Understanding Students' Reasoning While Comparing Expressions

Shweta Naik
Tata Institute of Fundamental Research
<shweta@hbcse tifr res in>

Rakhi Banerjee
Tata Institute of Fundamental Research
<rakhib@hbcse tifr res in>

K Subramaniam
Tata Institute of Fundamental Research
<subra@hbcse tifr res in>

In this paper, we analyse grade 6 students' reasoning in some tasks, which involved comparing two simple expressions The tasks allowed students to move from procedural to structural understanding of simple expressions Analysis of students' responses revealed various patterns of thinking and different ways of communicating the reasons Students were found to justify their responses using language or symbols The students moved from using predominantly language to using symbols predominantly, as the expressions became more complex

Studies have pointed out students' inability to deal with symbolic expressions in a consistent manner Many students do not make sense of the structure of the expressions and cannot use consistent rules to manipulate them successfully (Chaiklin & Lesgold, 1984; Kieran, 1989; Liebenberg, Linchevski, Sasman & Olivier, 1999) More recent efforts to understand students' ability to deal with symbols and symbolic expressions have shown that elementary school children are capable of making generalizations about properties of numbers and operations, representing and justifying them (Carpenter & Levi, 2000; Carpenter & Franke, 2001)

The traditional classroom culture emphasises procedural thinking in the form of routine algorithms without any focus on the meaning of the mathematical objects or operations This does not allow students to connect various mathematical procedures with the meaning of symbols or operation signs The students when first asked the meaning of an expression like 2+7 quickly respond by saying 'answer is 9'and only with some effort, they start looking at it as a relation 'two more than seven' or a representation for the number 9 Looking at an expression as a relation shifts students' attention from procedural to structural conception of an expression In the study being reported here, we capitalised on students' intuitive understanding of symbols and expressions to take them away from mechanical procedures and instead, focus their attention on the structure of expressions They were encouraged to communicate their understanding about these expressions using their own words, and gradually move to symbolic representations of their reasons

In this paper[1], we will discuss students' justification of their responses in certain tasks, which involved comparing simple arithmetic expressions This work is part of a larger study where one of the main objectives is to develop a sequence for teaching beginning algebra, using students' understanding of the structure of arithmetic expressions

Framework and Teaching Approach for the Study

Understanding mathematical objects (operations and symbols) and the properties and relationships that hold between them form an essential part of structural understanding (Kieran, 1989; Warren, 2001; Williams & Cooper, 2001) This is important in order to make the transition from arithmetic to algebra, when algebra is thought of as generalised arithmetic This understanding has been called by various names: relational thinking, structural thinking, and algebraic thinking (Stephens, 2004) In this context, the concept of 'term' can be seen as a structural element of an expression and the concept of 'equality' as a structural relation between two expressions Students' understanding of '=' has been explored by various researchers (e g Kieran, 1992) and has been found to be a cause of concern as many students tend to see it as a 'do something' signal

In the teaching approach adopted in the study, the students were made to focus on the structural aspects of the expressions They first learnt to see an arithmetic expression as a relation, that is, see 9 – 3 as a number which is '3 less than 9' and that it stands for the number 6 Further, students understand '=' as a relation between two expressions which have the same value They were asked to work on simple exercises of filling the box with <, =, > in questions of the type 12 + 5 □ 17 – 3 or to fill in the blank by a number so that the sentence is true, like 21 + 8 = ___ – 4 In the following tasks, the students were asked to work on similar exercises where the relation between the pairs of expressions is apparent (e g 24 + 47 and 25 + 48) and hence, the task can be completed just by looking at the expressions without recourse to calculations Students' intuitive understanding of symbols and operation signs and their expectations regarding the outcome of these operations play an important role in completing the tasks Successful completion of the task required them to correctly parse the expression and explore and identify the relationships between symbols and operation signs

Asking students to give reasons to justify their responses in such tasks formed another important aspect of the approach It compelled them to look for relationships among the mathematical objects and to make their understanding explicit by communicating their reasons to their peers or the instructor using either language or symbols These activities gave them a new way of looking at arithmetic expressions and provided opportunities to share their understanding with others The discussions provided them with immediate feedback and made it possible for them to see different ways of justifying their responses It also brought forward some of their implicit understanding of general rules of operations as well as their capacity to use some symbols to communicate and make sense of their arguments

Methodology

The main study, of which this is a part, is a design experiment study and is being conducted on 6^{th} grade students (11 to 12 year olds) Four cycles of the study have been completed and conducted between summer 2003 and autumn 2004, during the vacation periods of the students The students in the study come from low and mixed socio-economic strata from nearby English and vernacular medium schools (Marathi)

Cycle 1 was an exploratory phase and will not be reported in this paper The students in Cycle 2 and Cycle 4 were in the middle of grade 6 and had a brief exposure to integers Many of the students in Cycle 4 were also part of Cycle 3 There was a 6 month gap between each cycle We analyse the responses of 25, 53 and 28 English medium students and 34, 39 and 42 Marathi medium students from cycles 2, 3 and 4 respectively in the relevant tasks During these four cycles data was collected on these tasks by giving them tests at the beginning and at the end of the course

The tasks discussed during the course were of three kinds: (I) to compare two simple expressions (2 termed expressions with a + or a – sign in between) without calculation using <, =, > in the box (For example, 23 + 48 □ 24 + 47) (II) to fill in the blank by a term so that the expressions on both sides of the '=' sign are equal (For example, 36 – 19 __ = 35 – 20) (III) to find the value of an expression given the value of a related expression, without calculation (For example, if 237 + 498 = 735, then 238 + 499 = ?) Task (I) was intensely discussed (2 sessions of 45 minutes each) in Cycle 2 with instruction on how to communicate the reasons clearly but in the later cycles the time spent on this task was reduced to a large extent (1 session of 30-45 minutes), the purpose being mainly to expose the students to the task and observe the kind of responses they came up with In Cycle 4, the students were explicitly told to use symbols to communicate their reasons, rather than writing in their own language Task (II) was discussed explicitly in Cycle 2 but not in the later cycles Task (III) was discussed in cycles 3 and 4 briefly (1 session of nearly half an hour) and not in Cycle 2

Analysis of Data

Responses of the students in the following tasks will be discussed: (I) comparing two expressions without calculation (II) filling the blank with a term so that the expressions on both sides of the '=' sign are equal (III) to find the value of an expression given the value of a related expression The pairs of expressions used in all the tasks were of the following types: (a) expressions with one term constant (e g 37 + 58, 36 + 58), (b) expressions involving terms compensating each other completely (e g 53 + 38, 54 + 37) and (c) expressions with partially compensating terms (e g 53 + 38, 55 + 37) Similar types of expressions were posed with negative terms as well Figure 1 shows some examples of items in each kind of task Reasons, which are relevant, correct and are not based on calculation are counted as correct reasons

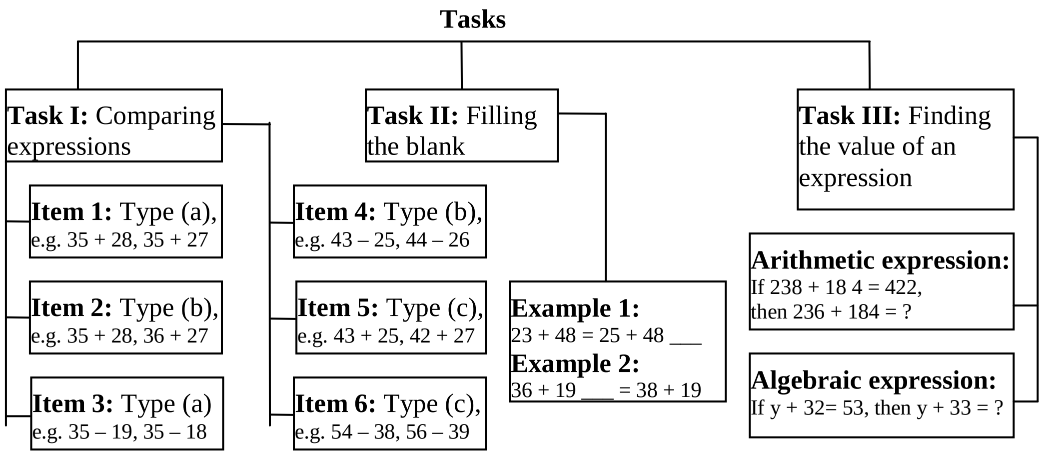

Figure 1 Examples of items in the tasks

Task (I): Comparing Two Expressions

The overall performance of students across all the cycles in items 1 and 2 is very high The number of students writing the reasons in all the cycles is nearly same, in spite of less instruction and discussions in the cycles 3 and 4 More students from Marathi medium gave reasons for their answers Table 1 below shows the performance of students in the 'comparing expressions' task in the post-test of the cycles 2 and 3 and the pre test of the cycle 4 The students in the pre test of Cycle 4 had attended the earlier course and therefore were aware of the requirements of the task

Table 1

Percentage of Correct Answers (A) and Correct Reasons (R) of English and Marathi Medium Students in Comparing the Expressions Task

		Item 1		Item 2		Item 3		Item 4	
		A	R	A	R	A	R	A	R
English	Cycle 2 (Post)	96	64	96	60	84	48	80	36
	Cycle 3 (Post)	83	62	87	53	79	40	57	15
	Cycle 4 (Pre)	96	75	86	57	68	46	54	18
Marathi	Cycle 2 (Post)	100	77	88	53	85	50	74	27
	Cycle 3 (Post)	100	90	100	77	69	44	63	41
	Cycle 4 (Pre)	95	86	88	86	62	45	60	38

Although the items 1 and 3 (type (a)) and 2 and 4 (type (b)) are of the same type, there is difference in students' performance both in giving the correct response and in writing reasons Students' performance is better in the expressions with positive terms (items 1 and 2) than with negative terms (items 3 and 4) These students had less facility in dealing with signed negative numbers as they are introduced to integers only in the middle of grade 6 The students in Cycle 3 were briefly exposed to integers in our course

Reasoning Items of type (b) (items 2 and 4) are more complex than items of type (a) (items 1 and 3), as in the former task, increase and decrease in both terms have to be taken into account to compare expressions This is also reflected in the decreased percentage of students writing reasons for these items 66% of students easily justified their responses for the first item (type (a)) by comparing the changed term For example, while comparing 37 + 58 and 36 + 58, the students say '*58 is same on both sides and 37 is more than 36*', concluding 37 + 58 > 36 + 58 Similarly in the third item, which is also of type (a), the most frequent strategy for all the students (26%) to justify their answer was to use the 'take away' model of subtraction Some students wrote while comparing 64 – 37 and 64 – 36 that '*in the right hand expression we are subtracting a smaller number than 37, so left side is smaller than right side*' Some students directly stated a general rule that '*subtracting more gives you less answer*' 21% of the students compared the changed numbers 37 and 36 as in item 1, and concluded that 64 – 37 is more than 64 – 36 These students ignored the negative sign before 36 and 37 and failed to anticipate the result of the subtraction operation However, in Cycle 2, 44% of the students correctly compared the terms –37 and –36 to justify their answer This could be a result of the discussions during the course as well as their exposure to integers in the school In the later cycles, very few students were found to use this strategy This is in spite of the fact that these students were also briefly introduced to the idea of integers during the course but no discussion on using order relations in integers as a strategy to compare expressions of the above type was initiated by the instructor

Students were more successful in writing correct reasons for the type (b) item with positive terms While comparing 54 + 67 and 52 + 69, Priyanka wrote '*54 is 2 more than 52 and 69 is 2 more than 67*' 42% of the students used similar strategy to justify their responses These students compare the terms and find that one term on each side has increased or decreased by the same amount They may be comparing both the expressions to a base expression, like 52 + 67 in the example above and adding 2 once to 52 and another time to 67 The base expression could also be 54 + 69, where students' write '*52 is*

2 less than 54 and 67 is 2 less than 69' Another way (13%) to justify their answer was to add and subtract the same number to an expression to get the other expression, which is being compared For example, while comparing the above problem, Tanmay wrote *'in the left side take 2 from 54 and add it to 67, do the same in the right side, 2 – 2 = 0'* This student seems to complement his 'adding and subtracting the same number' strategy with 'finding the difference between the terms' He also writes symbolic statement to clarify that the difference of the difference between the terms is zero A few students used only the strategy of 'finding the difference between the terms' Changing both the expressions to a third related expression was also a way used by a very few students to justify the answer For example, Pavan for the same item, wrote *subtracting 1 from 69 and adding it to 67 makes the same term [68], then subtracting 1 from 54 and adding it to 52 [making it 53], now both the terms are same [53 + 68]*

The number of reasons given by students was comparatively less in all the cycles in item 4 Although this item is of the same type (type (b)) as above, the students were seen to change their strategy to justify their answer Many of the students (17%) compared numbers leading to wrong answers Among the correct ones, the strategy used by most students was 'adding and subtracting the same number to an expression to get the other expression' 12% of English medium and 22% of Marathi medium justified their answers by using this strategy While comparing 85 – 38, 86 – 39 a few students wrote, *86 is one more and we are taking away one more [39 is one more than 38], so they are equal* This can be thought of as an extension of 'take away' model, used by the students in item 3

Some items in the Cycle 4 were more complex In all the previous cycles there were no items of partially compensating expressions (i e type (c)) Three of the items involved negative terms, with one each of the types (a), (b) and (c) Two other items consisted of only positive terms and were of type (b) and (c) The performance of both the groups in judging the correct sign for the box varied between 60% and 90%, depending on the complexity of the item The students did very well in the items with positive terms only (80% to 85%) and many could successfully give reasons using symbols for these Interestingly, in this case, more students gave correct reasons for the partially compensating expression (type (c)) than the completely compensating expression (type (b)) 60% to 70% of the students could correctly identify the sign for the box in the items involving negative terms Nearly half of the English medium students substantiated their responses by giving reasons 38% Marathi medium students at least once wrote the wrong sign for a correct reason The increased complexity of the items, to some extent, forced them to write reasons before identifying the sign, as it was no longer easy to base their answers solely on intuition It also seems that the ability to represent their reasons symbolically helped them to communicate their understanding about comparing such expressions Students had found it quite difficult to communicate their reasons about expressions with negative terms of even type (b) till the pre test of Cycle 4

In cycle 4, the reasons given by students were fully symbolic in nature, as per the instruction Following (Figure 2) are a few examples of reasons given by students for items of type (b) and (c)

52 – 37 = *53 – 38* +1 – 1 Priyanka	52 – 37 = *53 – 38* *-1 + 1* Prathmesh	74 – 26 < *75 – 29* *+ 1 + 3* Prajakta	74 – 26 < *75 – 29* – 1 + 3 Suraj	63 + 57 < *65 + 56* *-2 + 1* *+ 2 - 1* *= -1* *= +1* Saurabh

Figure 2 Examples of students’ symbolic reasoning

There are some differences in the manner the symbols are used in writing the reasons Priyanka tries to convey with her symbols that to get 53 from 52 we have to add 1 and to get –38 from –37 we need to subtract 1 leading to the expression +1 – 1 and hence the conclusion ‘=’ On the other hand, the second student Prathmesh, compares the terms and finds 52 to be 1 less than 53 and –37 is 1 more than –38 and therefore the expression –1+1 again leading to the same conclusion The last student Saurabh also compares the terms of the two expressions successfully and finds the difference between those to conclude one side to be smaller than the other side These instances of symbol use are not spontaneous and are influenced by the manner in which the instructor had used them during the short classroom discussions, but the fact that some of them were able to use symbols and interpret its meaning is an achievement for them Not all such efforts were successful as can be seen in the reasons of the third and the fourth student Prajakta compares the numbers and incorrectly finds the difference as +1 + 3 = +4 whereas Suraj wrote the correct reason but could not interpret the difference correctly In 7% of the instances, students could identify the difference between the terms correctly but failed to interpret the result especially in the partially compensating items (type(c))

Task (II): Filling the Blank with a Term to Make Two Expressions Equal

This task directly tested students’ understanding of the concept of ‘equality’ Students’ performance in this question was not as good as their performance in the task (I) Cycle 2 had three items, one of type (a) and 2 of type (c) The performance of students varied as the items became more complex The students succeeded (92% of English medium and 52% Marathi medium) to fill the blank, to some extent, when the expressions were of type (a), that is, when only one of the terms was changing Filling the blank was more difficult in expressions, where both the terms changed by unequal amounts (type (c)) For the item with positive terms 60% of English and 30% of Marathi medium students could fill the blank, but for the item with negative term only around 35% of the students succeeded Only half of those who wrote correct answers could give reasons for their answers As the blank was only on the left side of the ‘=’ sign, there were no responses which could be interpreted as ‘writing the answer of the sum’ but there were responses which did indicate that ‘=’ was being used as a symbol for association For example, in 35 + 26 + 1 = 35+ 25, the response +1 shows that 26 is 1 more than 25 In the type (c) item, 36 – 17 __ = 38 – 18, there were two kinds of responses, +3 and –3 The response +3 shows comparison of numbers without taking care of the sign, whereas the response –3 shows comparison of numbers together with the association error

In the later cycles, only around 50% of the English medium students and around 70% of the Marathi medium students could successfully complete this task In this case, there was only one item with the blank to the right of ‘=’ sign (e g 35 + 29 = 35 + 27 + __) The most common errors (~10% in each) have been to treat the ‘=’ sign as a signal to write the sum of the numbers (+64 or +62 or +126) or to use it as an indicator of association (writing –2 in the blank)

Task (III): Finding theVvalue of an Expression Given the Value of a Related Expression

This task was included in cycles 3 and 4 to give the students motivation to compare expressions and also to probe whether the students understand that the difference between the terms of the two expressions is reflected in the change of the value of the expression The items in the post-test of Cycle 3 and the pre test of Cycle 4 were of type (a) One involved an arithmetic expression and another involved an algebraic expression Table 2 shows the performance of students in these two tests in the two kinds of items

Table 2
Percentage of Students giving Correct Answer (A) and Correct Reason (R) in Finding the Value of an Expression given the Value of a Related Expression

		Arithmetic		Algebraic	
		A	R	A	R
English	Cycle 3 (Post)	55	32	45	19
	Cycle 4 (Pre)	75	46	54	32
Marathi	Cycle 3(Post)	67	64	69	56
	Cycle 4 (pre)	74	62	79	55

Fewer English medium students wrote reasons for their answers compared to Marathi medium students The reasons given by students were mainly by comparing terms For example, to find the value of 324 + 598, given the value of 326 + 598 = 924, many students wrote *'324 is 2 less than 326, therefore 924 – 2 = 922'* To find the value *y* + 34, given the value of *y* + 35 = 72, some students wrote *'If you add 35 to y you get 72, then if you add 34 to y you will get 71'* and some others wrote *'y + 34 + 1 = 72 therefore y+ 34 = 71'*

In Cycle 4, there were four items in this task involving a negative term Nearly all the students who gave correct response also gave correct reason 60% to 70% of the students could find the correct values and 50% to 60% of the students could write reasons for their answer But in 6% of the instances, students could not find the correct value in spite of finding the correct reason The reasons given by students are similar to the ones given by them for Task (I), by finding the difference between the terms

Discussion

Although tasks I and III forced students to look at the relation between the expressions, some students regressed to viewing the '=' as a 'do something' operator Type (a) and type (b) problems with positive terms were simple and the responses of students to the items of this type were spontaneous Items of type (b) with negative terms and type (c) items were more complex and influence of instruction and discussions could be seen in their responses

The reasons given by students for justifying their answers varied as the items became complex 21% of English medium and 4% of Marathi medium students were in the level of finding answers by calculations in Task I For the items of type (a) students gave reasons using language either by comparing terms or by using 'adding up' or 'take away' models

As the items became more complex, like type (b), some students started writing the difference between the terms as a symbolic expression together with verbal explanation A few students were spontaneous in using symbols while some others learnt it from their peers or instructor These students used plus sign to denote an increase in a term and minus sign to denote decrease in the term For example, in comparing 36 + 52 and 35 + 53, they wrote -1+1, concluding that the expressions are same This brings forth students' implicit understanding that in a pair of expressions if the terms in one expression increase or decrease by the same amount relative to the second expression, then the expressions are equal For the task of the type(c), more students tried to use symbols for justification, either spontaneously or when asked This might be because purely intuitive or verbal justifications were now difficult to process mentally Asking students to write the reasons symbolically might make it mechanical Students might find the difference between the terms but may not be able to interpret it correctly

Some students are seen consistent in writing reasons for the same type of items For the items 1 and 3 (type (a)) of Task I, 18% of English medium and 26% of Marathi medium students used same type of strategy while reasoning Also in items 2 and 3 (type (b)) of Task I, 23% of English medium students and 36% of Marathi medium students used the same strategy This indicates that these students were able to identify the same structure of these items and applied a consistent rule But this does not mean that using different strategies for the same type of items shows unawareness about the structure It might be the students' need for consistency that he/she uses the same strategy for the same type of items

1 Part of this paper was presented in the form of a poster in epi-STEME1 conference held in Goa during 13-17, December 2004

References

Carpenter, T , & Levi, L (2000) *Developing conceptions of algebraic reasoning in the primary grades* (Research Report) Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science www wcer wisc edu/ncisla/pubications/index html

Carpenter, T , & Franke, M L (2001) Developing algebraic reasoning in the elementary school: Generalisation and proof In H Chick (Eds), *The future of the teaching and learning of algebra*, (Proceedings of the 12th ICMI study conference) Melbourne, Australia: University of Melbourne

Chaiklin, S , & Lesgold, S (1984) Prealgebra students' knowledge of algebraic tasks with arithmetic expressions Paper presented at the annual meeting of the American Research Association

Kieran, C (1989) The early learning of algebra: A structural perspective In S Wagner & C Kieran (Eds), *Research issues in the learning and teaching of algebra* (pp 33-56) Reston, VA,: NCTM

Kieran, C (1992) Learning and teaching of school algebra In D A Grows (Ed), *Handbook of research on mathematics teaching and learning* (pp 390-419) New York: Macmillan

Liebenberg, R E , Linchevski, L , Sasman, M C , & Olivier, A (1999) Focusing on the structural aspects of numerical expressions In J Kuiper (Ed), *Proceedings of the 7th annual conference of the South African Association for Research in Mathematics and Science Education* (pp 249-256) Harare, Zimbabwe: SAARMSE

Stephans, M (2004) *Researching relational thinking* Japan: University of Tsukuba

Warren, E (2001) Algebraic understanding: The importance of learning in the early years In H Chick (Eds), *The future of the teaching and learning of algebra* (Proceedings of the 12th ICMI study conference, pp 633-640) Melbourne, Australia: University of Melbourne

Williams, A , & Cooper, T J (2001) Moving from arithmetic to algebra under the time pressures of real classrooms In H Chick (Eds), *The future of the teaching and learning of algebra*, (Proceedings of the 12th ICMI study conference, pp 665-662) Melbourne, Australia: University of Melbourne