)

Thus, $BC^2 = AC^2 - AB^2$

$BC^2 = 130 - 6^2 = 94$

Thus $BC = \sqrt{94}$

From Ptolemy's theorem, AB.CD + BC.DA = AC.BD

Substituting the above values in the theorem

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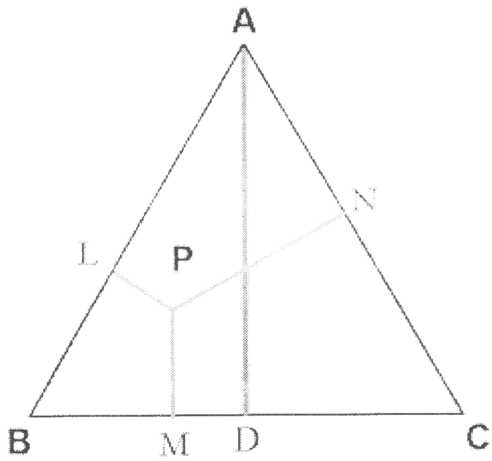
Altitudes from Interior Point in Equilateral Triangle

Given a point P in the interior of an equilateral triangle ABC, PL PM, PN and AD are perpendicular segments as shown in the figure alongside. Prove that PL + PM = PN = AD.

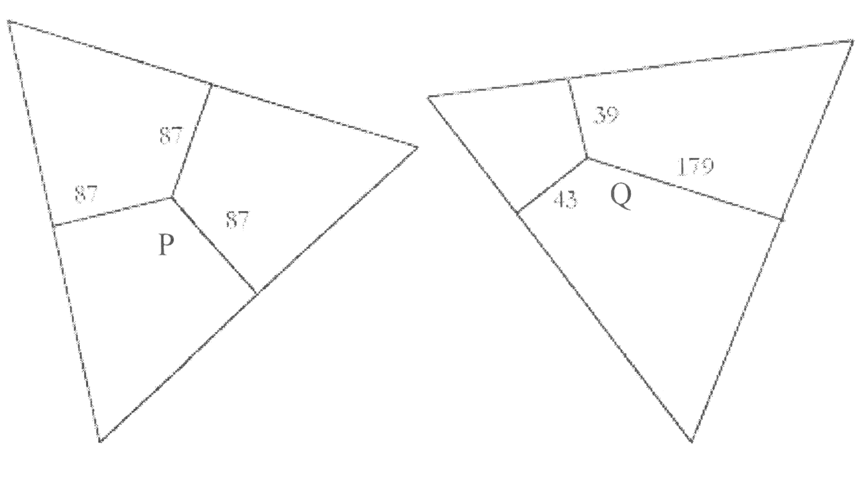


Hint: Use the concept of the area of a triangle.

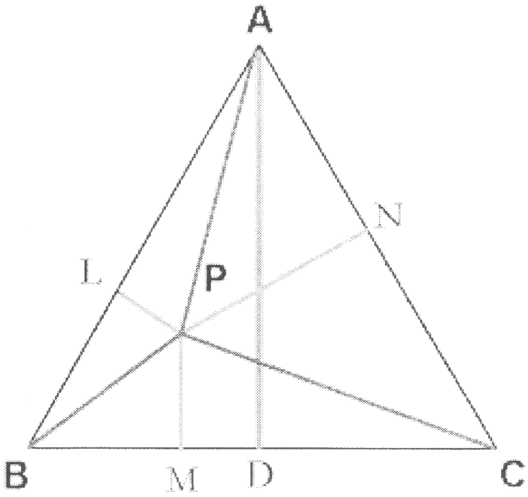
Comment: This is unique property of an equilateral triangle. Check if this property holds good for an isosceles or a scalene triangle.

Related problem:

If P and Q are points inside two equilateral triangles as shown with the perpendiculars having the given dimensions, check if the two triangles are congruent.



Solution to Altitudes from Interior Point in Equilateral Triangle



From P, we draw lines to the vertices A,B and C.

Now,

$$Area(\triangle ABC) = area(\triangle APB) + area(\triangle APC) + area(\triangle BPC)$$

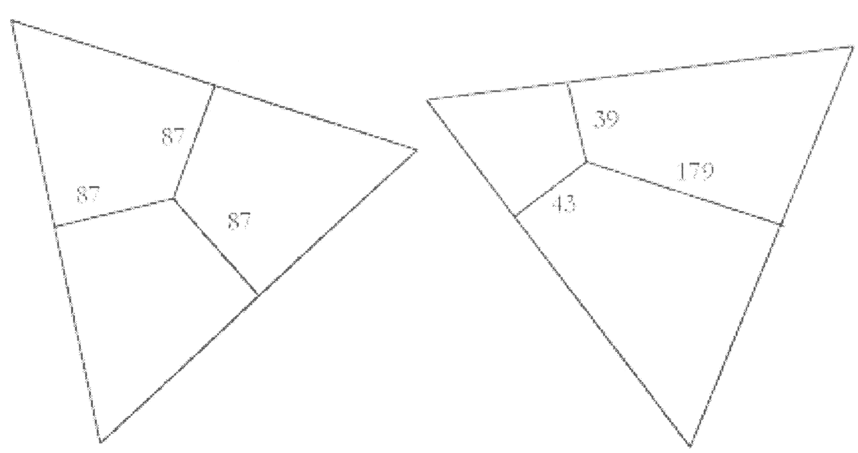
$\Rightarrow$ 1/2 (BC).(AD) = 1/2 (AB).(PL) + 1/2(AC).(PN) + 1/2(BC).(PM)

$\Rightarrow$ 1/2 (AB).(AD) = 1/2(AB).(PL) + 1/2(AB).(PN) + 1/2(AB).(PM) (since, AB = BC = AC)

$\Rightarrow$ AD = PN + PM + PN.

Therefore , in any equilateral triangle the sum the lengths of the perpendiculars drawn from any point inside the triangle onto the sides is equal to the length of any altitude of the triangle.

Solution to Altitudes in Equilateral Triangle Extension



From the previous problem, the length of any altitude of a triangle is equal to the sum of the lengths of the perpendiculars drawn from any interior point onto the sides.

Thus, the length of the altitude of the first triangle = (87 + 87 + 87)

= 261.

Similarly, the length of the altitude of the second triangle = (179 + 43 + 39)

= 261.

Thus as the lengths of the altitudes are the same, the two equilaterals are congruent. (Prove it!)

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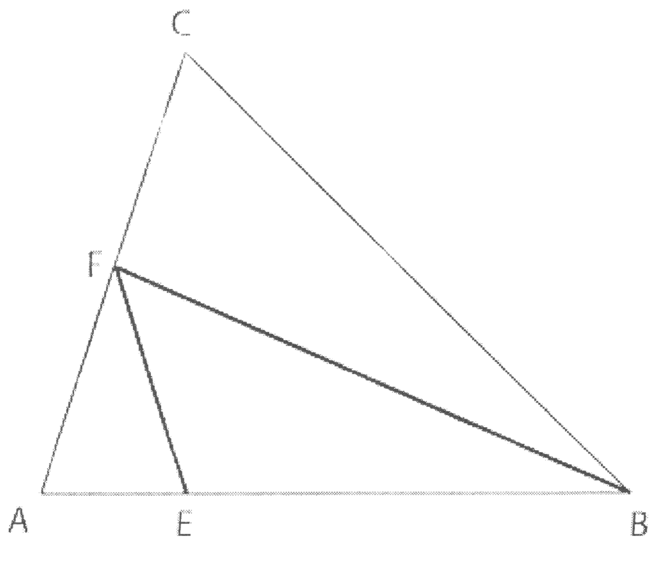
Area of Embedded Triangles



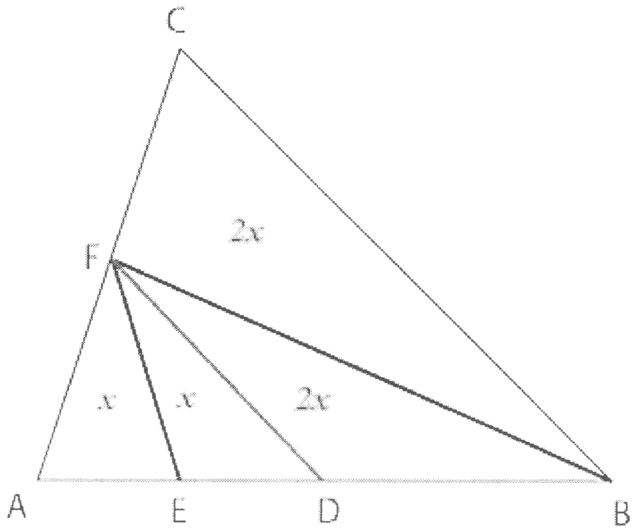
In $\triangle$ ABC, F is the mid-point of AC and E is a point on AB such that AE = (1/4) AB.

The area of $\triangle$ BEF is 63 square units.

Find the area of the $\triangle$ ABC.

Hint : Use the property of median of a triangle.

Solution to Area of Embedded Triangles



Triangles BFA and BFE in the figure have the same altitude from F. So their areas are proportional to the lengths of their bases.

Since BE =3/4 × BA, BA = 4/3 × BE

So area of $\triangle$ BFA = 4/3 × area of $\triangle$ BFE = 84 square units.

Since F is the midpoint of AC, the area of $\triangle$ ABC is twice the area of $\triangle$ BFA, that is, 168 square units.

Alternative Solution

In $\triangle$ ABC join the points F and D.

FD is the median of $\triangle$ ABF.

Now using the property that the median of a triangle divides the triangle into two triangles of equal area, we can say that

$\Rightarrow$ Area of $\triangle$ ADF = Area of $\triangle$ BDF---(1)

Similarly, since FE is the median of $\triangle$ ADF

Area of $\triangle$ AEF = Area of $\triangle$ DEF = x square units (say)

Therefore the area of $\triangle$ ADF = 2x = Area of $\triangle$ BDF.

Putting them together we can say that the area of $\triangle$ ABF = 4x sq. units.

In $\triangle$ ABC, BF is the median, meaning the two $\triangle$'s ABF and CBF are equal in area = 4x sq. units each.

Thus the area of $\triangle$ ABC = 4x + 4x = 8x sq. units----(2)

Now the area of $\triangle$ BEF = 3x sq. units (see figure) $\Rightarrow$ 3x = 63 $\Rightarrow$ x = 21

From (2), the area of $\triangle$ ABC = 8*21 = 168 sq. units

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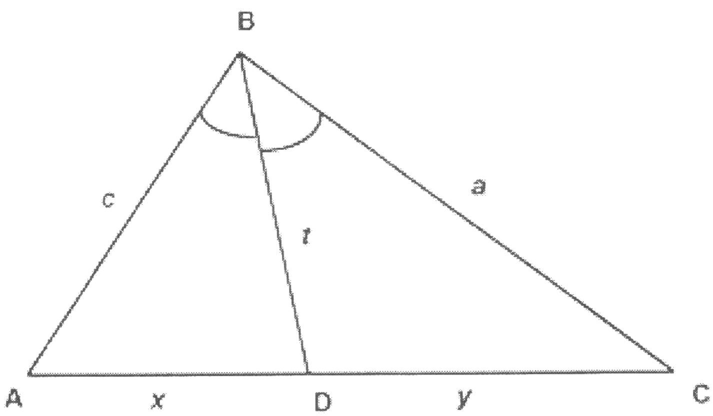
Angle Bisector and Sides of a Triangle



In $\triangle$ ABC, BD is the bisector of $\angle$B. If AD = x, DC = y, AB = c, BC = a, as shown in the figure alongside, then show that

$$a.c = t^2 + x.y$$

Note that the angle bisector divides the side AC in the ratio of the sides c:a. So x and y can be computed in terms of the sides of the triangle. (Here b is the length of side AC.)

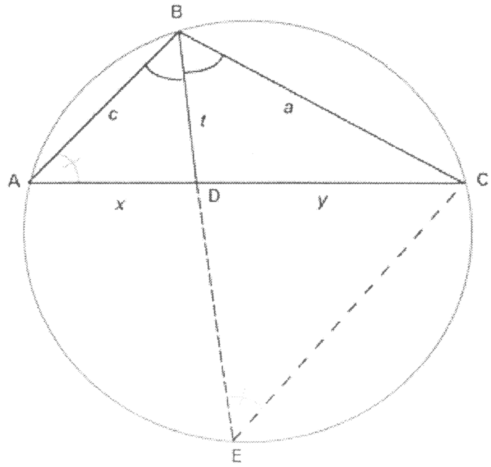
x=cb/(a+c)

y=ab/(a+c)

So the relation can be expressed as one between the angle bisector and the sides of the triangle.

Hint: Construct a circum circle and extend BD to meet the circum circle. Use similarity property of triangles.

Solution to Angle Bisector and Sides of a Triangle



Solution :

Construct a circumcircle of the triangle and extend BD to meet the circle at E. Join CE.

In $\triangle$'s BAD and BEC,

$\angle$ABD = $\angle$EBC (BD is the angular bisector)

$\angle$BAD = $\angle$BEC (angles in the same segment)

Thus $\triangle$ BAD $\sim$ $\triangle$ BEC.

$$\Rightarrow \frac{BA}{BD} = \frac{BE}{BC}$$

$$\Rightarrow \frac{c}{t} = \frac{BE}{a}$$

$$\Rightarrow BE = \frac{ac}{t} \text{ ------ (1)}$$

Again from similar triangles CDE and BDA (how are they similar?) we have

$$\frac{y}{DE} = \frac{t}{x} \text{ (How?)}$$

$$\Rightarrow DE = \frac{xy}{t} \text{ -------(2)}$$

Now from the figure BD + DE = BE.

Using (1) and (2)

$t + (xy/t) = ac/t$

Simplifying,

$t^2 + xy = ac$

***

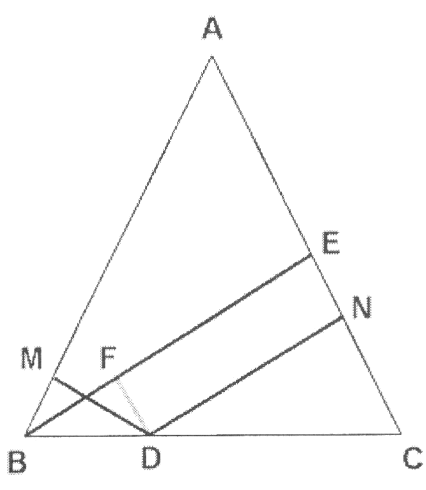
Altitudes of an isosceles triangle

Prove that the sum of lengths of the perpendiculars drawn from any point on the base of an isosceles triangle to the congruent sides is equal to the length of the perpendicular from the either end of the base to the opposite side.

Hints: There are at least two ways of doing this problem. One way involves dropping a perpendicular from the point on the base to the altitude. The other involves the construction of joining the point on the base to the opposite vertex.

Solution to altitudes of an isosceles triangle

First Solution:



Let ABC be the isosceles triangle with BC as the base and AB = AC. D is any point on the base BC and N and M are the feet of the perpendiculars drawn from D onto the equal sides of the triangle. BE is the altitude. From D we draw a line perpendicular to altitude BE meeting it at F.

FD is parallel to CA, since both are perpendicular to BE, So $\angle$FDB = $\angle$ACB. In triangles BFD and DMB, we have BD common, $\angle$BMD = $\angle$DFB = 90° and $\angle$MBD = $\angle$FDB

Thus the two triangles are congruent, meaning MD = BF--- (1)

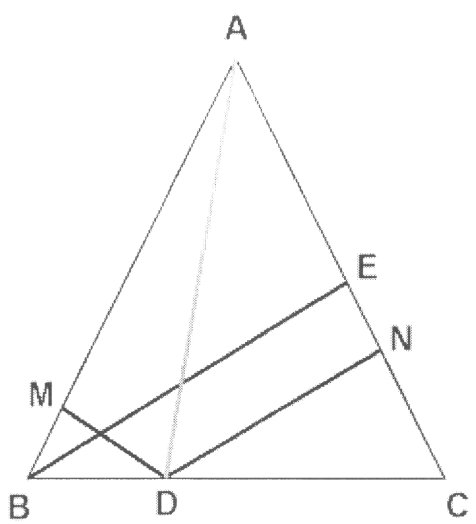
Now BE = BF + FE and FE = DN as FDNE is a rectangle (all angles 90°).

Using (1) we conclude,

BE = MD + DN

Which is what we are asked to show.

Second Solution:



Here we join AD. The area of $\triangle$ ABC being equal to those of $\triangle$'s ABD and ACD, we have

1/2 * AC * BE = 1/2 * AB * MD + 1/2 * AC * ND.

But as AC = AB (isosceles triangle ABC),

BE = MD + ND.

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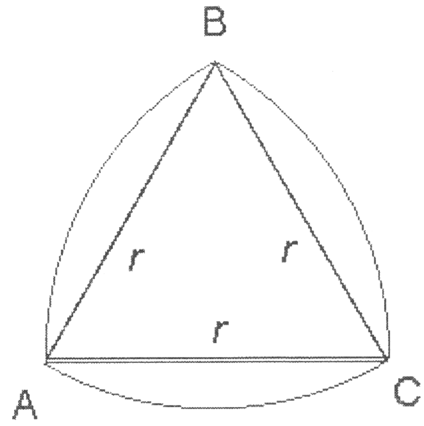
Reuleaux triangle



To construct a reuleaux triangle, first draw a circle with centre A and radius *r*. Now choose a point B on the circumference as the centre and with the same radius draw an arc AC passing through A and a point C on the circumference of the first circle. Now with C as centre and the same radius, draw an arc passing through A and B.

The reuleaux triangle is the figure bounded by the three circular arcs AB, BC and CA.

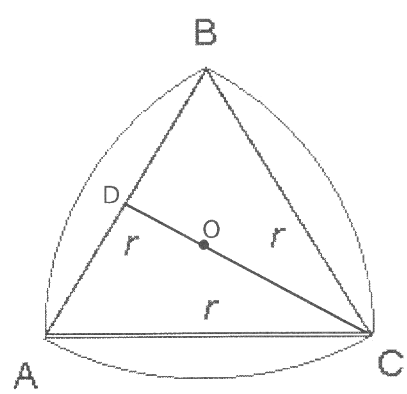
Find the area and the perimeter of the Reuleaux Triangle.

Hint: Apply the area of a sector and the length of an arc in a circle.

Comments: The Reuleaux Triangle is a closed curve of constant width! (Show it). When in a square, all its sides will touch the square sides, like a circle which is also a closed curve of constant width. For more on Reuleaux Triangles, go to the following websites:

[1] (*http://whistleralley.com/reuleaux/reuleaux.htm*)
[2] (*http://mathworld.wolfram.com/ReuleauxTriangle.html*)

Solution Reuleaux Triangle



We connect C to the centroid O and extend it to meet the AB at D. Now area of the sector CAEBC is

A(sector) = $\frac{\theta}{360}\pi r^2$.

But θ = 60°.

$\Rightarrow$ A(sector) = $\frac{\pi r^2}{6}$

Area of the triangle:

$A_t = \frac{1}{2}r^2 sin\theta = \frac{1}{2}r^2.\frac{\sqrt{3}}{2} = \frac{\sqrt{3}r^2}{4}$.

Thus the area of the segment AEB, on chord AB

$A_s = (\frac{\pi}{6} - \frac{\sqrt{3}}{4})r^2$

Therefore the area of the Reuleaux Triangle is

$A_{ReuleauxTriangle} = 3\, A_s + A_t$

$= \frac{1}{2}(\pi - \sqrt{3})r^2$

Length of the arc AEB = $\frac{\theta}{360}2\pi r$

As $\theta = \frac{\pi}{3}$ = 60°

$l_s = \frac{\pi r}{3}$

Therefore the perimeter of the Reuleaux Triangle

P = $3l_s$ =

$P = \pi r$

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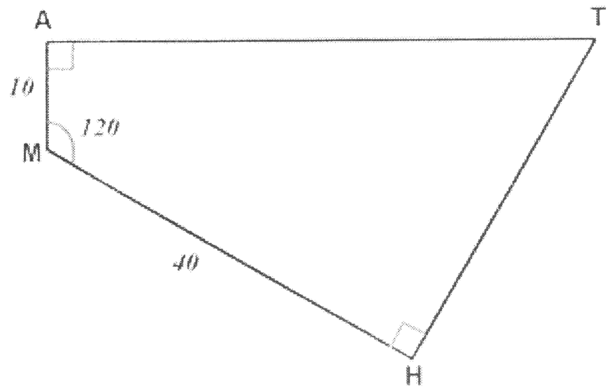
Cyclic Quadrilateral with Two Right Angles



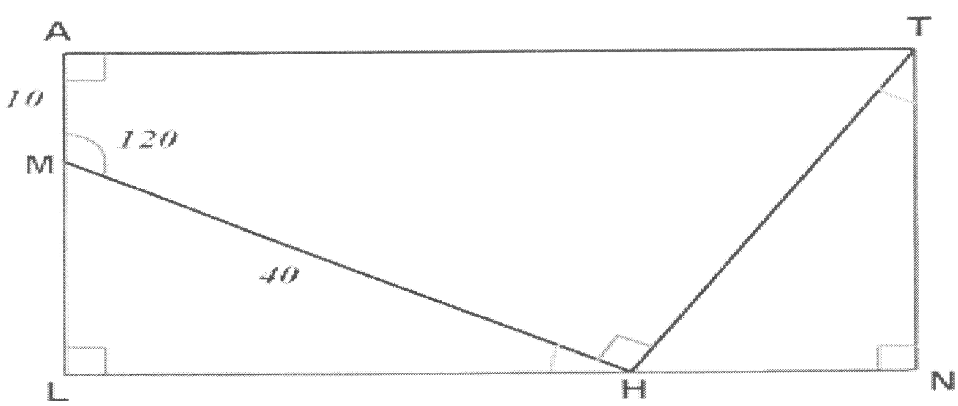
In the quadrilateral given in the figure above, $\angle A = \angle H = 90^\circ$ and $\angle M = 120^\circ$.

MA = 10 and MH = 40 units.

Find TH.

Hint : Construct appropriate auxiliary element.

Solution to Cyclic Quadrilateral with Two Right angles



We extend AM to L and join LH such that $\angle MLH = 90^\circ$. We extend LH to N and join TN such that $\angle TNH = 90^\circ$.

Now in $\triangle$ MLH, $\angle LMH$ is 60°.

Therefore $\frac{ML}{MH} = \cos 60° = \frac{1}{2}$.

$\Rightarrow$ ML = 20, as MH = 40.

$\Rightarrow$ AL = AM + ML = 30 = TN.

Now in $\triangle$ THN, $\angle$THN is $60°$. (Prove it!)

Therefore, $\frac{TN}{TH} = \frac{\sqrt{3}}{2}$ as $\sin 60° = \sqrt{3}/2$.

$\Rightarrow$ TH = $\frac{30 * 2}{\sqrt{3}}$

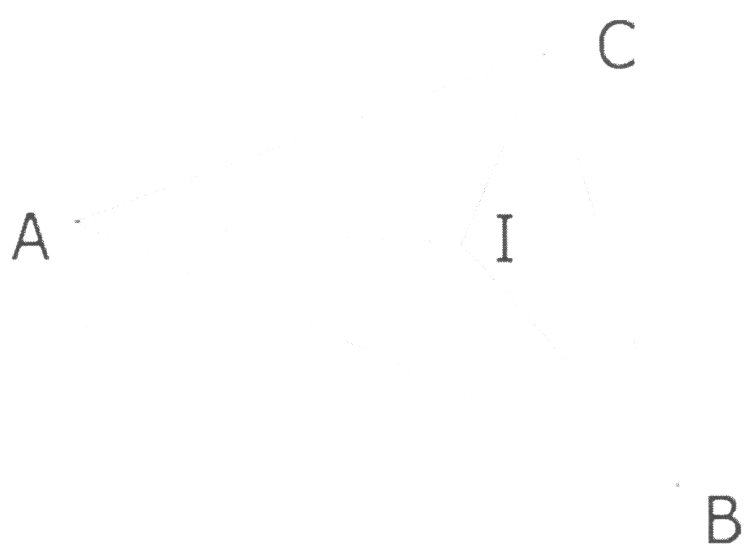
Or, TH = $20\sqrt{3}$ units

***

Interior Point in Triangle Farthest from Sides

An island in the sea has a triangular shape. Describe the point on the island that is farthest from the island's shore.

Solution to Interior Point in Triangle Farthest from Sides



We know the (shortest) distance from a point to a line is given by the length of the perpendicular from the point onto the line. We also know that all the angle bisectors of the sides of the island meet at the incenter I of the triangle. The incentre is at equal distance from all the three sides of the triangle.

Join the incentre I of the triangle to the vertices of the triangle. The original triangle is now divided into three smaller triangles.

Choose one of the smaller triangles. Consider the side belonging to the original triangle as the base and the Incentre I as the vertex. Any point inside the triangle or on its edges, is closer to the base than the vertex I.

Making the same argument for the other two small triangles allows us to see that any point other than I will be closer than I to one of the three sides of the triangle. Hence the incenter of the island will be the farthest point from the island's shore.

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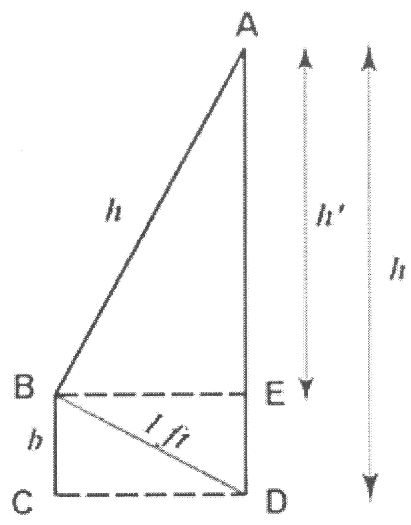
Rope problem

A rope attached to the top of a vertical pole standing upon horizontal ground just reaches the ground. If I take hold of the end of the rope, and, keeping it perfectly taut, and pull it out feet from the foot of the pole, I found that the end of the rope is b feet from the ground. How high is the pole?

Hint : Draw the diagram and use the Pythagorean theorem.

Comment: Good problem for the translation of a problem into a diagram and then apply the Pythagorean theorem.

Solution to Rope problem



We have AD as the rope (or the pole) and CD as the ground level. The rope was pulled so that it's end was a feet from the foot of the pole, AB is it's new position. B is b feet from the ground.

Using Pythagorean theorem in the triangle BCD, we get the length of CD as $\sqrt{1-b^2}$.

Therefore, BE = $\sqrt{1-b^2}$ = CD.

In $\triangle$ AEB, if we have the length of AE as h', then using the Pythagorean theorem again, we have,

$BE^2 + AE^2 = AB^2$

$\Rightarrow (\sqrt{1-b^2})^2 + h'^2 = h^2 \Rightarrow h' = \sqrt{h^2+b^2-1}$ = AE

Now AD = AE + ED

$\Rightarrow h = h' + b = \sqrt{h^2+b^2-1} + b$

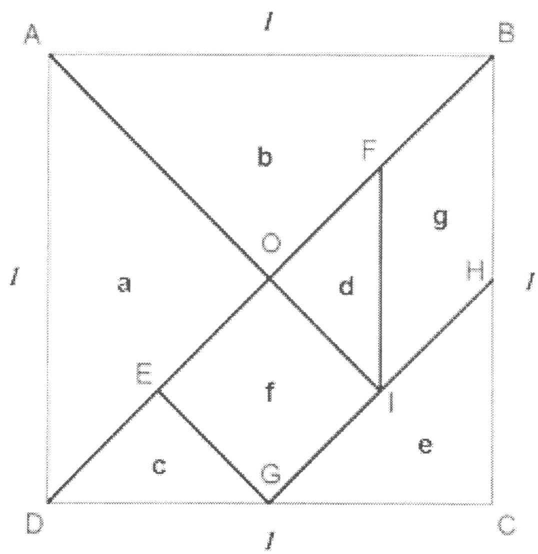
$\Rightarrow (h-b)^2 = h^2 + b^2 - 1$

$\Rightarrow 2hb = 1$

$$h = \frac{1}{2b}$$

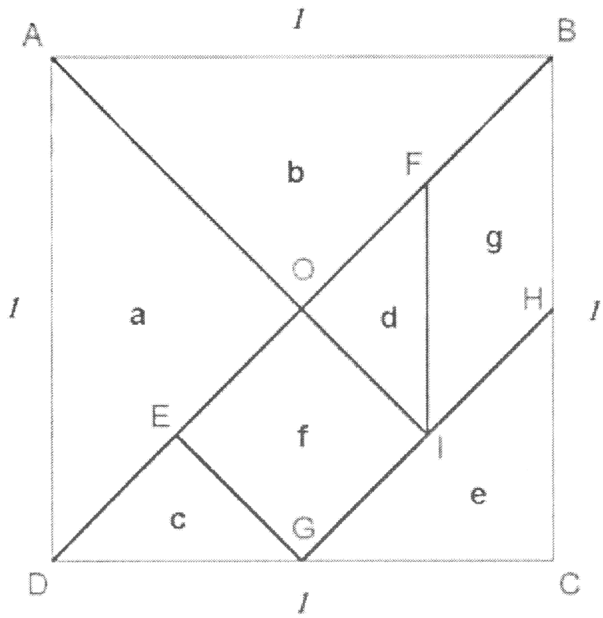
***

Area of Tan gram Pieces



A tan gram is an ancient Chinese puzzle in which a square is cut into seven pieces -- five triangles, one square and one parallelogram. If the area of entire tan gram is one square unit, what is the area of each piece?

Solution to Area of Tan gram Pieces



In the diagram, the pieces are shown, consisting of five right isosceles triangles, a square and a parallelogram. The small letters in each of the figures indicate their areas.

It is easy to see that the areas a and b are the same and together make up half the area of the unit square.

Thus $a = b = 1/4$ sq. units

Also evident is BH = HC = 1/2 units = GC.

Thus the area of triangle HGC = 1/2*1/2*1/2 = 1/8 square units. or $e = 1/8$ sq. units.

Now GH = $\sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$

Thus GI = 1/2 GH = $\frac{1}{2\sqrt{2}}$

Area of the square = $(\frac{1}{2\sqrt{2}})^2 = \frac{1}{8}$, giving $f = \frac{1}{8}$ sq. units.

We note that the parallelogram BFIH has a base of IH = GI and a height EG = GI meaning that it has the same area as the square, giving $g = \frac{1}{8}$ sq. units.

The two triangles EDG and OIF have the same areas as they have bases equal to 1/2 units and heights of 1/4 units.

Thus their areas are 1/16 sq. units each, giving $c = d = 1/16$ sq. units.

***

Interior angles and diagonals of polygons

Two regular polygons have 17 interior angles and 53 diagonals in all. How many sides does each one have?

Hint: Apply the formula for the number of diagonals in a polygon.

Solution Diagonals in a Regular Polygon

If the number of sides/vertices/interior angles/ of a polygon is *n*, then the total number of diagonals of the polygon is given by

$$d = \frac{n}{2}(n-3)$$

If we take the number of sides of the two polygons as *m* and *n*, then we have

m + n = 17 ------------- (1)

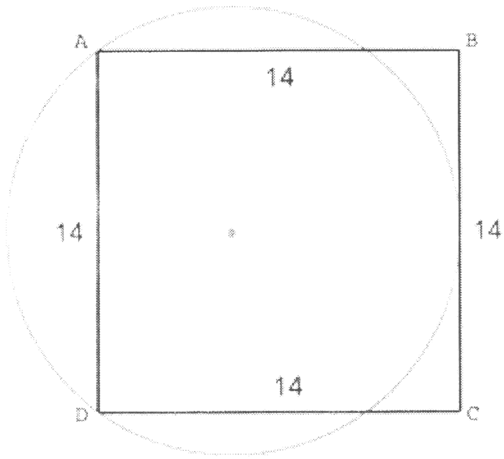
&

$$\frac{1}{2}[n(n-3) + m(m-3)] = 53$$

Solving equations 1 and 2, we get

m = 11, n = 6

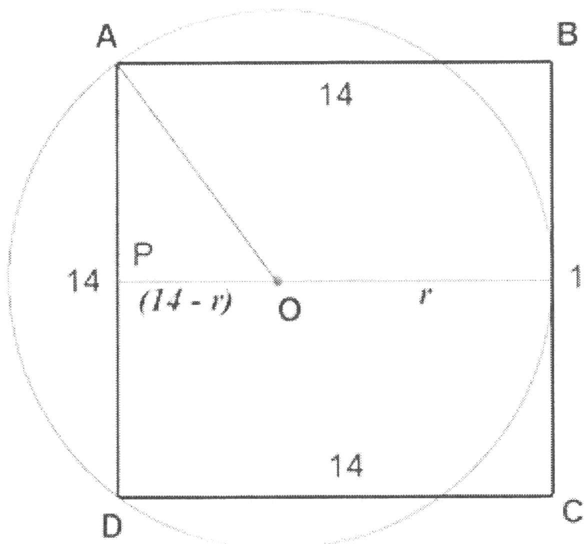
Overlapping Square and Circle



Square ABCD has sides of length 14 units. A circle is drawn through A and D, so that it is tangent to BC as shown in the figure. What is the radius of the circle?

Hint: Draw the diameter through the centre and apply Pythagoras' theorem.

Solution to Overlapping Square and Circle



Let O be the center of the circle. We draw a diameter passing through the point of contact of the square and the circle to intersect the side AD of the square at a point P (say). Join A and O (the radius)

In $\triangle$ APO, $\angle APO = 90^\circ$

AP = 7, AO = r and PO = (14 - r) as is evident from the figure.

$AO^2 = AP^2 + PO^2$

$\Rightarrow r^2 = 7^2 + (14 - r)^2$

$r^2 = 49 + 196 + r^2 - 14r$

14r = 245

r = 17.5 units.

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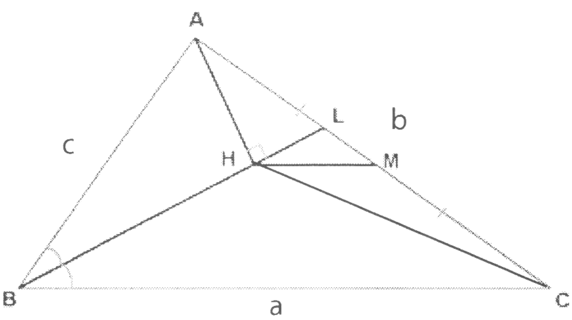
Median-angular bisector-altitude



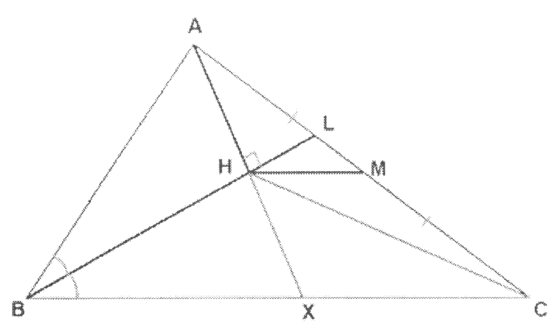
In $\triangle$ ABC, *a*, *b* and *c* are the lengths of the sides and a > c.
It is given that BL is an angle bisector in $\triangle$ ABC, AH is an altitude in $\triangle$ ABL and HM is the median in $\triangle$ AHC.

Find the length of median HM in terms of a, b and c.

Hint:Extend AH to X lying on BC and use the mid - point theorem.

Comment: A nice problem in seeing and applying the known concepts of school geometry.

Solution to Median angular bisector altitude



We extend AH to meet BC at X. We find AXB to be an isosceles triangle as the angular bisector of B is perpendicular to AX which is the base. Thus BX = c.

Therefore the length of XC = a-c.

Looking at triangles AHM and AXC, we note that M is the mid point of AC (given) and H is the midpoint of AX (an angular bisector is also the perpendicular bisector in an isosceles triangle). Using the mid - point theorem, we can say that HM is both parallel and half in length to XC. Thus,

$$HM = \frac{a-c}{2}$$

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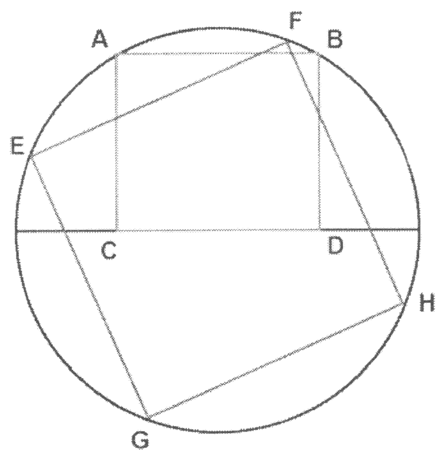
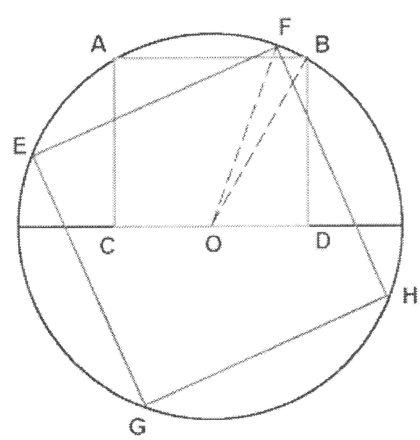
Area Ratio of Squares Inscribed in Circle and Semicircle



If the area of a square inscribed in a semi circle is x and the area of the square inscribed in whole circle is y, calculate the value of (y/x).

Solution to Area Ratio of Squares Inscribed in Circle and Semicircle



The length of a side of the smaller square is $\sqrt{x}$. Join BO. Now in $\triangle$ OBD, using the Pythagorean theorem,

$$OB = \sqrt{\frac{x}{4} + x} = \frac{\sqrt{5x}}{2}.$$

Now OB = OF, radius of the circle.

Twice the length of OF gives the diagonal FG of the square EFGH.

Therefore FG = $\sqrt{5x}$

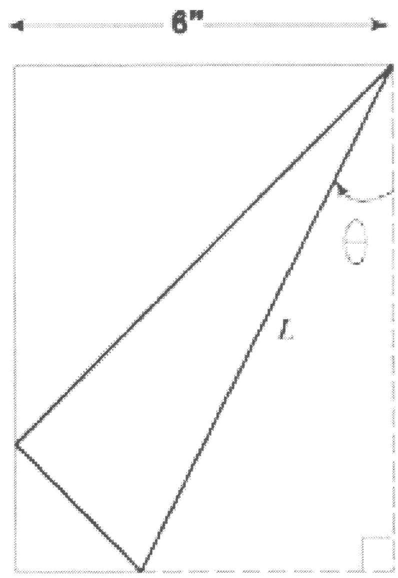
Now the area of a square with diagonal d is $d^2 / 2$.

Therefore the area of square EFGH = y = $\frac{5x}{2}$. Therefore,

$$\frac{y}{x} = \frac{5}{2}$$

* * *

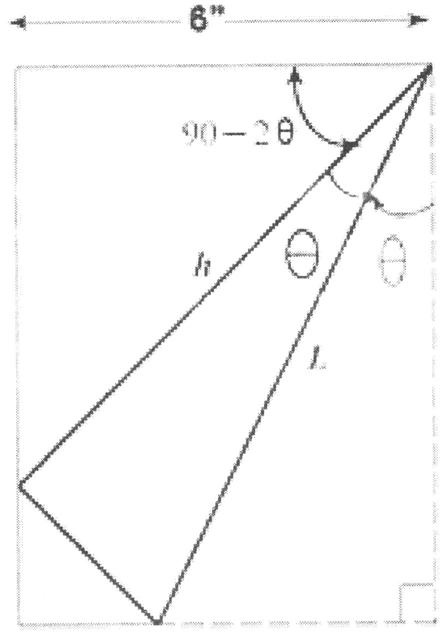
Page Folding



To mark his page in the book he is reading, Anand always folds as shown in the diagram so that the bottom right corner always touches the opposite side of the same page. In this book, the pages are 6 inches wide. What is the length in inches of the crease ***L*** that Anand makes, in terms of the angle **θ**?

Hint: Take the length of the page to be some h and apply trigonometric identities. Remember that a page will have the same vertical angle through which it is folded as well as maintain it's other properties.

Solution to Page Folding



From the diagram, $\sin 2\theta = \dfrac{6}{h}$ ---------(1)

Also from the diagram : $\cos\theta = \dfrac{h}{L} \Rightarrow L = \dfrac{h}{\cos\theta}$ --------(2)

From (1), $h = 3cosec\theta sec\theta$.

Therefore $\dfrac{h}{\cos\theta} = 3cosec\theta sec^2\theta$.

Therefore, from (2),

L = $3cosec\theta sec^2\theta$.

ALGEBRA

Augustus de Morgan

The mathematician AUGUSTUS DE MORGAN lived his entire life during the 1800`s. In the last year of his life he announced, "once i was x years old in the year x^2". In what year was he born?

Solution to 'Augustus de Morgan'

Augustus Morgan age was x years in the year x^2during the 1800`s

The only perfect square between 1800 and 1900 is 1849.

$x^2 = 1849$

$\Rightarrow x = \sqrt{1849}$

x = 43 years

He was 43 years (x years) old

$\Rightarrow$ His birth year is 1849 - 43 = 1806.

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Triangular number

Let, t_n be the nth triangular number,

$$t_n = \frac{n(n+1)}{2},$$

Find

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \ldots + \frac{1}{t_{2005}}.$$

Hint :Try to re-arrange the triangular numbers in a sequence as difference of two numbers.

Solution to 'Triangular number'

Let, be t_n the nth triangular number,

$$t_n = \frac{n(n+1)}{2}$$

For n = 2005

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} + \ldots\ldots + \frac{1}{t_{(2005)}}$$

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{1}{2005}$$

$$\Rightarrow \frac{2}{1.2} + \frac{2}{2.3} + \frac{2}{3.4} + \frac{2}{4.5} + \ldots. + \frac{2}{2005.2006}$$

$$\Rightarrow [\frac{2}{1} - \frac{2}{2}] + [\frac{2}{2} - \frac{2}{3}] + [\frac{2}{3} - \frac{2}{4}] + \ldots. + [\frac{2}{2005} - \frac{2}{2006}]$$

$$\Rightarrow \frac{2}{1} - \frac{2}{2006}$$

$$\Rightarrow 2\left(\frac{1}{1} - \frac{1}{2006}\right)$$

$$\Rightarrow \frac{2005}{1003}.$$

***

Remainder of a function

Let f(x) = $x^4 + x^3 = x^2 + x + 1$, what is the remainder, when $f(x^5)$ is divided by f(x)?

Hint

Use following relation

$x^n - y^n = (x - y)(x^{n-1}.y + x^{n-2}.y^2 + ... + y^n)$

Solution to 'Remainder of a function'

This problem can be solved by two methods

The first method is to directly divide the polynomial (x^5) by the polynomial f(x).

$f(x) = x^4 + x^3 + X^2 + x^2 + 1$

$f(x^5) = x^{20} + x^{15} + x^{10} + x^5 + 1$

$$\frac{f(x^5)}{f(x)} = \frac{x^{20} + x^{15} + x^{10} + x^5 + 1}{x^4 + x^3 + x^2 + x + 1}$$

Carrying out the polynomial division, we find that

$x^{20} + x^{15} + x^{10} + x + 1 = (x^4 + x^3 + x^2 + x + 1)(x^{16} - x^{15} + 2x^{11} + 3x^6 - 3x^5 + 4x - 4) + 5.$

The remainder is 5.

Method – II

We know that $(x^n - 1) = (x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + ... + x + 1)$ for all integers n.

$$\Rightarrow (x^{kn} - 1) = (x^k - 1)(((x)^k)^{n-1} + ((x^k)^{n-2} + ...(x^k) + 1.$$

$$\Rightarrow ((x)^{5n} - 1) = (x^5 - 1)((x^5)^{n-1} + (x^5)^{n-2} + ... + (x^n) + 1)$$

So, $(x^5 - 1)$ is a factor of each of the polynomials

$(x^{20} - 1)$, $(x^{15} - 1)$ and $(x^{10} - 1)$...

$f(x) = x^4 + x^3 + x^2 + x + 1$

$f(x^5) = x^{20} + x^{15} + x^{10} + x^5 + 1$

We can rewrite this as

$f(x^5) = (x^{20} - 1) + (x^{15} - 1) + (x^{10} - 1) + (x^5 - 1) + 1 + (1 + 1 + 1 + 1)$---(since we subtracted 4 ones so added 4 ones)

Now we each $(x^{20} - 1)$, $(x^{15} - 1)$, $(x^{10} - 1)$ and $(x^5 - 1)$ is a multiple of$(x^5 - 1)$

So, $f(x^5)$ is the sum of four multiples of f(x) and 5.

And they are also multiples of f(x)

The remainder when $f(x^5)$ is divided by f(x) is 5.

***

Credit Card number

Twelve positive integers are written in a row on a credit card. The fourth number is 4 and the twelfth number is 12. The sum of any three neighboring numbers is 333. Determine the Credit card number. (Integers)

Hint : Use the condition of sum of three numbers as 333 and repeat the numbers.

Solution to 'Credit Card number'

The twelve positive integers are written in a row, such that the fourth number is 4 and the twelfth is 12.

The sum of any three neighboring numbers is 333.

The fourth number is 4 (given) and let the fifth number be "x".

The sixth number will become (329 - x), since it is given that the sum of three consecutive numbers is 333.

The numbers are,

--, --, --, 4, x, (329-x), x, 4, (329-x), 4, x, 12.

But, as per the series the twelfth number is (329 - x) = 12

x = 317

The twelve positive integers are

4, 317, 12, 4, 317, 12, 4, 317, 12, 4, 317 and 12.

***

Find 'x'

If,

$$a = \frac{xy}{x+y}; b = \frac{yz}{y+z} andc = \frac{zx}{z+x}$$

When a ,b and c are not equal to zero. Find the value of x in terms of a, b and c.

Hint : Find the reciprocals and try to eliminate y and z by basic operation.

Solution to 'Find 'x'

It is given that, for a, b, and c ... 0

$$a = \frac{xy}{x+y}$$

$$\Rightarrow \frac{1}{a} = \frac{x+y}{xy}$$

$$\Rightarrow \frac{1}{a} = \frac{1}{x} + \frac{1}{y}$$

□ $b = \frac{yz}{y+z}$

$$\Rightarrow \frac{1}{b} = \frac{y+z}{yz}$$

$$\Rightarrow \frac{1}{b} = \frac{1}{y} + \frac{1}{z}$$

$$\square\ c = \frac{zx}{z + x}$$

$$\frac{1}{c} = \frac{z + x}{zx}$$

$$\frac{1}{c} = \frac{1}{z} + \frac{1}{x}$$

Let us find the value of

$$\frac{1}{a} - \frac{1}{b} + \frac{1}{c}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} - \frac{1}{y} - \frac{1}{z} + \frac{1}{z} + \frac{1}{x}$$

$$\Rightarrow \frac{2}{x} = \frac{1}{a} - \frac{1}{b} + \frac{1}{c}$$

$$\Rightarrow \frac{2}{x} = \frac{ab + bc - ac}{abc}$$

$$\Rightarrow x = \frac{2abc}{ab + bc - ac}$$

***

Problem solving camp

All the students in a problem solving camp took a 100 point test. Five students scored 100, each student scored at least 60, and the mean score was 76. What is the smallest possible number of students in the class?

Solution to 'Problem solving camp'

Method - I:

Let the number of students in the class be x.

Mean score = 76

Total score = 76 .x = 76x

Five students scored 100

Five students total = 500

Each student scored at least 60

Total marks = (76x - 500) $\geq$ 60(x-5)

$\Rightarrow x = 12.5 \approx 13$

The smallest number of students who took part in the test x = 13.

Method - II:

There are 5 students who scored 100.

Mean score = 76

The marks above the mean score = 5(100 - 76) = 120

But remaining scored at least 60.

The difference of 76 and 60 is 16

The marks scored above the mean must be compensated by students who scored below the average.

The minimum number of the students below average = 120/16 = $\approx$ 8

So the smallest number of students who took part in the test = 5 + 8 = 13.

***

Sum = Product = Quotient?

Find two numbers x and y such that the sum, product and quotient of the two numbers are equal.

Solution to 'Sum=Product=Quotient?

Given that sum of the numbers x and y =

Product of the numbers x and y = quotient of the numbers x and y

$$\Rightarrow (x + y) = (x.y) = (x/y)$$

Case I

$x + y = x.y$

$\therefore\ xy - x = y$

$\therefore\ x(y - 1) = 0$

$\Rightarrow x = 0$ or $y = 1$ -----------------(1)

Case 2

$$x + y = \frac{x}{y}$$

$$xy + y^2 - x = 0$$
$$\Rightarrow y + y^2 = 0$$
$$\Rightarrow y(y+1) = 0$$
$$\Rightarrow y = 0, y = -1$$

From (1) we have

$$x(y-1) = y,$$

$$x = \frac{y}{y-1}$$

$$\Rightarrow x = 0, x = \frac{1}{2}$$

$x = 0$ and $y = 0$ are trivial solutions and $\frac{x}{y}$ is not defined. So the solutions are $x = \frac{1}{2}$ and $y = -1$.

***

Sum of cubes

If $m + n = 3$ and $m^2 + n^2 = 6$. Find the numerical value for $m^3 + n^3$

Hint: Use the algebraic identities $(a^2 + b^2)$ and $(a^3 + b^3)$ or use substitution method.

Solution to 'Sum of cubes

Given $m + n = 3$ and $m^2 + n^2 = 6$

We know by identity,

$m^2 + n^2 = (m + n)^2 - 2mn$

$\Rightarrow 6 = (3)^2 - 2mn$

$\Rightarrow 6 = 9 - 2mn$

$\Rightarrow mn = 3/2$

We know by identity, $m^3 + n^3 = (m + n)^3 - 3mn(m + n)$

On substituting the numerical values of $(m + n)$ and mn we obtain

$$m^3 + n^3 = \frac{27}{2}$$

***

Grand mother

My son is five times as old as my daughter. My wife is five times as old as my son. I am twice as old as my wife. Grand mother, who is as old as all of us put together, is celebrating her eight-first birth day today. How old is my son?

Solution to 'Grand mother'

Let the age of the son be x years. (Say)

Daughter's age will be $1/5(x)$ years. (Since, son's age is five times the daughter's age)

Wife age will be $5(x)$ years (since wife age is equal to five times the age of the son).

Father's age = 2(wife's age) = $2(5x) = 10x$ years.

Grand mother's age = (sum the ages of all)

$\Rightarrow 1/5(x) + x + 5x + 10x = 81$ years (since she is celebrating her 81st birth day)

$x = 5$ years, therefor the son's age = 5 years.

***

Year 1901

If 1st Jan 1901 was a Tuesday, then what day of the week will be 1st Jan 2001?

Solution to 'Year 1901'

1st Jan 1901 was a Tuesday.

The difference between 2001 and 1901 is 100 years.

In one year we have 365 days, $365 \div 7$ is 52 with a remainder of 1 day.

For leap years there is an extra day. There are 25 leap years between 1901 and 2001.

So number extra of days over and above 52×100 weeks is = 100 + 25 =125 days

$125 \div 7 = 17$ remainder 6

6th day from Tuesday is Monday.

$\Rightarrow$ 1st Jan 2001 will be MONDAY.

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Problem by Shakuntala Devi

The dates -- 5.01.05 and 1.05.05 are interesting dates because the product of the date and the month number equal to year number. Can you find the year in the twentieth centaury that gives the greatest number of dates of this kind.

Solution to 'Problem by Shakundala Devi'

5 x1 = 5, 1 x 5 = 5

24.01.24==24 x 1 = 24

12.02.24 ==12 x 2 = 24

08.03 .24 == 8 x 3 = 24

03.08.24 == 3 x 8 = 24

06.04.24 == 6 x 4 = 24

04.06.24 == 4 x 6 = 24

The year 2024 has greatest number of dates of this kind.

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Euler's Problem

Three persons play together --in the first game, the first loses to each of the other two as much money as each of them has. In the next, the second person loses to each of the other two as much money as they have already. Lastly, in the third game, the first and second person gain each from the third as much as money as they had before. They, then leave off and find that they have all an equal sum, nearly 24 rupees each. Find with how much money each sat down to play.

Solution to 'Euler's Problem'

This problem can be solved in two ways--- one by using simultaneous equations and other by backward counting.

First method:

Assume that A, B and C had Rs. a, Rs. b and Rs. c respectively at the start of the game.

At the end of the first game

A lost to B and C as much money as they had at the start.

$\Rightarrow$ The money left out with person A is (a - b - c),

Money left with B is b + b = (2b) and

Money left with the C is c + c = (2c)

At the end of the second game

A gained same amount as what he had at beginning, so A has, 2(a - b - c)

B lost as money as they had with them, so B has, 2b - (a - b - c) - 2c = 3b - a - c.

C gained as much money as he had, so C has, 2c + 2c = 4c

At the end of the third game

A gained as much money as he had at beginning of the third game, so he has,

2(a - b - c) + 2(a - b - c) = 4(a - b - c)

B also gained as much money he had at beginning of the third game, so he has,

(3b - a - c) (3b - a - c) = 2(3b - a - c)

C lost as money as the other person had, so C has, [illegible] = (7c - a - b)

But, at the end of all the three of them had the same amount = 24

$\Rightarrow$ 4(a - b - c) = 24 --------------- (1)

2(3b - a - c) = 24 --------------- (2)

7c - a - b = 24 --------------- (3)

On solving the equations we have a = 39, b = 21 and c = 12.

A started the game with Rs 39, B with Rs.21 and C with Rs. 12.

***

Free baggage

When Mr and Mrs Abhishikth took the airplane, they had together 94 kg of baggage. He paid Rs.75 and she paid Rs.100 for excess weight. If Mr.abhishikth made the trip by him self with the combined baggage of both of them, he would have to pay Rs.675. How many kg` of baggage can one person take along with out charges?

Solution to 'Free baggage'

Let the baggage allowed per person be x kg.

Let t be the extra charge per kg of excess luggage.

The extra charge that Mr. And Mrs. Abhishikth together paid was Rs. 175.

So $t(94 - 2x) = 175$ ----------------(1)

The extra charge that Mr. Abhishikth alone paid was Rs. 675 --------------(2)

Subtracting (1) from (2), we have

$tx = 500$

$$or\ x = \frac{500}{t}$$

substituting in 2 we have

$$t(94 - \frac{500}{t} = 675$$

or $94t - 500 = 675$

or $t = \frac{1175}{94} = 12.5$

$$x = \frac{500}{12.5} = 40kg$$

So the maximum luggage allowed per person without extra charge is 40kg.

***

Number of pages

In numbering the pages of a book, a printer used 3289 digits. How many pages were there in the book, assuming that the first page in the book was numbered from one.

Hint :Use the concept of digits used in expressing a number.

Solution to 'Number of pages'

To number the pages from 1 to 9 printers will use 9 digits (One digit on each page)

To number the pages from 10 to 99 the printer will use (90 X 2) = 180 digits.(two digits on each page)

To number the pages from 100 to 999 the printer will use (900 X 3) = 2700 digits.(three digits on each page)

To number the pages from 1 to 999 total number of digits used =

9 + 180 + 2700 = 2889 digits.

The left out digits = 3289 - 2889 = 400 digits, these digits are

Used to number the pages from 1000 and above. (Four digits on each page)
The number of pages which contains 4 digits = 400/4 = 100 pages.

The total number of pages = 9 pages of one digit + 90 pages of two digits + 900 pages of three digits + 100 pages of four digits

The total number of pages of the book = 1099 pages.

***

Goats and Chickens

A farmer has some goats and some chickens. He sent his son and daughter to count how many of each he has. 'I counted seventy heads 'said the son and 'I counted two hundered legs" said the daughter. How many goats and how many chickens does the farmer actually have?

Hint: Frame the equation and solve OR use trial and error method

Solution to 'Goats and Chickens'

Let the number of goats be x and chickens be y.

The total number of heads = x + y = 70 ---(1)

The number for legs goats = 4

x goats will have 4x legs,

The number of legs for chickens = 2

y chickens will have 2y legs

Total number of legs = 4x + 2y = 200 ---(2)

On solving the two equations we get x = 30 and y = 40

The farmer had 30 goats and 40 chickens.

***

Difference of two Squares

There is a number (an integer) which yields a square; if you add 100 to it and another square if you add 168 to it. What number is it?

Solution to Difference of two Squares

$$100 + x = y^2$$

$$168 + x = z^2$$

$$\Rightarrow z^2 - y^2 = 68$$

$$but, z^2 - y^2 = (z + y)(z - y)$$

$$\Rightarrow z^2 - y^2 = 1 \times 68 = 2 \times 34 = 4 \times 17$$

Among above factors of 68 only 2 and 34 can be factors that satisfy the condition of integer.

$$\Rightarrow z^2 - y^2 = 2 \times 34$$

$$(z+y)(z-y) = 2 \times 34$$

$$z - y = 2, z + y = 34$$

On solving, we have $z = 18$ and $y = 16$

$$\Rightarrow x = 156.$$

***

Ball point pens

A certain make of ball point pens was priced rupees 50, in the store opposite a high school, but found few buyers. When, however, the store had reduced the priced, the whole remaining stock was sold for Rs. 3189.What was the reduced priced? Is the condition sufficient to determine the unknown? (Assume that the reduced priced was in whole rupees.)

Solution to 'Ball point pens'

Let the reduced price of the pen be Rs. x and number of pens be y.

Since the reduced priced was Rs.50,

Total stock was sold for Rs.3189

The prime factors of 3189 are 31 and 103

$\Rightarrow x.y = 3189$

$\Rightarrow x.y = 1 \text{ x } 3189 = 31 \text{ x } 103$

$\Rightarrow$ x can have any of the values 1, 31 103 and 3189

But we know that $x < 50$ and $x > 1$

$\Rightarrow x = 1$ or31

$\Rightarrow$ The reduced priced of the pen is therefore Rs. 1or Rs.31. From the context, it is clear that Rs. 31 is a more acceptable solution.

***

Appu and Subbu

Two boys APPU and subbu, twice ran a race of 200 m. In the first heat, Appu gave subbu a head start of 8 m and two seconds running time and then proceeded to beat him by

2 seconds any way. The next time, he gave Subbu 16 m and 5 seconds. but this was too much and Subbu beat him by 20 m. How fast do the boys run? Assume that their speeds did not change through both the races, find how fast the boys run.

Hint : Hint to Appu and Subbu

Find the time taken by each to cover distance Equate the time and solve the equations.

Solution to 'Appu and Subbu'

Let Subbu`s speed be x m/sec and Appu`s speed be y m/sec.

A nice way getting two equations in x and y is to determine for each heat, the length of time the boys were simultaneously engaged in running, that is from the time Appu finally started running to the time the first finished.

In the first heat,

Subbu is down the track at B by the time Appu strata at E, but Appu

Covers the entire 200mts EF while Subbu only makes it as for as C,

Where the reaming distance CD would take Subbu another two seconds.

Now, since it takes Subbu 192/x seconds to go to 192mts from A to D,

it only takes him $t(BC) = (\frac{192}{x} - 4) sec.$ to

go from B to C. Since, this all the time Appu needs to cover the

whole course,

The time taken by Subbu to cover the distance BC =

Time taken by Appu to cover the distance EF.

$$\Rightarrow t(BC) = t(EF)$$

$$\Rightarrow \frac{192}{x} - 4 = \frac{200}{y} \text{ --- (1)}$$

Similarly in the second heat Subbu is at B when Appu stars at E, and Subbu reaches D when Appu is only at G, which is 20 m short of finish line.

The time take by Subbu to cover the distance BD = time taken by Appu to the distance EG

$$\Rightarrow t_(BD) = t_(EG)$$

$$\frac{184}{x} - 5 = \frac{180}{y} \text{ ---- (2)}$$

On solving the eqs. (1) And (2) we get

the speed of Subbu = x = 8 m/sec and the speed of Appu = y = 10 m/sec.

***

Water melons

A farmer harvested ten tonnes of water melons and sent them by river to the nearest town. It is well known that a water melon, as reflected in its name, it is made almost entirely of water. When the barge left the content of the watermelons was 99% water by weight. On the way to the town, the watermelons dried out some what their water content dropped to 98%.

What was the weight of the watermelons when they reached the town?

Hint

Think the quantity which is constant

Solution to 'Water melons'

At beginning watermelons have 99% water and 1% of pulp.

$\Rightarrow$ Amount of pulp = 1% of 10 tonnes= 100kg.

By the time water melons reached the market water was reduced to 98%

Let the weight of the watermelons on reaching the market be x tonnes.

$\Rightarrow$ The amount of pulp has become 2%of the x tonnes of watermelons.

But we know the amount of pulp = 100kgs

$\Rightarrow$ 2% of x tonnes = 100kgs (since, only water evaporates but not the pulp)

$\Rightarrow$ The weight of the watermelons by the time they reached the market

x = 5000kg = 5tonnes.

Comment: The answer is surprising. But the change in the percentage from 1% to 2% is a doubling. Since the amount of pulp has remained the same, the total weight must have reduced by half.

***

Remainder of a polynomial

A polynomial has a remainder of 3, when divided by (x-1) and a remainder 5 when divided by (x-3). What is the remainder, when the polynomial is divided by (x-1) (x-3)?

Hint

Use Remainder theorem

Solution to 'Remainder of a polynomial'

Suppose that p(x) is the polynomial, and let r(x) be the remainder when p(x) is dividedly (x-1)(x-3). Then

$p(x) = (x-1)(x-3)q(x) + r(x)$, for some polynomial q(x), and r(x)

Has a degree 2 or less. That is, $r(x) = ax + b$, for some numbers a and b.

We know, by Remainder theorem, when p(x) is divided by (x-t) the remainder is equal to p(t).

$p(x) = (x-1)(x-3)q(x) + (ax+b)$ (since, $r(x) = ax + b$)

$P(1) = (1-1)(1-3)q(x) + a(1) + b$

$p(1) = a + b = 3$ (given)---(eq.1)

$P(3) = (3-1)(3-3)q(x) + a(3) + b$

$p(3) = 3a + b = 5$ (given) -- (eq.2)

on solving the eqs (1) and (2) we get $a = 1$ and $b = 2$

$\Rightarrow r(x) = ax + b = x + 2.$

The remainder is (x + 2).

***

Positive Integers

How many numbers of the three digit positive integers are 12 times the sum of their digits?

Solution to 'Positive Integers'

Let the three digits of the number be a, b and c

$\Rightarrow$ The number = 100(a) + 10(b) + 1(c)

As per the condition, the number = 12times the sum of the digits

$\Rightarrow$ 100a + 10b + c = 12(a + b +c)

$\Rightarrow$ 88a – 11sc = 2b

$\Rightarrow$ 11(8a - c) = 2b

In above equation L.H.S is a multiple of 11 $\Rightarrow$ R.H.S (2b) also to be a multiple of 11

Since b<9, the only possibility is R.H.S = 0

$\Rightarrow$ L.H.S = (8a - c) = 0 = R.H.S

$\Rightarrow$ 8a = c

Since $1 \leq a \leq 9$ and $0 \leq c \leq 9$

$\Rightarrow$ If a = 1 and c = 8

$\Rightarrow$ Only one number is possible, that is 108.

Since 108 = 12 (1 + 0 + 8).

***

Digit

The left most digit of an integer of length 2000 digits is 3. In this integer, any two consecutive digits must be divisible by 17(or 23. The 2000th digit may be either a or b. What is the value of (a + b)?

Solution to 'Digit'

Two digit multiples of 17 are 17, 34, 51, 68 and 85

Similarly two digit multiples of 23 are 23, 46, 69 and 92.

The left most digit of 2000 digit number is given is 3, so the only

Multiple that starts with 3 is 34.

The second digit is 4

Similarly the third digit is 6, since the multiple that starts with 4 is 46.

The fourth digit can be either 8 or 9, since there are two multiples that start with 6 that is 68 or 69.

Now we have two cases,

Case--(1)

If fourth digit is 8 then fifth digit is 5, since the multiple that start with 8 is only 85.

The number will be 3468517 and terminates here, since there is no multiple of 17 or 23 which start with 7. But it is given that the number has 2000 digits; therefore the fourth digit cannot be 8.

Case --- (2)

Let the fourth digit be 9, the number will be 34692 3469234....

The five digits continue to repeat till the end of 1995 digits.

For the last five digits we have two choices that are they can be either 34692 or 34685.

$\Rightarrow$ The 2000th digit can be either 2 or 5.

$\Rightarrow$ a = 2 and b = 5

$\Rightarrow$ a + b = 2 + 5 = 7.

***

Men and Work

"x men work x hours a day for x days to dig a x meters long tunnel"." If y men work y hours a day for y days". What length of the same tunnel would you expect them to dig?

Solution to 'Men and Work

Given that x men work x hours a day for x days to dig a x meters long tunnel.

$\Rightarrow$ One man working x hours a day in x days can dig (x/x) = 1 m long tunnel

One man working one hour a day in x days he can dig 1/x m long tunnel.

one man working one hour a day in one day he can dig $\frac{1}{x^2}$ m long tunnel.

Y men working for one hour a day in one day they can dig $y \cdot \frac{1}{x^2}$ m long tunnel

Y men working for y hours in one day they can dig $y^2 \frac{1}{x^2}$ m long tunnel

y men working for y hours for y days they can dig y $\frac{y^2}{x^2}$ m long tunnel

$\Rightarrow$ y men working for y hours in a day for y days they can dig $\frac{y^3}{x^2}$ meters long tunnel.

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Hypotenuse from Area and Perimeter

The area and perimeter of a right angled triangle being given, find the hypotenuse.

------ NEWTON.

Solution to 'Newton's Problem'

Let the sides of the right triangle containing the right angle be a and b, and the hypotenuse be c units.

The area of the right triangle is A sq. uints and perimeter be p units.

$\Rightarrow a + b + c = p$ ---- (1)

The area of the right triangle = 1/2.a.b = A

a.b = 2A --- (2)

$a^2 + b^2 = c^2$ --- (3) by Pythagoras theorem.

We know, a + b + c = p

$\Rightarrow a + b = p - c$

Squaring on both sides

$(a + b)^2 = (p - c)^2$

$a^2 + b^2 + 2ab = p^2 + c^2 - 2pc$

$\Rightarrow c^2 + 2(2A) = p^2 + c^2 - 2pc$ ------------(From eq. 2 and 3)

$\Rightarrow$ Hypotenuse $c = \dfrac{p^2 - 4A}{2p}$.