PROBLEM SOLVING IN GEOMETRY

Lectures by K. Subramaniam
Lecture notes (draft) Written by K. M. Bhatt

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PROBLEM SOLVING

Problem solving is the process of accepting a challenge and striving to resolve it. The teaching of problem solving, then, is the action by which a teacher encourages students to accept challenging questions and guide them in their resolution. Problem solving requires students to engage in a process and hopefully to become skilful in selecting and identifying relevant conditions and concepts, searching for appropriate generalisations, formulating plans, and employing previously acquired skills.

IMPORTANCE OF TEACHING PROBLEM-SOLVING IN MATHEMATICS

Most mathematics educators believe problem solving is an important instructional activity. It is because problem solving enables the students to become more analytic in making decisions in life and also helps them to know the importance of value judgement that has paramount importance in learning mathematics. Few mathematicians regard problem solving as the basic mathematical activity and other activities such as generalisation, abstraction, theory-building, and concept formation are based on problem solving.

In the problem solving sessions, conducted by Dr. K. Subramaniam, we have worked out the problems from the book "Mathematical Discovery" by G. Polya. These sessions were attended by Dr. H.C. Pradhan, Sri K.M. Bhatt, Shekar Agre, Dhruva, Kum. Swati and Kum. Yogitha. The discussions in each session helped us to explore New Mathematics.

PROBLEM SOLVING

Session – 1 :-

DR. SUBRAMANYAM.

TOPIC :- PATTERN OF TWO LOCI. FROM: Mathematical discovary by GEORGE POLYA.

PROBLEM: 1:

The given Qn is about Constructing an $\triangle ABC$, Given that a, b, c, the measure of the sides.

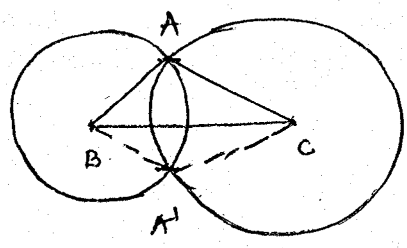
Method:- use one condition and drop the other two.

→ Draw a line segment BC = b.

→ Keeping B as centre & a radius of a describe a circle.

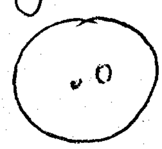
→ Repeat the same procedure with C as centre & a radius of c.

→ The intersection of the circles gives vertex A. Thus Complete the $\triangle ABC$ or $\triangle A'BC$



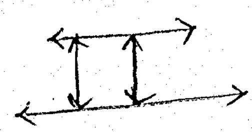
Prob: 1.1:

→ The locus of a variable point that has a given distance from a given point is the [illegible] circle with given point as centre.



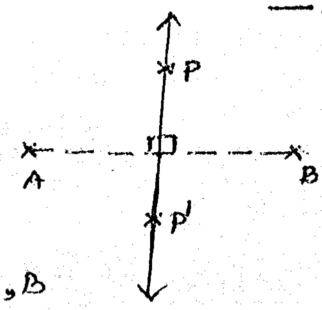
1.2.

→ The locus of a point with a given distance from a straight line is the line parallel to the given Straight line.



Discussions are stressed more on pattern & relationship.

1.3: The Locus of a variable point that remains equidistant from two given points:

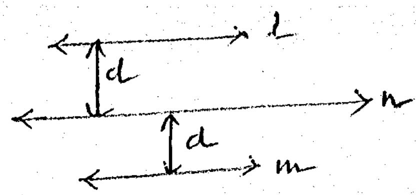


→ The diagram is plotted showing A, B

→ Join AB. & Draw ⊥lar bisector of AB

→ The Locus is ⊥lar bisector of AB.

1.4:

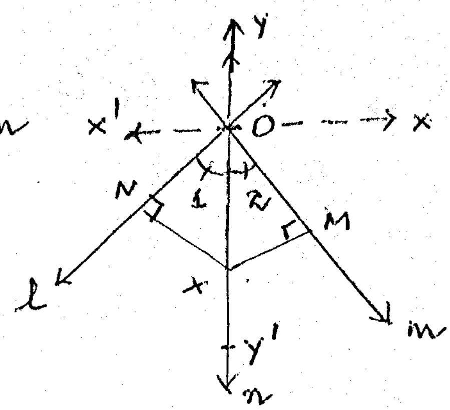


→ Draw straight line n such that distance 'd' between l, & n = d

→ And distance between m, n = d.

→ Thus the locus is the line n such that $l \parallel n \parallel m$.

1.5.



→ Draw the intersecting lines l, m intersecting at O.

→ Draw a line n through O such that XN = XM where XN ⊥ l, XM ⊥ m

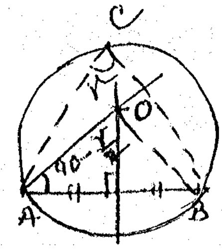
→ Now ∠1 = ∠2 by RHS Congruency condition.

→ So the locus is the angular bisector of ∠MON.

→ It is possible to have the line at other pair also.

→ So Locus is $(XOX') \cup (YOY')$.

1.6



→ First the given vertex AB are plotted ⇒ the Linesegment AB is obtained

→ Then it is necessary to draw ⊥lar bisector of AB.

→ At A Construct an angle $90° - \gamma$ to intersect the ⊥lar bisector at O.

→ With O as centre and OA as radius, Draw circum. circle.

→ So the Locus is the major arc with AB as Chord.

Some useful notations discussed by polya:

→ Angles as α, β, γ and sides as a, b, c.

→ h_a, h_b, h_c are altitudes from A, B, C.

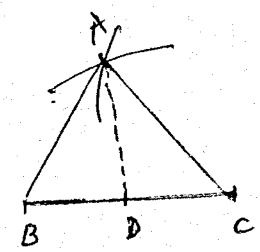
→ d_a, d_b, d_c are angular bisectors.

→ R: radius of circum circle

→ r: radius of incircle.

→ m_a, m_b, m_c are medians from A, B, C.

Prob: 1.8:

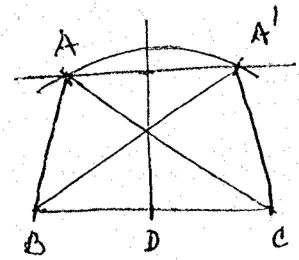


→ When median m_a and two sides are given as a, b → better draw $BC = a$

→ identify its mid point say D.

→ Draw $AC = b$ from C and $AD = m_a$ from D, to complete the Δ.

Prob: 1.9:



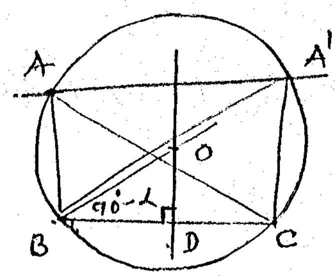
→ When only one side 'a' is given, So plot $BC = a$ & Mark its mid point D.

→ At D Construct an altitude of length h_a & then we need to draw a line parallel to BC.

→ Now with the measure of the median

intersect the parallel line at two points A & A', Keeping D as centre. This will give us ΔABC & $\Delta A'BC$.

1.10:

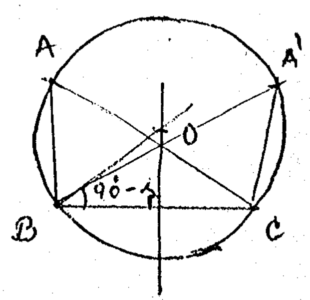


→ In the construction of the Δ with a, ha and α, we need as in the earlier case to draw BC first.

→ Bisecting BC ⊥ly at D and constructing ∠B = 90° − α to intersect the ⊥lar bisector at O. This will lead to the drawing of circum circle through B, C with O as centre.

→ further a line ∥ to BC to the height ha from D will give us point A, A'.

1.11:-

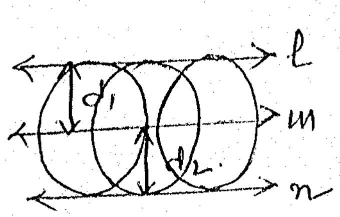


→ In the Construction of the Δ with a, ma and α the above said procedure in problem 1.10 is repeated to get the circum circle.

→ further with median length the points A, A' are obtained on the circle

1.12:-

Possibility - 1 :



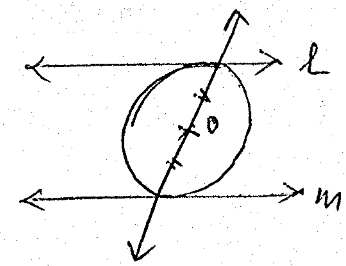
→ When the three lines l, m, n are parallel to each other and $d_1 = d_2$

→ Infinite circles can be constructed with the centre is on the line m.

Possibility 2:

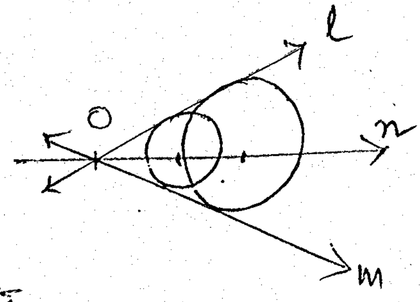
→ Two lines l, m are parallel to each other and the third line n is the transversel.

→ The centre of the circle O lies on the tranversal.



Possibility : 3

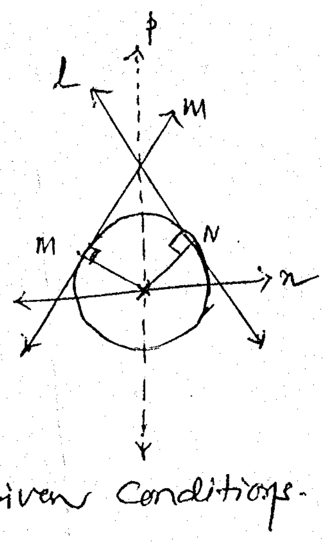
→ Two of the lines l, m intersect at a point O and n also passes through O such that it acts as the angular bisector of vertically opposite angles.



→ In this case also we can draw infinite no. of circles with their centre is on n

Possibility 4:

→ Two lines l, m are intersecting at a point O and line n is intersecting both l, m and the angular bisector as shown in the fig.

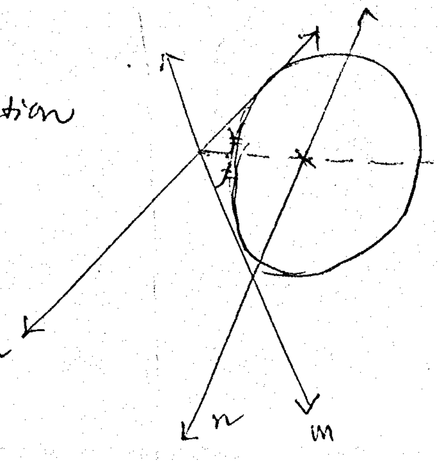


→ In this case also we can have many circles constructed satisfying the given conditions.

Possibility 5:

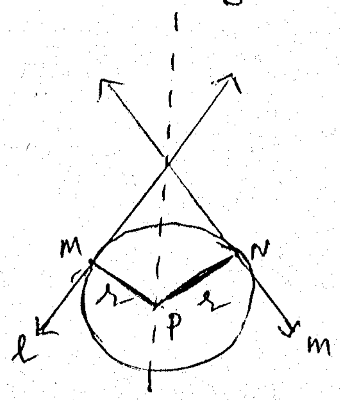
→ It is possible to have the construction of the circle with the lines l, m, n intersecting as shown in the figure

→ By varying the position of n we can draw many circles.



1.13

→ Let us say that l, m are the given intersecting lines, intersecting at the point O.

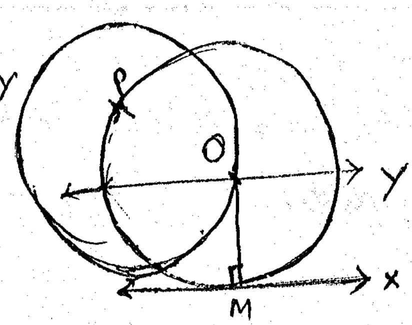


→ First of all Draw the angular bisector of the v. opposite angles as shown. This will help in obtaining the required line segment of length r and it is the ⊥er distance from the angular bisector.

→ Now the circle with P as centre and r as radius will have the necessary condition satisfied.

1.14

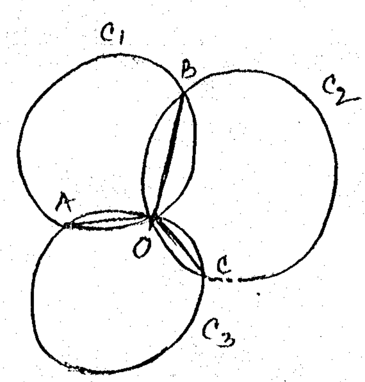
→ To begin with let us have a line Y And now from the point 'P' which is given and a radius r which is also given draw the circle. Let this circle intersect the line Y at O.



→ Now keeping O as centre complete another circle with radius r

→ from O, drop OM and at M construct MX ⊥ OM. This gives X which is tangent to the circle.

1.15

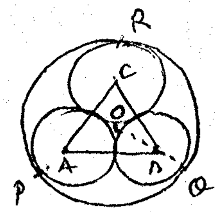


→ first we need to identify the centre of circle C_1 taking the distances OA, OB as the chords of this circle.

→ Similarly we repeat it for circle C_2 and C_3.

→ The point of concurrency of all circles i.e O will give the position of the ship!

1.16 :-

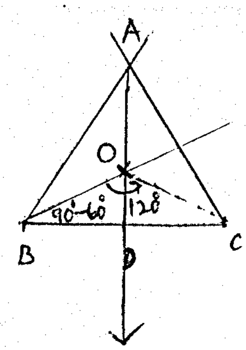


OQ = 2 OB .

→ Since all the inner circles are of equal r & their centres joined together form an equilateral Δ.

→ Now a circle constructed with the radius, twice that of the R (radius of circum circle) of the equilateral Δ give us a circle with all conditions satisfied.

1.17



→ Let BC be any arbitrary linesegment.

→ The mid point BC then is identified as D

→ at D we will draw a ⊥er which is inter sected by an angular ray of 90°- 60°

→ This gives measure of $\angle BOC = 120°$. And hence BC is seen under 120°.

→ Same is repeated for other sides of the Δ also.

→ The Δ must be equilateral & the point is common for orthocentre, incentre, circum centre, Centroid

Problem Solving

Session – 2:

DR. K. Subramanyam

→ Revision of the problems discussed in Session 1, which is followed by the discussion of problem 1.16.

Problem: Within a given circle, describe three equal circles so that each shall touch the other two and also the given circle.

Case-1:- Three circles inside a circle.

→ What is known or given

→ The circle of given radius

→ So first construct the circle of given radius

→ Then shall we have an equilateral Δ inscribed in the circle?

→ Yes, let us consider the Δ ABC with AB = BC = AC inside the circle.

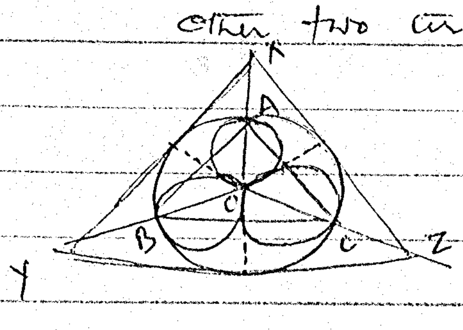
→ Join the vertices of the Δ to the centre of the circle ~~and~~ to get three angles & bisect these angles.

→ Then draw a tangent at the point where the angular bisector touches the circle.

→ Further produce this tangent (if necessary) to meet the OA and OC produced at X & Z.

→ Then construct incircle for the Δ XOZ.

→ Now we can repeat this process for obtaining other two circles.



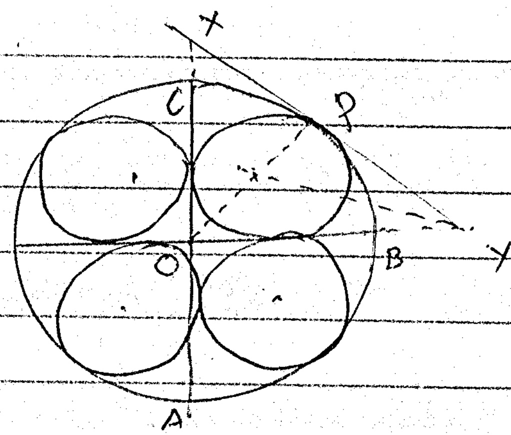
Here $r = \frac{\sqrt{3}R}{\sqrt{3}+2}$

R = radius of ~~small~~ large Ωle

r = radius of small Ωle.

Case: 2 : Inscribing 4 circles within a circle.

→ Given data is the circle.
→ So draw the circle
→ Then divide it into 4 quarters.
→ Draw the angular bisector of one of the quarter from O meeting the circle at P.
→ Now we need to draw a tangent at P to meet the OA and OB produced at X & Y.
→ Construct in circle to ΔXOY, which will satisfy the necessary conditions.
→ We have to repeat the same for other quarters also.



Here

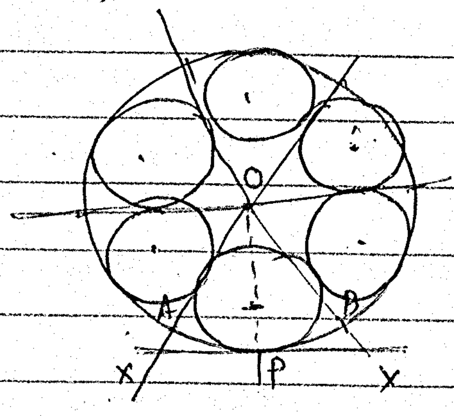
$$r = \frac{R}{\sqrt{2}+1}$$

Case: 3 : Inscribing 6 circles within a given circle.

→ Given for us is the circle.
→ So let us complete the circle of convenient radius first.
→ Further, since we need six circles to be inscribed, divide the circles into six equal parts.
→ Construct the angular bisector of one part to meet the circle at P.
→ Consider a tangent at P to meet the two sides OA & OB produced at X, Y.

→ Now construct incircle to $\triangle OXY$, which satisfy the given conditions.

→ finally we have to repeat the process for remaining five segments.



Here $r = \frac{1}{3}R$.

Discussion on $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\sin\theta\left(\sqrt{1-\cos^2\theta}\right)$$

We can put it as $\quad k = 2x\sqrt{1-x^2}$

$$\frac{k^2}{4x^2} = 1-x^2$$

put $x^2 = y$

$$\frac{k^2}{4y} = 1-y$$

This reduces to quadratic form: $k^2 = 4y - 4y^2$

—o✗—

PROBLEM SOLVING

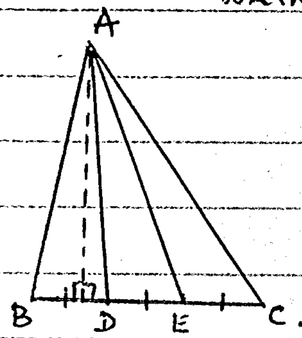
SESSION - 3

DR. SUBRAMANYAM

08.01.1997.

Problem: 1.18 : Trisection of a given triangle.

method → 1.

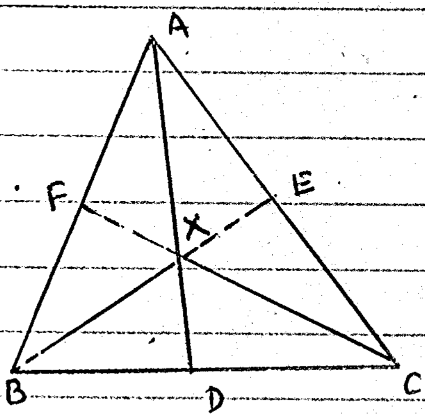


→ Construct the Δ of given dimension

→ Trisect the base into three equal parts.

→ join these into the vertex to get the Δ's having equal area.

→ But this method is not excepted as the condition given in the question rules it out.



Method : 2

→ Construct the given triangle.

→ Drop the three medians from the opposite vertices to meet at X.

→ Now ar ΔXAB = ar ΔXBC = ar ΔXAC

→ reason:

→ AD divides Δ ABC into two equal parts.

ie ar Δ ABD = ar Δ ACD.

→ Also in ΔXBD, ar Δ XBD = ar Δ XCD. by similar argument

→ Therefore ar Δ ABD − ar ΔXBD = ar ΔACD − ar ΔXBD

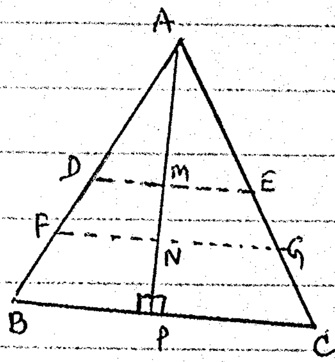
→ That gives ar Δ XAB = ar ΔXAC.

→ By similar argument we can prove that ar ΔXAB = ar ΔXBC.

→ Therefore ar ΔXAB = ar ΔXBC = ar ΔXAC.

Prob: 1.18 Continued:

* To divide the Δ into three ~~equal~~ parts having the same area such that the lines are parallel to the base.



→ Construct the Δ ABC

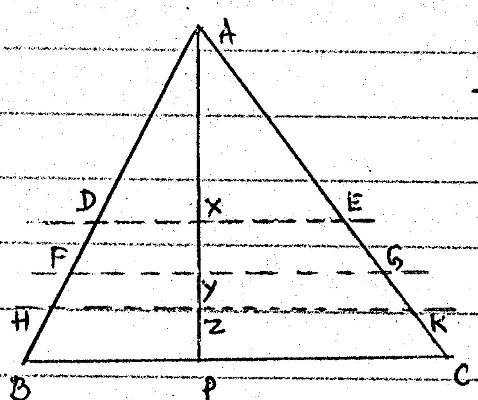
→ Put the altitude AP to BC.

→ Cut AP at M such that $AM = h/\sqrt{2}$

→ Cut AP at N such that $MN = h/\sqrt{3}$

→ Now ar Δ ADE = ar trap DEGF = ar trap FGCB.

* To divide a triangle into four equal parts:



→ Construct the given Δ ABC

→ put the altitude AP to BC.

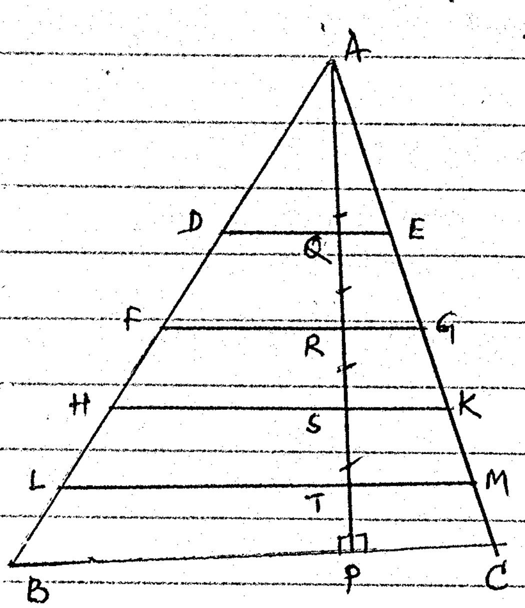
→ Divide AP into two equal parts at X

→ Draw another line FG ∥ DE to cut AP at Y such that $XY = h/\sqrt{2}$.

→ Similarly Draw HK ∥ FG, such that $YZ = \frac{\sqrt{3}h}{2}$.

→ Now all the four parts have equal area.

* To divide the triangle into 5 parts of equal area.



→ Construct the given triangle ABC

→ Draw perpendicular AP to BC

→ Draw DE ∥ BC at cut AP at Q, such that $AQ = h/\sqrt{2}$.

→ also draw FG ∥ BC, such that $QR = \frac{\sqrt{2}h}{\sqrt{5}}$

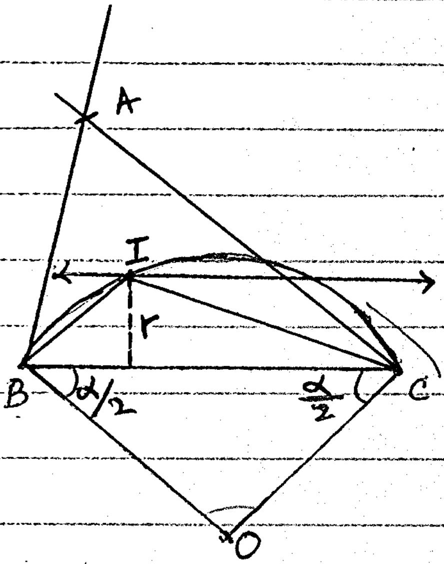
→ draw HK ∥ BC, such that $RS = \frac{\sqrt{3}h}{\sqrt{5}}$.

→ draw LM ∥ BC, such that $ST = \frac{2h}{\sqrt{5}}$

→ Now all the five parts have equal area

Prob: 1.19

TO Construct a triangle from a, α and r.



→ discussions went on for about 20 minutes about different way of the construction.

→ So the solution emerged with the following steps.

→ draw BC as per given measurement

→ at B and C, below BC, construct the angle $\alpha/2$, to meet at O.

→ with O as centre, draw an arc through B, C.

→ Now draw a line parallel to BC through the arc. at a distance equal to r, to cut the arc at I.

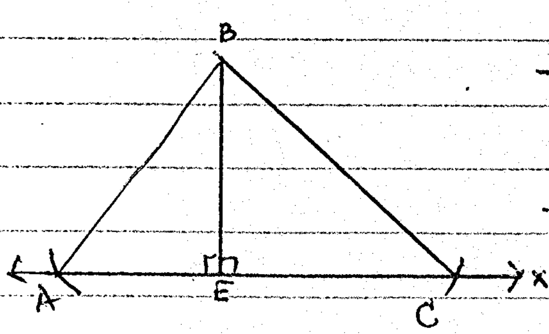
→ join IB and IC.

→ Now at B measure an angle equal to twice that of $\angle IBC$ & at C measure an angle equal to twice that of $\angle ICB$, such that both meet at A.

→ Δ ABC is the ~~eqe~~ required Δ.

Problem: 1.20:

TO Construct a Δ given a, h_b, c.

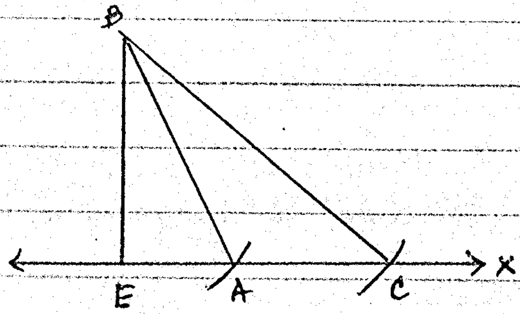


→ put any Line X and mark a point E on it.

→ At E draw the perpendicular BE equal to h_b.

→ from B Cut on either side of X with a and c, to obtain A, C. Complete the Δ ABC.

→ If we cut a, c on the same

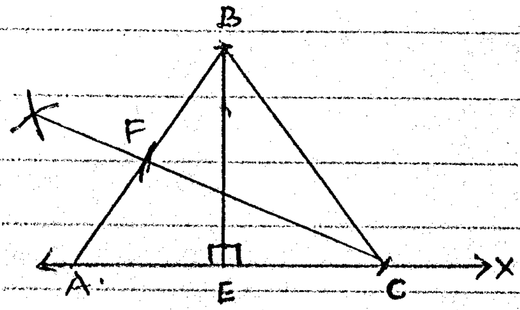


side of BE, then we get an **obtuse angled** Δ.

→ So there are two types of Δs formed.

Problem: 1.21:

To construct a triangle with a, h_b and d_γ.



→ draw any line X and mark a point E on it.

→ at E draw altitude $BE = h_b$.

→ Now from B put an arc for the length $BC = a$ to cut X at C.

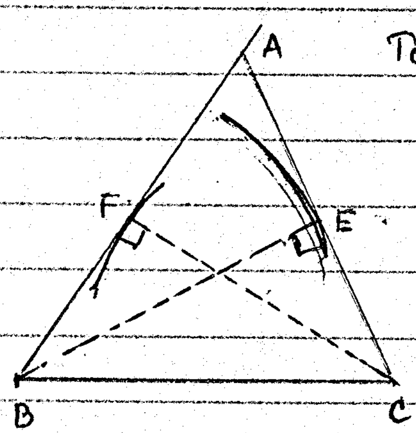
→ further construct the angular bisector of $\angle C$ and cut it for a length of d_γ at F

→ Join B through F to meet X at A.

Δ ABC is the required Δ.

→ only one such Δ can be constructed.

Prob: 1.22:



To construct a Δ from a, h_b, h_c.

→ Construct $BC = a$.

→ From B, put an arc of radius h_b.

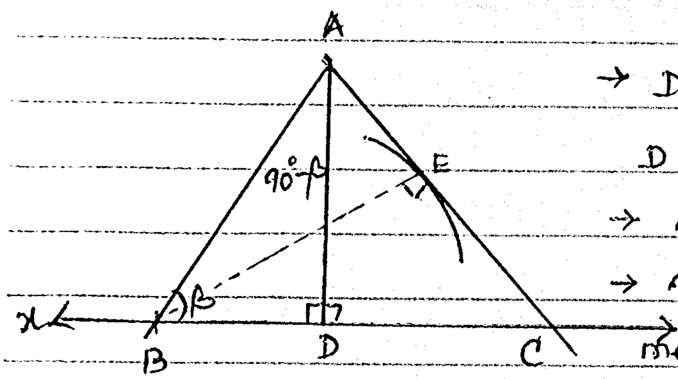
→ From C, put an arc of radius h_c

→ Now draw tangents from C and B to these arcs to meet at A.

→ Δ ABC is the required triangle

Prob: 1.23. (a)

To construct a Δ, given h_a, h_b, β.



→ Draw any line x and mark a point D on x.

→ At D, draw perpendicular $AD = h_a$.

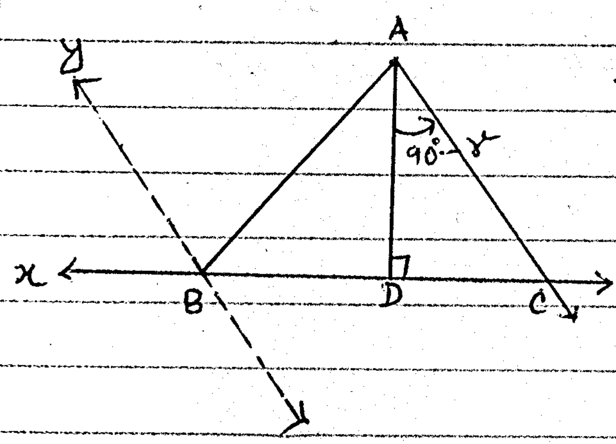
→ At A construct an angle $90-\beta$ to meet x at B.

→ from B, with a radius of h_b draw an arc.

→ Now from A, draw a tangent to the arc to meet x at C.

Δ ABC is the required triangle.

(b) To construct a Δ, given h_a, h_b, γ



→ Draw any line x and mark a point D on it.

→ at D, draw AD perpendicular equal to h_a.

→ at A measure an angle $90°-\gamma$ to meet x at C.

→ Now draw $y \parallel AC$ to intersect x at B.

→ Δ ABC is the required triangle.

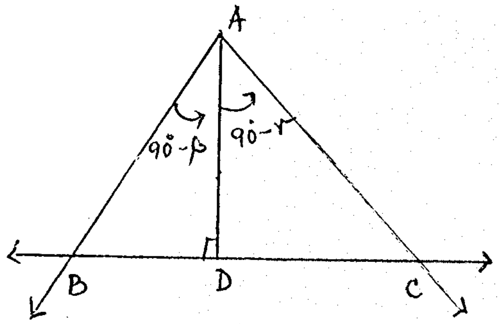
Problem Solving

14/1/97

By DR. K. Subramanyam.

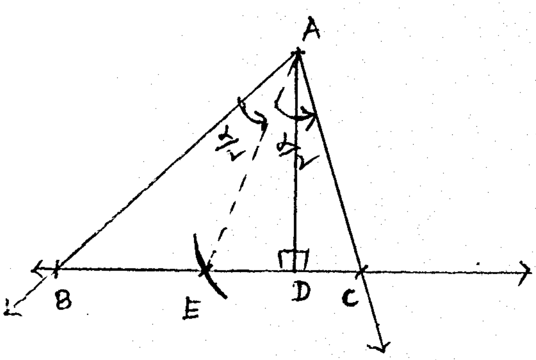
Session - 4:

Problem: 1.24. Construction of a triangle from h_a, β, γ.



- Draw a line XY.
- Mark some point D on XY.
- At D. Construct a perpendicular AD for a length equal to h_a.
- At A, to one side construct an angle $90-\beta$ and to the other side $90-\gamma$ such that both intersect at B, C.
- $\triangle ABC$ is the required triangle.

Problem: 1.25: Construction of a triangle from h_a, d_α, α.

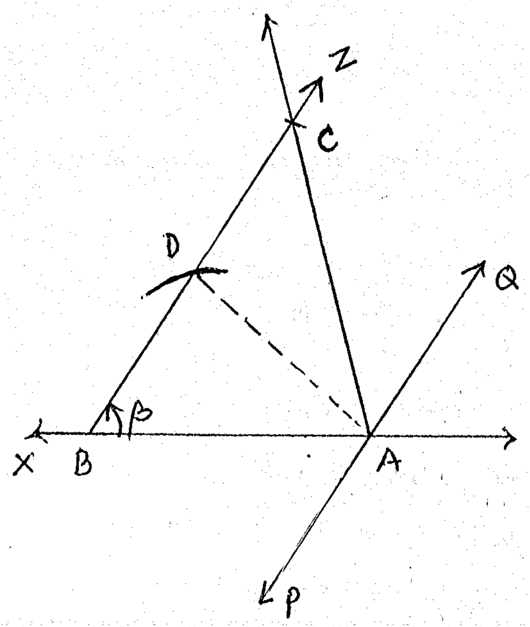


- Draw a line XY.
- At any point D on XY, Draw perpendicular AD of length h_a.
- Further with a radius of d_α put an arc to intersect at E.
- Now At A, Construct angle $\alpha/2$ on both the sides of AE, to intersect at B and C.
- $\triangle ABC$ is the required triangle.

Cont'd in: Next page -

Problem 1.25 (S_1)

To construct a triangle with h_a, d_α, β.



- First draw a line XY and mark the point B on it.
- At B construct an angle equal to β.
- Now draw a line PQ parallel to XY at a distance of h_a to intersect XY at A.
- From A, sweep an arc to cut XY at D
- Now construct an angle equal to $\angle BAD$ on AD to intersect ~~XY~~ BZ at C.
- Δ ABC is the required triangle.

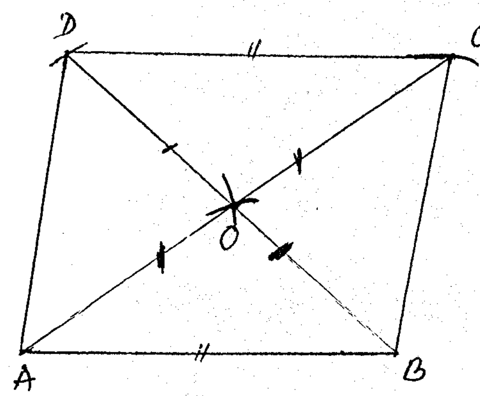
Problem 1.25 (S_2)

To construct a triangle given h_a, d_β, α

Contd — 3.

Problem 1.26 :

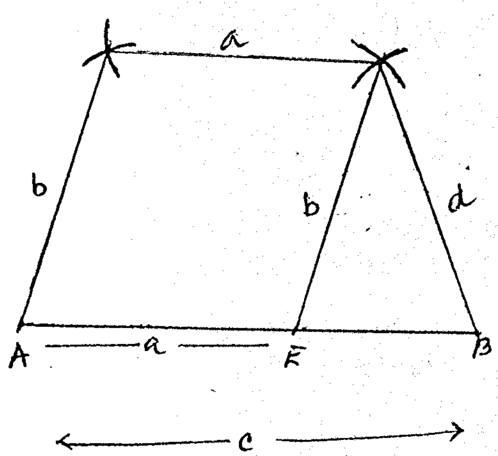
To construct a parallelogram given one side and length of two diagonals.



- Draw a line AB equal to the length of one side given.
- from A put an arc for a radius equal to ½ of one diagonal & from B repeat the same method with ½ of another diagonal to get the Δ AOB.
[Here the property of parallelogram i.e diagonals bisect each other is used].
- Now produce AO to C such that the diagonal AC is obtained
- Also repeat the process at BO to get BD.
- Join A to D, C to D & B to C to get the parallelogram ABCD.

Problem : 1.27

To construct a trapezoid given the foursides a, b, c, d and a||c.

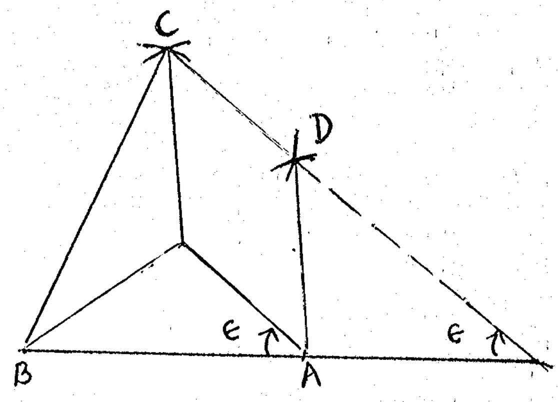


→ Draw a line AB = c
→ Mark a point E on AB such that AE = a
→ from E put an arc of radius of b and from B put the arc of radius d, to ~~meet~~ intersect arc b at C
→ Now with C as centre put an arc of radius a & cut this arc from another arc drawn from A with radius b. Name the intersection point as D.
→ ABCD is the required trapezoid.

Cont'd — 4

Problem 1.28

Construct a quadrilateral given a, b, c and d and angle ∈ which is the angle included by a, c produced.



→ Draw a line AB = c units.

→ At A make an angle ∈.

→ Cut this ray making the angle ∈ by an arc of radius equal to a to get the triangle ABO.

→ Now put two arcs, one from B with the radius BC = d and one from O with the radius b, to intersect at C.

→ Keeping C as reference point put an arc of radius a & intersect this arc at D by another arc put from A with radius equal to b.

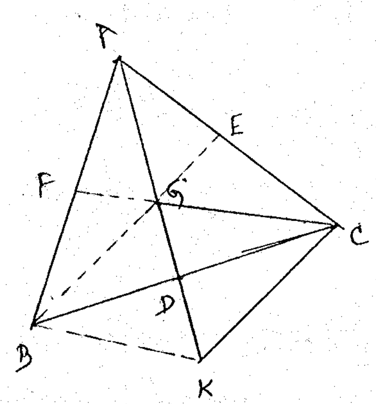
→ Now ABCD is the required Quadrilateral

Session: 05 — Problem Solving — 22-01-1997

DR. K. Subramanyam.

Problem: ① : To construct a triangle, given its three medians.
EXAMPLE'

Method: 1:



→ Construct a triangle GCK with $GC = \frac{2}{3}CF$, $CK = \frac{2}{3}BE$ and $GK = \frac{2}{3}AD$. (where BE, CF, AD are medians).

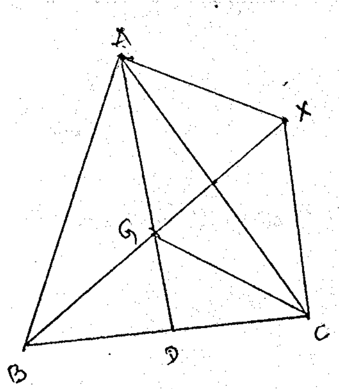
→ Mark D the midpoint of GK and produce it to B such that BD = DC

→ Now Join B to G and K to complete the parallelogram BGCK.

→ Produce KG to A such that AG = GK

→ Join AB, AC to get the triangle ABC.

Method: 2:

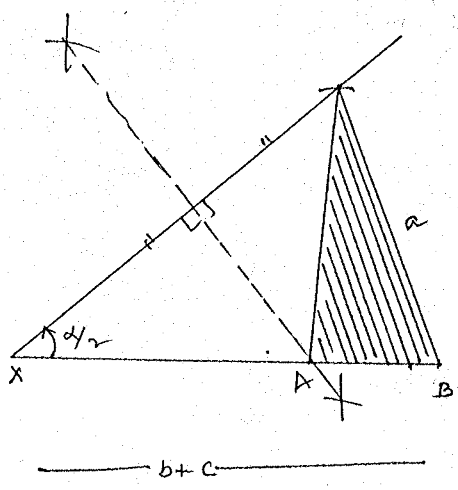


→ First complete the parallelogram AGCX with $GC = \frac{2}{3}CF$, $CX = \frac{2}{3}AD$ and diagonal $GX = \frac{2}{3}BE$ (where AD, BE, CF are medians)

→ produce XG to B such that BG = GX.

Join AB, BC and AC, to get the required ΔABC.

Problem: 1.29 : To construct a triangle given, a, b+c and α.



→ Draw a linesegment, $XB = b+c$

→ At X mark the angle $\alpha/2$ such that $\angle YXB = \alpha/2$

→ Now from B, put an arc of length a on XY, call it as C

→ Join BC.

→ Further Draw perpendicular bisector of XC to intersect XB at A.

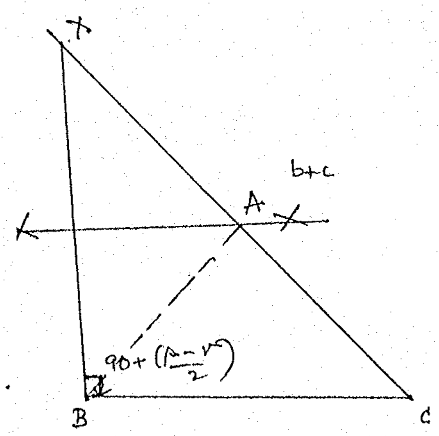
→ Join A to C to get the required triangle.

Problem: 1.30.

To construct a triangle from a, b+c and $\beta - \gamma$.

→ Analysis:

$$\alpha+\beta+\gamma = 180^\circ$$
$$\alpha = 180^\circ - (\beta+\gamma)$$
$$\frac{\alpha}{2} = 90^\circ - \left(\frac{\beta+\gamma}{2}\right)$$
$$\frac{\alpha}{2}+\beta = 90^\circ - \beta/2 + \beta - \frac{\gamma}{2}$$
$$= 90^\circ + \left(\frac{\beta-\gamma}{2}\right)$$



Steps:

→ Draw line segment $BC = a$

→ At B measure angle $90^\circ + \left(\frac{\beta-\gamma}{2}\right)$.

→ Intersect this line from an arc of length b+c to get X.

→ Draw perpendicular bisector of BX to meet XC at A.

→ Join A to B & A to C, to get the required triangle ABC.

Problem: 1.31: Triangle from: $a+b+c$, h_a and α.

Analysis:

$$\beta/2 + \alpha + \frac{\gamma}{2} = \frac{\beta+\gamma}{2} + \alpha$$

$$\text{But } \beta+\gamma = 180^\circ - \alpha$$

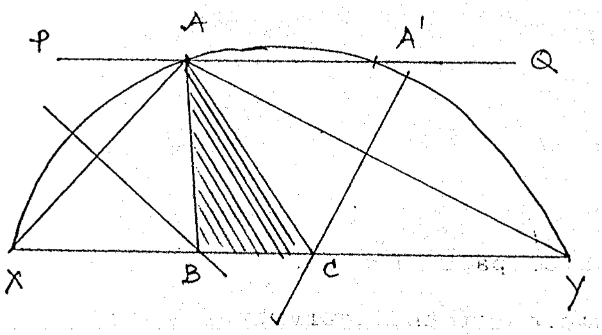
$$\frac{\beta+\gamma}{2} = 90^\circ - \alpha/2$$

$$\therefore 90^\circ - \alpha/2 + \alpha = \underline{\underline{90^\circ + \alpha/2}}.$$

$$\Rightarrow \beta/2 + \alpha + \frac{\gamma}{2} = 90^\circ + \frac{\alpha}{2}.$$

Steps:

→ Draw $xy = a+b+c$



→ Now construct an arc through xy, such that any angle on the arc measure $90^\circ + \frac{\alpha}{2}$.

→ With h_a height draw a line $PQ \parallel xy$ intersecting the arc at A, A'.

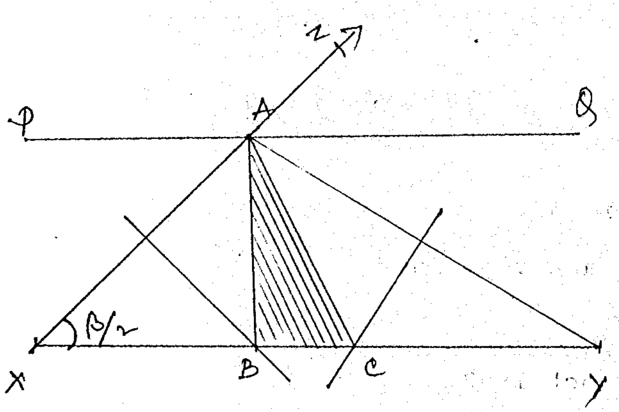
→ Join AX, AY.

→ Draw ⊥ler bisector of AX and AY to cut xy at B, C.

→ $\triangle ABC$ is the required triangle.

Problem: 1.31 (a) Triangle from $a+b+c$, h_a and β.

→ Draw a line $xy = a+b+c$.



→ at x construct the angle $\frac{\beta}{2}$.

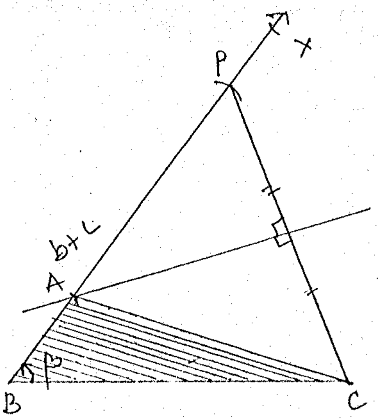
→ Draw $PQ \parallel xy$ to cut, ~~[illegible]~~ such that the distance is h_a such that it cuts xz at A

→ Join AY.

→ Draw perpendicular bisector of AX, AY to meet xy at B, C.

→ $\triangle ABC$ is the required triangle.

Problem: 1.30 (a). Construct a Δ with a, $b+c$ and β.



→ Draw the segment $BC = a$.

→ at B, measure angle β such that $\angle XBC = \beta$

→ Cut of BX to a length of $a+b$ at P.

→ Join PC and draw its perpendicular bisector such that it intersects BP at A.

→ Join AC, to get the required Δ.

Problem: From Example: Given two circles of different radius, draw direct common tangent to them. (Direct common tangent)

→ Draw the two circles & name their centre as O and O'.

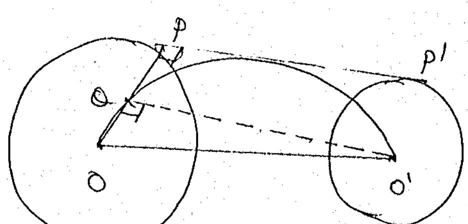
→ Join OO'.

→ Draw a semicircle with OO' as diameter.

→ From O, cut the semicircle at Q, such that $OQ = r_2$, radius of smaller circle.

→ Join QO', also produce OQ to P.

→ Draw $PP' \parallel QO'$, to get the tangents.



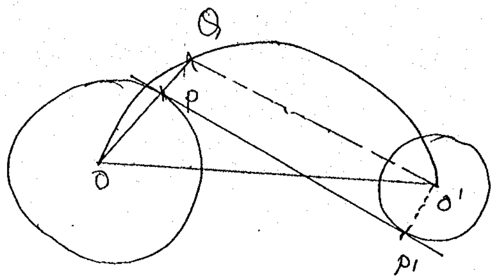
Problem: 1.32: Given two circles, Draw indirect common tangents to them.

→ Draw two circles with O and O' as centre.

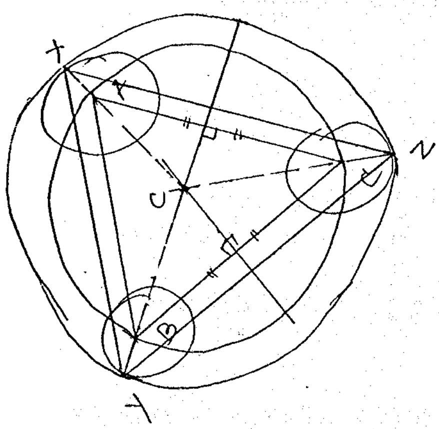
→ Draw a semicircle with OO' as diameter.

→ Cut the semicircle at Q from O, such that $OQ = r_1 + r_2$ (r_1, r_2 are radius of larger and smaller circle).

→ Join Q to O', & Draw a line $PP' \parallel QO'$, which is the required indirect common tangent.



Problem : 1.33 Given three equal circles, construct a circle which contains the three circles and tangent to it.



→ Draw the given circles on the plane.

→ Join their centres to get the △ABC

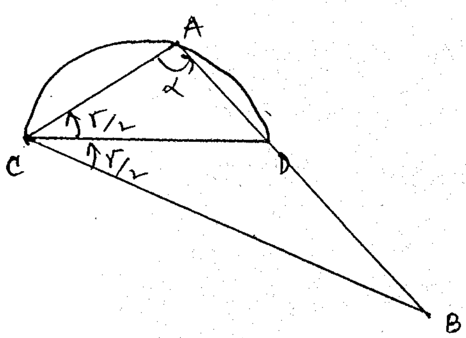
→ Draw a circum circle to this △ABC.

→ Now produce the circum centre through each of A, B, C to meet the circles at X, Y, Z.

→ Draw circumcircle to the △XYZ, which satisfies the required circle. (Red coloured circle)

Session – 6 Problem Solving Dr. Subramanyam.

Problem no : 1·34: To construct a triangle from α, β and d_r.



→ Draw a line segment $CD = d_r$.

→ Now on CD draw an arc such that any point on the arc make the angle α.

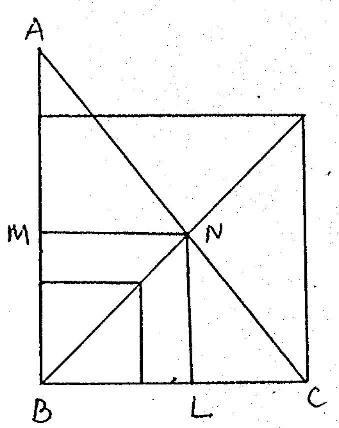
→ At C measure the angle $\frac{\gamma}{2}$ such that it cuts the arc at A.

→ Also on the other side of C measure the angle $\frac{\gamma}{2}$

→ Now produce AD to B such that AB and CB intersect at B.

→ Δ ABC is the required triangle.

Problem : 1·35:- To put a square inside a right triangle.



→ Construct the right angled Δ Given.

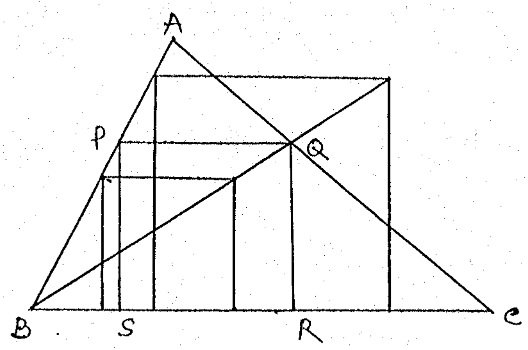
→ Draw a square inside the Δ such that the two sides and one vertex of the square are on the Δ.

→ Now construct another square outside the triangle.

→ Draw a line through the fourth vertex of the two squares to cut the hypotenuse of the Δ at some point.

→ Now from this point draw parallel lines to the two other sides of the Δ. This gives the required square. BMNL.

Problem: 1.36 TO inscribe a square inside any triangle.



→ First let us analyse that: one side of the square lies on side BC, one vertex on the side AB and other vertex on AC.

→ So, first construct the given triangle ABC.

→ Now, draw a small square inside the Δ

→ Also draw two more squares satisfying the conditions discussed in the analysis.

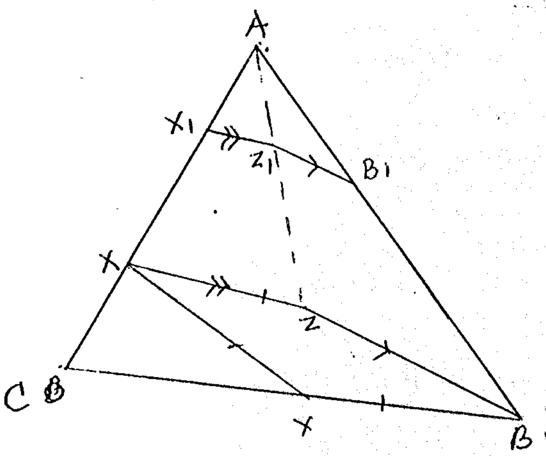
→ Now join all the free vertices such that it cuts AC of the Δ at some point.

→ From this point complete the square. PQRS is the required square.

PROBLEM SOLVING: SESSION: 7 4. 2. 1997

DR. SUBRAMANYAM:

PROBLEM:- Given three points A, B, C., draw a line intersecting AC in X and BC in Y so that AX = XY = YB.



→ Mark the three points A, B, C and join them to form the triangle.

→ Now Draw X_1Z_1 such that $X_1Z_1 \parallel BC$ and $X_1Z_1 = AX_1$, let it meet the line AZ at Z_1.

→ Draw $Z_1B_1 = X_1Z_1$

→ Then Draw $ZB \parallel Z_1B_1$ and $XZ \parallel X_1Z_1$

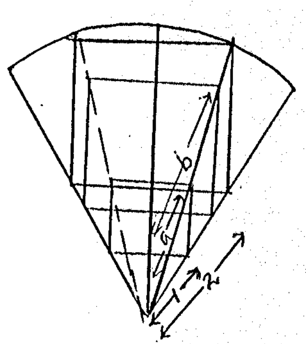
→ Now Draw $XY \parallel ZB$

→ Since XYBZ constitutes a Rhombus, XY = YB and By Construct AX = XY.

Therefore AX = XY = YB.

Problem: 1.37:

Given Sector of a Circle, inscribe a square in it such that two vertices lie on arc and one corner each on the radii that joins the arc.



→ Draw any arc and draw the radius joining the mid point of the arc.

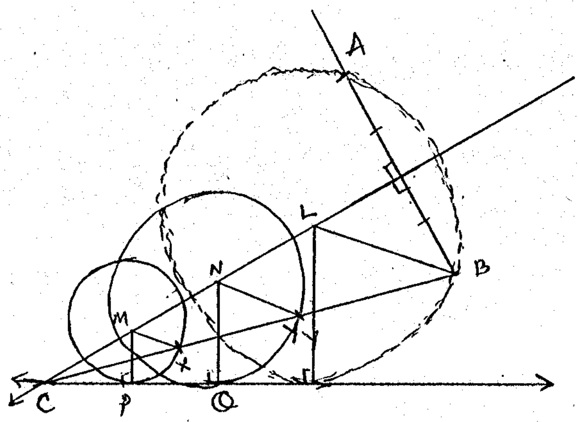
→ Draw two smaller squares inside the sector such that the two ~~corners~~ corners lie on the radius joining the arc.

→ Now Draw a line from the centre through the corners of these squares to meet the arc to get two points.

→ With these two points as reference, complete the square which is the required square for us.

Proof: $\frac{a}{b} = \frac{1}{2} = \frac{2a}{2b}$.

1.38: Construct a circle given two points on it and a straight line tangent to it.



Proof: $\frac{CP}{CQ} = \frac{CX}{CY} = \frac{MX}{NY} = \frac{CM}{CN}$

→ Mark two points A, B and join them.

→ Draw any line x.

→ Construct the perpendicular bisector of AB and if required, produce to cut x at C.

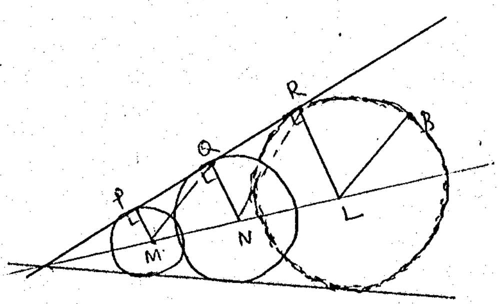
→ Now mark points M, N on the bisector of AB and Draw MP, NQ perpendiculars to x.

→ Further complete the circles with M, N as centres.

→ Now Draw BL || NY || MX

→ Keeping L as centre and BL as radius Complete the circle, which satisfies our Condition.

1.39. Construct a circle with one point on it and two lines tangent to it. (lines are non parallel).



→ Draw the two lines and produce them if necessary to intersect. Mark this point as O.

→ Construct the angular bisector of the angle formed at the intersection.

→ Mark the points M, N on the bisector line and draw MP, NQ perpendiculars to any one line

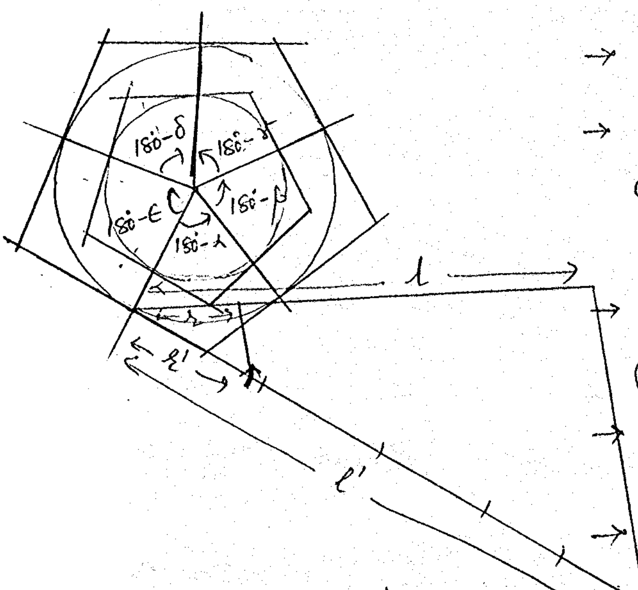
→ Complete the circles with MP, NQ as radius.

→ Now draw perpendicular LR such that LR = LB.

→ With LR as radius Complete the circle, which is the required circle.

DR. SUBRAMANIAM: **Problem solving** Session - 8 -1 11.02.1997

PROB: 1.40: (a) Construct a pentagon circumscribable about a circle given four angles $\alpha, \beta, \gamma, \delta$ and length of the perimeter.



→ The five angles be $\alpha, \beta, \gamma, \delta$ and ϵ.

→ Now each ~~[illegible]~~ the angle at the centre is calculated to be $180°-\alpha, 180°-\beta \ldots\ 180°-\epsilon$.

→ Construct these angles with 'O' as the reference point.

→ Now draw a circle with any convenient radius with 'O' as centre.

→ Draw tangents to the radii to get the pentagon.

→ ~~Repeat the process for any other [illegible]~~

→ Now obtain the radius of the required circle by the method of proportion.

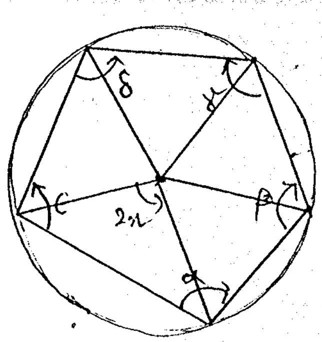
l = Perimeter of pentagon given

r' = radius of the circle drawn

l' = Perimeter of pentagon obtained

r = radius of the required circle.

Problem: 1.40 (b). To construct a pentagon inscribed in a circle.

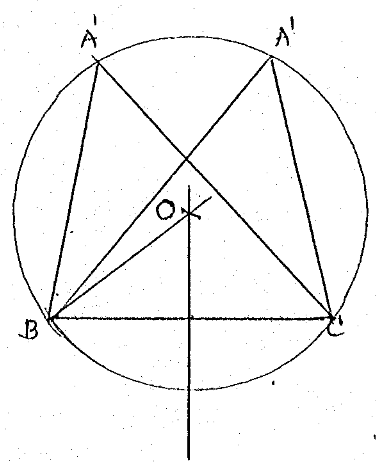


We know:
$$x = 360° - \beta - \gamma - \epsilon$$
$$= 360° - (540 - \delta - \alpha)$$
$$= (\alpha + \delta) - 180°$$
$$2x = 2(\alpha+\delta) - 360°$$

By this way we can calculate the central angle and construct the pentagon required.

Problem from 1.41 : Construct a triangle from h_a, h_b, h_c.

Problem: 1.43 : construction of a triangle from a, α, and R.



→ Draw a line segment $BC = a$ units.

→ Construct perpendicular bisector of BC.

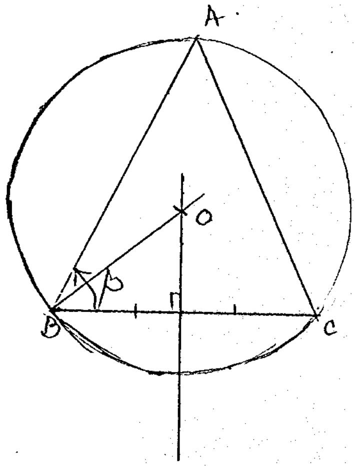
→ from B put an arc on the bisector for the length R units. at O.

→ with O as centre & R as radius complete the circle.

→ Now join any point on the major segment to BC, which gives the required circle.

→ We can draw many circles.

Prob 1.44: (a) Construction of a triangle from a, β, R.



→ Draw a line segment $BC = a$ units.

→ Construct perpendicular bisector of BC.

→ With a radius of R put an arc to intersect the $\perp^r$ bisector at O.

→ With O as centre and R as radius complete the circle.

→ Now at B, measure an angle β such that the line segment meets the circle at A.

→ Join AC to get the required triangle.

1.44(b): To construct a triangle from a, r and R.

→ Draw a line segment $BC = a$ units.

→ Construct perpendicular bisector of BC.

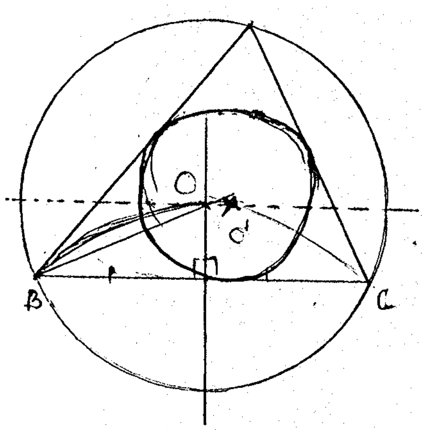
→ With R as radius put an arc to intersect the perpendicular bisector at O.

→ With O as centre and R as radius draw the circle

→ Now draw a line parallel to BC at a distance of r.

→ Now since the angle at the circle 'α' is known draw an arc to form angle of $90^\circ + \alpha/2$.

→ This gives the ~~radius~~ centre of the incircle. With this complete the triangle.



05/03/97. Problem Solving - Session - 09. Dr. Subramanyam. -1

from Session 10, the second chapter is taken for discussion.

Prob: 2.1: There are 10 paise and 5 paise coins and total 50 coins are there valuing upto Rs 3.50. To find the no of 10 paise & 5 paise coins.

Sol: Algebraic Method
Let the 5 paise coins be x & 10 paise coins be y

Given $5x + 10y = 350$
$x + y = 50$

solving we get $x = 30$, $y = 20$.

so 5 paise coins are 30 & 10 paise are 20.

Trail and Error:

Let all the coins be 10 paise coins.

$\Rightarrow$ Total value $= 50 \times 10 = 500$ paise.

but given total value $= 350$ paise

$\Rightarrow$ we have to reduce $500 - 350 = 150$ paise

$\Rightarrow$ $150/10 = 15$ coins of 10 paise are to be put as 5 paise

$\Rightarrow$ $15 \times 2 = 30$ coins of 5 paise are there.

Problem: 2.2. There are Hens and Rabbits, Together 50 heads and 140 feet are there. To find the no of Hens & Rabbits.

Sol by Trail and Error: - If all the 50 are Hens.
Then no of feet $= 50 \times 2 = 100$.
But there are 140 feet.

$\Rightarrow$ There are $140 - 100 = 40$ more feet.

$\Rightarrow$ $40/2 = 20$, therefore there are 20 Rabbits & $50 - 20 = \underline{\underline{30}}$ Hens.

Example:- ① One pipe can fill a tank in 15 m
other one in 20 m
Third pipe in 30 m
when all are opened in How many minutes the fill the tank.

Solution:- Let all pipes are opened for 5 minutes.
Then first pipe will fill $1/3$,
Second pipe will fill $1/4$
Third pipe will fill $1/6$ of the tank

$\Rightarrow$ together $1/3 + 1/4 + 1/6 = \frac{3}{4}$ th tank is filled

$\therefore$ time required to fill the whole tank $= \dfrac{1}{\frac{3}{4} \times \frac{1}{5}}$

$= \dfrac{20}{3}$ m.

Example ② Of Income I a person spends $1/3$, $1/4$ & $1/6$ for different purposes. How long he can live.

Solution: Amount spend $= \frac{I}{3} + \frac{I}{4} + \frac{I}{6}$

$= \dfrac{3I}{4}$.

Therefore $3I/4$ is spent for 1 year
Then I for how many years?

$\Rightarrow \dfrac{I}{\frac{3I}{4}} = \underline{\underline{\frac{4}{3}}}$ year.

Problem 2.4:– A patrol plane moves with a speed of 220 miles ph. and has fuel for 4 hours. It takes against the wind speed 20 miles ph. How far it can fly and return safely?

Solution:– Let the distance travelled be x miles.

Time for going $\dfrac{x}{200 \text{ miles}}$ $(220 - 20 = 200)$

Time for coming $\dfrac{x}{240 \text{ miles}}$ $(220 + 20 = 240)$

But $\dfrac{x}{200} + \dfrac{x}{240} = 4$

$$x\left(\frac{6+5}{1200}\right) = 4$$

$$\therefore x = \frac{4800}{11} \text{ miles.}$$

Symbolically $\dfrac{x}{V-W} + \dfrac{x}{V+W} = T$

$$\frac{x\{V+W+V-W\}}{V^2-W^2} = T$$

$$\frac{2xV}{V^2-W^2} = T$$

$$\text{or} \quad x = \frac{T(V^2-W^2)}{2V}.$$

Substituting $x = \dfrac{4(220^2-20^2)}{2\times 220}$

$$= \frac{4(220+20)(220-20)}{2\times 220}$$

$$= \frac{4800}{11} \text{ miles.}$$

Problem:– A dealer has kinds of dry grapes, one costs Rs 90/- a kg and other costs Rs 60/- a kg. Total cost for 50 kg is Rs 72/- per kg.

Solution:– By algebraic method:

Let x be the wt. of one kind & y be the wt. of other.

We have
$$90x + 60y = 72 \times 50$$
$$x + y = 50$$

$$\Rightarrow \quad \underline{\underline{y = 30 \text{ kg}}}$$

$$\Rightarrow \quad \underline{\underline{x = 20 \text{ kg}}}$$

By Trial and error.

Let all 50 kg be of type x

$\Rightarrow$ total cost $=$ Rs (90×50)

$=$ Rs $\underline{\underline{4500}}$/-.

Given total cost $=$ Rs 72×50

$= \underline{\underline{3600}}$

$\Rightarrow$ There is an increase of Rs $\underline{\underline{900}}$.

$\Rightarrow$ Less $\dfrac{900}{90} \times 3$ kg

$\Rightarrow$ Less 30 kg

$\therefore$ Wt of x type $= 50 - 30 = 20$ kg

& wt. of y type $= 50 - 20$

$= \underline{\underline{30 \text{ kg}}}$.

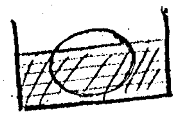
General form: $ax + by = cv$

$$x + y = v$$

$$\Rightarrow \quad ax + by = cv$$
$$bx + by = bv$$

$$\Rightarrow \quad x = \left(\frac{b-c}{b-a}\right)v \quad \& \quad y = \left(\frac{c-a}{b-a}\right)v.$$

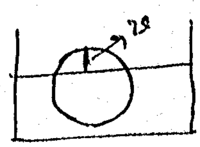
Problem: An iron sphere is floating in mercury. Pour water and find out what happens to the ball.



Ans:- On pouring water on the ball which is partially dipped in mercury, the ball raises up. This happens as the total thrust on the ball increases.

The discussion also was on the density of the ball and the density of the liquid.

Problem: Given the density of Iron 7.84, water 1.00 & Mercury 13.60 find the amount of iron ball immersed in Mercury.



Solution: Let v be the volume of the ball floating and V is the volume of the ball.

By Archemedis Law, wt. of Hg displaced $= (V - v) \times 13.6$

$$7.84V = (V-v)13.6$$

$$\Rightarrow \quad 13.6\,v = (13.60 - 7.84)V$$

$$\Rightarrow \quad v = \frac{5.76}{13.6}V$$

$$\Rightarrow \quad = 0.423 \simeq 0.42.$$

II case: On pouring water, we have

$$7.84V = (V-v)13.6 + v(1)$$

$$= (V-v)13.6 + v$$

$$7.84V = 13.6V - 12.6v$$

$$\therefore v = \frac{5.76}{12.6}V$$

$$v = 0.457 \Rightarrow v = 0.46.$$

So the ball has risen.

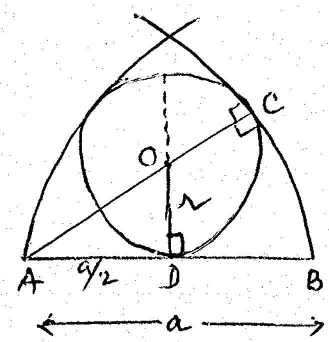
Putting it in the general form: with D for water as a
D for Hg as b
D for Iron as C

In case ① we have $v = \left(\frac{b-c}{b}\right)V$

case ② we have $v = \left(\frac{b-c}{b-a}\right)V$

Construction Problem:-

Two arcs are put with AB as radius & A, B as centres. Inscribe a circle in it. Find the centre and radius of the circle.



Solution: Let $AC = AB = a$

$OD = r$ = radius.

$AO = (a - r)$

$AD = a/2$

In right triangle ADO.

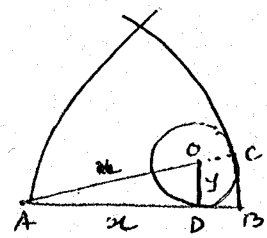
$$(a-r)^2 = r^2 + \frac{a^2}{4}$$

$$\Rightarrow a^2 - a^2/4 = 2ar$$

$$\Rightarrow r = \frac{3}{8}a$$

Contd

Special case:— The circle touching two of the sides in the Gothic curve.

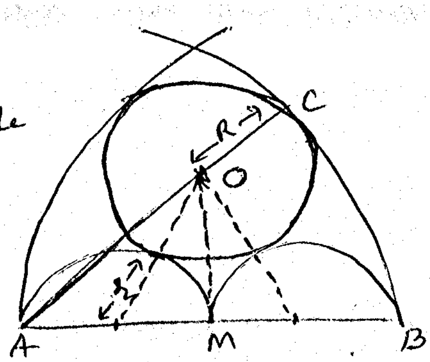


In the fig: $x^2+y^2=(a-y)^2$

$$y=\frac{a^2-x^2}{2a}$$

which is a parabola & hence cannot be constructed.

Problem :— Find the centre of the circle that touches four circular arcs forming a curvilinear quadrilateral.



Solution:— Let $AB = AC = a$

R = radius of the circle

$OM = x$

we have $(a-R)^2 = \left(\frac{a}{2}\right)^2 + x^2$

$$\Rightarrow a^2+R^2-2aR-\frac{a^2}{4} = x^2$$

$$\Rightarrow \frac{3a^2}{4} - 2aR + R^2 = x^2 \quad \text{—(1)}$$

$$(R+r)^2 = \left(\frac{a}{4}\right)^2 + x^2$$

$$R^2+r^2+2Rr = \frac{a^2}{16} + \frac{3a^2}{4} - 2aR + R^2 \quad \text{(Sub for } x \text{ from (1))}$$

$$r^2+2Rr = \frac{a^2+12a^2}{16} - 2aR$$

$$2Rr+2aR = \frac{a^2+12a^2}{16} - r^2$$

$$\Rightarrow R(2r+2a) = \frac{12a^2}{16}$$

$$r = a/4$$

$$2R\left(\frac{a}{4}+a\right) = \frac{12a^2}{16}$$

$$\Rightarrow 10R = 3a \quad \text{or} \quad R = \frac{3a}{10}$$

Substituting for R in ①

$$x^2 = \frac{3a^2}{4} - 2a\left(\frac{3}{10}a\right) + \frac{9a^2}{100}$$

$$= \frac{150a^2 - 120a^2 + 18a^2}{200}$$

$$x^2 = \frac{48a^2}{200}$$

$$\therefore \quad x = \sqrt{\frac{6a^2}{25}}$$

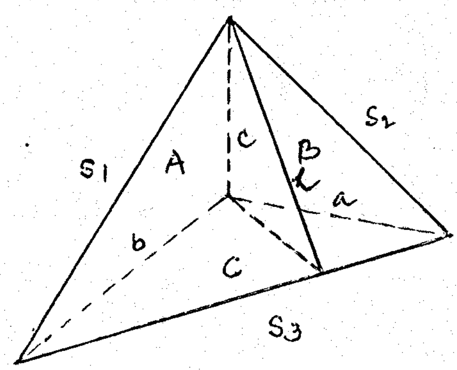
$$x = \frac{\sqrt{6}\,a}{5}$$

Analogy of Pythagoras theorem in Three Dimension.

We know $A = \frac{1}{2}bc$

$B = \frac{1}{2}ca$

$C = \frac{1}{2}ab$.



Let D be the area of the figure.

Then $D = \sqrt{(s)(s-s_1)(s-s_2)(s-s_3)}$

$s = \frac{s_1+s_2+s_3}{2}$

$s_1^2 = b^2+c^2$

$s_2^2 = c^2+a^2$

$s_3^2 = a^2+b^2$

$s-s_1 = s_2+s_3-s$

$s-s_2 = s_3+s_1-s$

$s-s_3 = s_1+s_2-s$.

Sub. we get.

$$D = \sqrt{\frac{s_1+s_2+s_3}{2}\,(s_1+s_2-s)\,(s_2+s_3-s)\,(s_3+s_1-s)}$$

$$D = \frac{1}{4}\sqrt{(s_1+s_2+s_3)\,(s_1+s_2-s)\,(s_2+s_3-s)\,(s_3+s_1-s)}$$

$$D^2 = \frac{1}{16}\left[(s_2+s_3)^2 - s_1^2\right]\left[s_1^2-(s_2-s_3)^2\right]$$

$$= \frac{1}{16}\left[2a^2+2s_2s_3\right]\left[-2a^2+2s_2s_3\right]$$

$$= \frac{1}{4}(s_2^2s_3^2-a^4)$$

$$= \frac{1}{4}(a^4+a^2b^2+a^2c^2+b^2c^2-a^4)$$

$$= \frac{1}{4}(4A^2+4B^2+4C^2)$$

$$D^2 = A^2+B^2+C^2$$

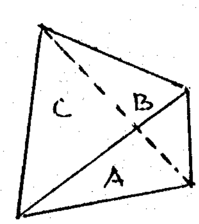
Alternate method.

$$4D^2 = d^2h^2$$
$$4A^2 = b^2c^2$$
$$4B^2 = c^2a^2$$
$$4C^2 = a^2b^2$$

$$d^2l^2 = 4C^2$$
$$\therefore (a^2+b^2)l^2 = 4C^2$$
$$d^2 = h^2+c^2$$
$$l^2d^2 = 4C^2 = a^2b^2$$
$$l^2 = \frac{4C^2}{a^2+b^2}$$
$$d^2 = \frac{4C^2}{a^2+b^2}$$
$$d^2 = \frac{4C^2}{4C^2} \times a^2+b^2$$
$$d^2 = a^2+b^2$$

$$a^2+b^2 = h^2+c^2$$
$$h^2 = a^2+b^2-c^2$$
$$4D^2 = (a^2+b^2-c^2)(a^2+b^2)$$
$$4c^2 = d^2(h^2-c^2) = 4D^2 - c^2(a^2+b^2)$$
$$\therefore 4D^2 = 4A^2+4B^2+4C^2$$
$$\Rightarrow \underline{D^2 = A^2+B^2+C^2}$$

Problem: 2.9 find the volume V of a tetrahedron that has a trirectangular vertex O, being given the areas A, B and C of the three faces meeting in O.



Solution:
$$V = \frac{A \times h}{3}$$
$$A = \frac{ab}{2}$$
$$B = \frac{bh}{2}$$
$$C = \frac{ah}{2}$$

$$\frac{abh^2}{4} = BC$$

$$h^2 = \frac{4BC}{ab}$$

$$h = 2\sqrt{\frac{BC}{ab}}.$$

$$V = \frac{A(2)}{3}\sqrt{\frac{BC}{ab}}$$

$$= \frac{2A}{3}\sqrt{\frac{BC}{2A}}$$

$$= \frac{1}{3}\sqrt{\frac{4A^2BC}{2A}}$$

$$= \frac{1}{3}\sqrt{2ABC}.$$

Q.10: Find the volume of the tetrahedron, given a, b, c.

Solution: $V = \frac{\sqrt{2}}{3}\sqrt{\frac{ab}{2}\cdot\frac{bh}{2}\cdot\frac{ah}{2}}$

$$= \frac{\sqrt{2}}{3}\sqrt{\frac{a^2b^2h^2}{8}}$$

$$V = \frac{abh}{6}.$$

Q.10: Analog to Hero's theorem. (Find the volume to the above said tetrahedron)

Solution:

$$h_1^2 + h_2^2 = a^2$$

$$h_2^2 + h_3^2 = b^2$$

$$h_3^2 + h_1^2 = c^2$$

we have $V = \frac{\sqrt{2ABC}}{3}$

$$= \frac{1}{6}\sqrt{h_1^2 \cdot h_2^2 \cdot h_3^2}$$

$$h_2^2 - h_1^2 = b^2 - c^2$$

$$h_2^2 = \frac{a^2+b^2-c^2}{2}.$$

$$= \frac{a^2+b^2+c^2}{2} - c^2$$

Substituting we get.

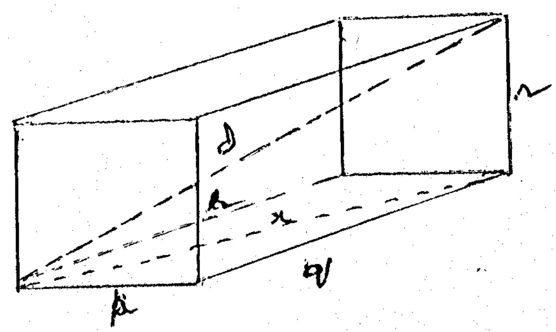
$$V = \frac{1}{6}\sqrt{(s^2-c^2)(s^2-b^2)(s^2-a^2)}$$

where $s^2 = \dfrac{a^2+b^2+c^2}{2}$

Here the volume is fixed as the height is fixed.

2.11. Find the length d of the diagonal of a rectangular box.

Solution:



we know:

$$x^2 = p^2+q^2$$

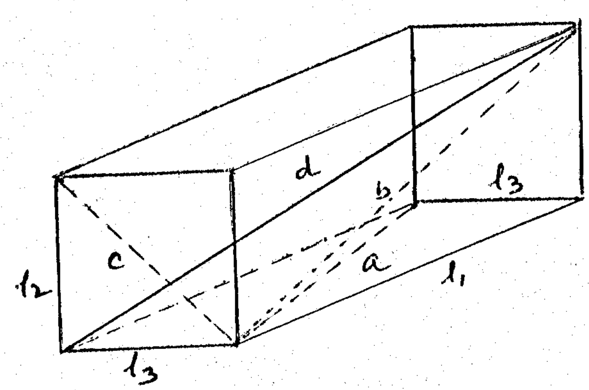
but $d^2 = x^2+r^2$

$$d^2 = p^2+q^2+r^2$$

or $d = \sqrt{p+q+r}$.

2.12. Find the length d of the diagonal of a box, given a, b, c, the lengths of the diagonals of the three faces having the corner common.

Solution: Let l_1, l_2, l_3 be the sides.



$$b^2 = l_1^2+l_2^2$$

$$c^2 = l_2^2+l_3^2$$

$$a^2 = l_3^2+l_1^2$$

$$a^2+b^2+c^2 = 2(l_1^2+l_2^2+l_3^2)$$

$$= 2d^2$$

$$\Rightarrow \quad d^2 = \frac{a^2+b^2+c^2}{2}$$