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# UNDERSTANDING TEACHERS' KNOWLEDGE OF AND RESPONSES TO STUDENTS' MATHEMATICAL THINKING

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*Knowledge about students' thinking can help teachers take informed decisions about teaching and learning. Little is known about the nature of knowledge that teachers possess about their students and ways in which it enters the classroom. The paper reports a case study which is aimed at understanding teachers' knowledge about students' mathematical thinking in situ. Teacher's response to students' mathematical thinking was characterized using classroom observations, task-based interviews, complemented with the anticipation of and reflection on students' responses to 'proportion' problems. The design and implementation of classroom-based task with a focus on students' thinking helped in facilitating teacher reflection and learning.*

**Key words:** students' mathematical thinking, task-based interviews, teacher knowledge about students, teacher learning

## INTRODUCTION

The past few decades have seen an emerging interest in understanding teachers' knowledge of mathematics and how it informs teacher learning (Zaslavsky & Peled, 2007). Hill, Ball & Schilling (2008) highlight the significance of distinguishing teacher's knowledge of mathematics from the knowledge of *mathematics for teaching.* Mathematical knowledge for teaching implies thinking of suitable pedagogies using which mathematics can be communicated to students at different cognitive levels. Shulman (1986) called the specialised form of teacher knowledge as *Pedagogical Content Knowledge* (PCK), complementary to and distinct from content knowledge and general pedagogical knowledge of a teacher. PCK is 'an understanding of what makes the learning of specific topics easy or difficult; the conceptions and pre-conceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons' (Even & Tirosh, 2008). Hill et al. (2008) suggested a comprehensive model for mapping teachers' knowledge that includes PCK and helps us understand (a) teachers' knowledge about students' mathematics as an important aspect in the scheme of teachers' knowledge; and (b) ways in which this knowledge interacts with the other inter-related aspects of teachers' knowledge base. Teachers' knowledge about students would be incomplete without conscious reflection on aspects significant to teaching and learning in classroom. In fact, Schön (1984) calls teacher a *reflective practitioner,* a professional capable of knowledge-in-action, reflection-in-action and reflection-on-action. The two strands of knowledge and reflection can be tied meaningfully to facilitate teacher learning.

The organizational framework offered by Cochran-Smith & Lytle (2000) relates knowledge and practice to understand teacher learning. They classify the relation of knowledge and practice as *knowledge-for-practice, knowledge-in-practice,* and *knowledge-of-practice. Knowledge-for-practice* is the knowledge generated by the professionals and shared with teachers through professional development workshops. This knowledge is then used by teachers in classrooms. *Knowledge-in-practice* is the craft knowledge of teaching that competent teachers gain from their experience and practice. Teachers deliberate on their teaching and learn from it. *Knowledge-of-practice* is generated by teachers when they work in communities and relate insights from their practice with the larger social, cultural and political contexts of inquiry. In the present study, an attempt is made to support teachers' learning by developing their *knowledge-of-practice.* There is a need to encourage teachers to focus on students' mathematical thinking and develop a critical perspective towards their teaching in the light of their experiences and wisdom that exists in the field.

Knowledge about students' thinking is an integral part of teacher education. Knowledge of students' mathematical thinking includes knowing about students' (alternate) conceptions, their conceptual difficulties, potential learning trajectories; and developing sensitivity to what students think and do in a mathematics classroom. The sources of teacher's knowledge about students' thinking could be teachers' shared experiences, their own and peer reflection on students' conceptual difficulties and insights drawn from research literature in the field. The knowledge of mathematics along with knowledge about students' learning mathematics guides teachers in planning and taking in-the-moment decisions in classroom. Knowing about students’ mathematical thinking supports opportunities for asking questions linked to students’ ideas, eliciting multiple strategies, drawing connections across strategies, and so on (Franke, Kazemi & Battey, 2007). Unfortunately, knowledge of content and students' thinking are separately dealt with in the teacher preparation and teacher education programmes in India. The psychology courses deal with the components of students' thinking and learning. The concept-related discussions are confined to the methods courses such as Pedagogy of Mathematics. It is believed that the experience of teaching would help teachers to integrate the two knowledge pieces together and blend them in their teaching. Discussions on concept-specific students’ thinking and learning in teacher education need exploration in the Indian context.

Another issue at hand is the scarcity of interventions where teachers are engaged with research on students' concept-related thinking and analyse its potential for teaching. Cognitively Guided Instruction (CGI) (Carpenter, Fenemma, Peterson, Chiang, & Loef, 1989) has been an intensive attempt where teachers were provided with research-based knowledge about student trajectories in whole number concepts through use of semantic problem type framework. But CGI misses the analysis of teachers' knowledge about students gained from their diverse experiences and building on it through research-based materials. Another consideration is that programmes like CGI tell us nothing about whether teachers who are not involved in such professional development process possess such knowledge and if so what shape it takes (Hill et. al, 2008). Researches try to relate teacher knowledge with students' thinking but 'missing are the analysis that take into account the complexity of actual mathematics instruction that needs to consider various (and sometimes conflicting) factors, facets and circumstances' (Even & Tirosh, 2008).

## MOTIVATION

Despite the extensive work done in the field of developing teachers' knowledge, there are difficulties in identifying its nature and extent. Teachers know the most about their students and their ways of thinking and learning. They make conjectures about students' learning, listen and respond to them in the classroom and share intellectual and affective moments with them. All this helps in formulating teacher knowledge which remains largely unexplored and unchallenged. Ball, Hill & Bass (2005) question whether this is due to the nature of methods that we use or the nature of (teacher) knowledge that remains tacit and unarticulated. In this paper, an attempt is made to characterise the complexity of teacher's knowledge about students' mathematics and its significance for teacher learning.

## SIGNIFICANCE OF THE STUDY

In the Indian context, there has been little focus on subject-specific (mathematical) knowledge required for teaching. The significance of this specialised knowledge is also neglected by the teacher professional development programmes. With changing trends in the last few years, there has been a growing realisation of the relevance of teachers' knowledge. The focus on 'active' learning on the part of children has raised issues concerning teacher knowledge. Teacher learning in classrooms is still elusive. The National Curriculum Framework (2005) has articulated a vision for teaching mathematics that includes 'engaging students in problem solving, mathematical communication, systematic reasoning and making connections'. It is emphasised that teachers need to develop ways in which learners develop sophisticated ways of solving (mathematical) tasks (NCFTE, 2009/10). Although such a concern has been acknowledged in recent curricular and policy documents, we are far from identifying pathways through which these concerns can be realised. Exemplars or models through which students' thinking can be utilised as a tool for teacher learning in various teacher education programmes are yet to be explored.

## RESEARCH OBJECTIVES

The research aimed to investigate:

- nature of teacher's knowledge about students' mathematical thinking and learning
- relation between teacher's knowledge and her responses to students' mathematical thinking in and outside classroom
- teaching practices which reflect manifestations of knowledge about students' thinking

## THE STUDY

This study is a part of a larger ongoing study which tries to utilise students' mathematics as an authentic context for teacher learning. This paper reports a pilot study to inquire about teacher's knowledge of and responses to students' mathematical thinking while teaching in classroom. There was a need to design a suitable context within which teacher's knowledge of students' mathematical thinking could be elicited and explored. The aspects focussed in the design and implementation of a *classroom-based task* for the teacher were: students' mathematical thinking and the mathematical concept being discussed in the classroom. Teacher's knowledge about the concept and students, and its manifestations in classroom while teaching were the object of study.

## METHODOLOGY

Exploratory Case Study was considered as appropriate methodological design to probe deeper into teacher's knowledge, thinking and decisions while teaching. Classroom observations were used to capture the dynamics of classroom teaching and learning in the context. Task-based interviews were conducted with the teacher and the students to understand their perspectives on the concept being taught. Task-based interviews involve a subject, an interviewer, interacting in relation to one or more tasks (questions, activities, problems), generally used in psychological studies in making inferences about mathematical thinking, learning, and problem solving (Goldin, 2000). They are used to focus subjects' attention on the process of mathematical tasks rather than the final answers.

### Participants and settings

Students participating in the study were from an English-medium private school in Mumbai that follows an Indian Certificate of Secondary Education (ICSE) curriculum. Unlike most schools in India, students in this school address teachers by their name indicating equality of respect. Initially in this study, four mathematics teachers from this school were followed in their classrooms. However, practical limitations and attempt to focus on one teacher across her classrooms was found to be suitable for the objectives of the study. A Grade VII teacher who unlike the other mathematics teachers in that school, allowed students to talk and ask questions while teaching was chosen for the study. The teacher was followed for a period of three months (21 sessions of 30-90 minutes each[1]) as she taught in two Grade VII classrooms, with 34 students in each class. The concept focused in interactions was Proportions, as the teacher taught this during the period of study. Some other mathematical concepts (like geometry and ratio) and projects were also observed by the researcher to gain familiarity with the setting, students and the teacher.

### Data collection and analysis

Teacher's understanding of students' mathematical thinking was gained through classroom observations and task-based interviews with students and the teacher (constitutes Phase 1 of the study). Classroom observations were video, audio-recorded and were supplemented with field notes by the researcher. All interviews with the teacher and students were audio-recorded and transcribed. The teacher was interviewed prior to and after every lesson while teaching 'Proportions'. The nature of questions posed to the teacher were related to objectives of teaching the lesson, considerations for lesson planning, connections with the

1 The sessions include classroom observations, task-based interviews with students and the teacher throughout the study

previous lesson(s), etc. This was followed by classroom observation of her teaching. Instances where she deviated from the plan, her responses to students' questions/ responses in class, questions posed to students, etc. were focused. (Any) Five students were interviewed after each lesson. Questions posed were related to how they solved the problems in class, their questions or suggestions. Teacher was interviewed after each lesson to elicit more about her thinking behind the instances highlighted in classroom observations. Specific cases of students' questions and responses (on proportion problems) were taken up in this after-class discussion with the teacher. Phase 2 of the study included the design and implementation of a task. Six problems on 'proportional thinking' were created or modified from literature (Lamon, 1993, 2006). The problems were discussed with the teacher for their suitability with students. The teacher was then requested to anticipate students' responses to these proportion problems. 11 students, selected by the teacher as representative of the range of ability in her class, were asked to solve these problems and justify their solutions. Their verbal and written explanations were taken up as reflections with the teacher. Thus, the data sources included observations, interviews, discussions with students and the teacher. Written documents like teacher's lesson plans, assessment records, students' notebooks and test papers, background of the teacher and students, etc. were also studied.

The thick descriptions of teacher's teaching in classroom and her responses to students' mathematical thinking from Phase 1 enabled the creation of a teacher profile. Patterns in teacher's responses to students' errors, alternate solutions, justifications, etc. while teaching, were identified. Interviews helped in triangulating teacher's responses and also served as instances for reflection. The questions posed by the teacher, students and their responses were analysed. The data from Phase 2 was organised in categories as depicted in Table 1.

Table 1: Analysis of first Proportion Problem

[T: Teacher, R: Researcher, S1,2..11: Students]

| Proportion Problem | Teacher's Anticipation | | Students' Responses | | Reflection with the Teacher |
|---|---|---|---|---|---|
| | *Strategy* | *Error* | *Strategy* | *Error* | |
| The cost of 10 pens is Rs. 42. What will be the cost of 15 and 20 such pens? | Cross-Multiplication | Cancellation Errors | Halving cost of 10 and adding to the cost of 10 for 15 pens. Then, doubling the cost of 10 pens to find the cost of 20 pens (S1,2,3,4,5) | None | The methods are good but they are commonsen-sical. I don't know how far will these methods |
| | Algebra method (beginning with the unknown as x) | Calculation Errors | Cross- multiplication (S2) | | |

| | | | |
|---|---|---|---|
| Unitary Method (find the cost of 1 pen and multiply it with the number of pens needed) | Writing the ratios incorrectly like 10/42 : x / 15 | Finding the cost of 5 pens (as it is a common factor of 10,15,20) and using it to find the cost of 15 and 20 pens (S6,11) | help them. See this person (pointing to S10) has done it using algebra. They [students] need to work systematically like this. |
| R: Don't you think students might use halving or doubling to solve this problem?<br><br>T: I don't know if they know that much. If I would have been at their place I would not have used this method. There is a direct method of working the proportion method, so why go for some long or complicated method. They might use unitary method but not doubling and all. | | Unitary method to find the cost of 15 pens, doubling cost of 10 to find for 20 pens (S7,8,9)<br><br>Algebra Method (unknown as x and finding its value) (S10)<br><br>Unitary Method (finding the cost of 1 pen and then multiplying it with 15 and 20) (S11) | |

## ANALYSIS AND FINDINGS

It was found that teacher's notion of what constitutes formal mathematics aligned with her goals of teaching mathematics. In clarifying the difference, the teacher stated that "formal mathematics is about algorithms and routes to problem solving which are precise, while commonsensical or out-of-school mathematics includes using strategies like halving and doubling". Therefore, the goal of school mathematics teaching is to make "students learn these algorithms for better performance in standard examinations". In light of this goal, "once the students have been taught the algorithms, they are expected to use them while solving problems". The teacher was found carefully selecting students to answer the questions posed by her based on their attention in class. The decisions on which student should respond depends on her personal knowledge of the student (as quiet, shy, participative, hyperactive, etc.). Teacher's knowledge of students is also guided by these personal qualities attributed to students. Teacher thought it to be part of her responsibility to respond to students' questions. Students' questions were not revoiced or discussed in whole class. The same was true for the strategies and errors made by students (refer to the excerpt below). The teacher also believed that students cannot solve a problem "correctly unless they are taught". She did not consider that students' knowledge and thinking needs articulation and sharing in classroom.

**Excerpt from classroom observations** [T-Teacher, S-Student]

T: How to find the square root of 2025 [which is the product of 25 and 81]. To remove a square we put a square root on the other side. Use factorisation method

S: There is a easy method

T: I know

S: Can I show you the method?

T: No [*Teacher shows factorisation on board]*

S: J (*calling the teacher*) you can directly do it

T: Wait [*Teacher completes factorisation and leaves the class as the time gets over*]

**Student's strategy shared during the interview after class**

S: We made 2025 from 25 & 81. 5 x 5 & 9 x 9, so 5 x 9= 45 [is the square root], then why factorisation?

**Teacher's response to this student's strategy in the after-class interview**

> These are common-sense answers. They [students] are in school to learn algorithms. These answers will not help them in board examinations.

The knowledge about students' conceptual understanding was justified through criteria like "attentiveness, listening to the teacher, and his/ her personal interest in mathematics". For instance, while discussing the responses of two students, the teacher remarked

> He is intelligent as he goes by what is being taught. He pays attention and listens to me in the classroom [S10]

> I don't know how he is going to cope further [in board exams] because he is not listening to all the topics so the basics is not being dealt with, now he is managing to get a B (grade) because see he has solved most of the questions using logic but these things don't work later, he is using his common sense to find answers, that's very good but I don't think how long will he be able to do this... [S6]

Discussions centred around students' work served as a context for probing deeper into teacher's knowledge of students' mathematical thinking. Teacher's understanding of students came from sources like responses to teacher's questions in classroom, written tests and occasionally students' notebook. In the written tests, the incorrect responses from students were marked by the teacher, but were not accompanied with descriptive feedback. When a majority of students do a particular problem incorrectly, teacher knows that the students have not understood and therefore the response is to repeat and revise the concept.

An important finding is that although the teacher possessed a sound content knowledge of mathematics and showed a concern about students in her approach, her idea of what students need to learn in mathematics was restricted to reproducing the algorithms. This finding makes us think that teacher's knowledge of students is not a by-product of the sound knowledge of content. The teacher in this case misunderstood knowledge of students' thinking as being able to predict their performance. Though the teacher provided space for students to talk in classroom, attempts where students' mathematical responses were noticed, revoiced, or used for discussion were missing.

It was interesting to notice the enthusiasm with which students approached the proportion word problems posed by the teacher in class and researcher in Phase 2 of the study. Students were also keen to share different ways used to solve the problems. The set of proportion problems (given in Phase 2) elicited different student strategies and their justifications. Some of the strategies used were halving and doubling, estimation, using the common factor, unitary method, proportion method, generalising the relation or solving algebraically, comparison with half, and sometimes a combination of two or more strategies. An example to substantiate this is as follows. The students were given a problem on inverse proportion. The teacher anticipated that none of the students will be able to solve this problem since it was not taught in class. On the contrary, all the students solved this problem correctly and using diverse strategies. Some students' ways are shown in Figure 1.

*Problem*: It takes four people 3 days to wash all the windows of the K-Star mall. How long will it take for 8 people to do this job?

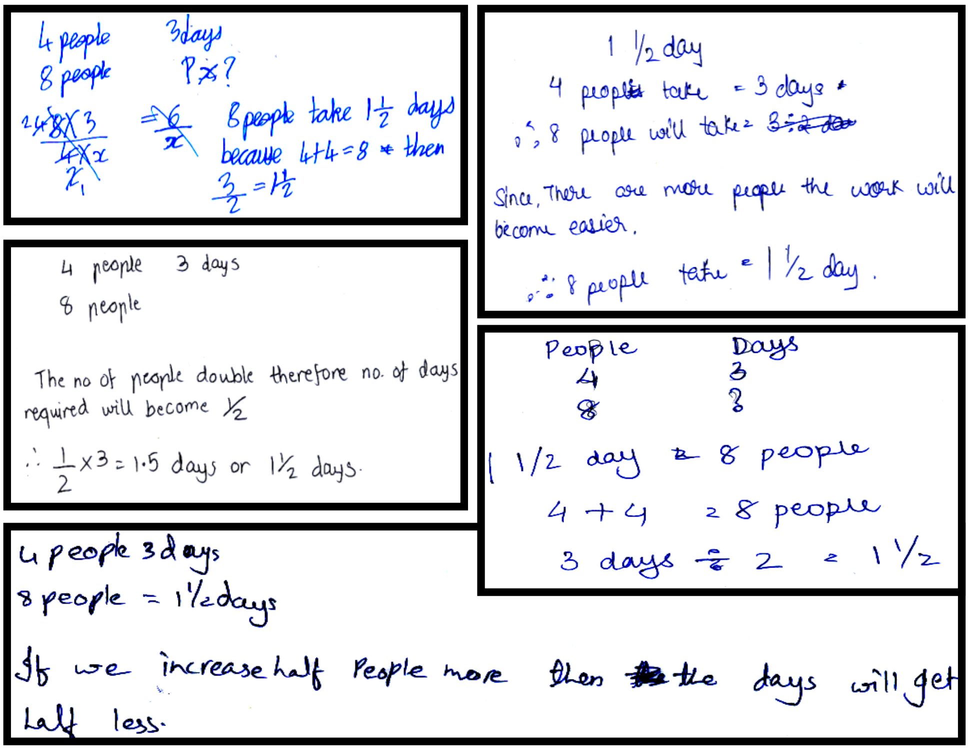
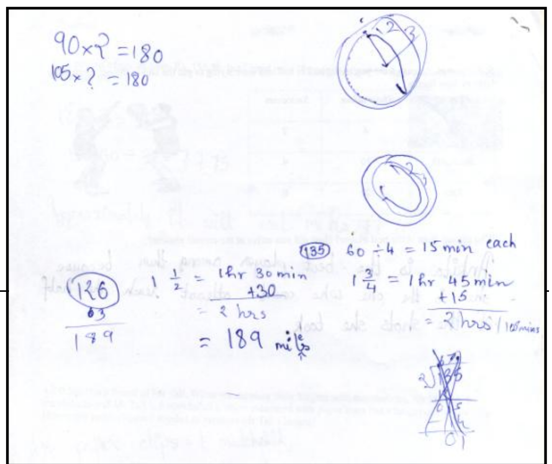



Figure1. Student Responses (S 5,7,2,3,8)

Analysis of data from phase 2 of the study (a glimpse is shown in Table 1) showed a clear mismatch between the problem solving strategies or errors anticipated by the teacher and those by students. On reflection, the teacher rationalised this gap by classifying students' strategies as logical but distant from algorithms and therefore "unacceptable in school". However, some other unanticipated evidences from students' strategies like the potential of a students' response, their conceptual understanding, ability to relate mathematical concepts, solve problems by reading the context despite not being taught (as in case of the problem above) were conflicting with teacher's knowledge about students' thinking. A case in point was the discussion around a student S11, whom the teacher classified as weak, inattentive, and generally possessing no understanding of ratios and proportions.

*Problem:* Which vehicle has faster average speed- a truck that travels 126 miles in 1½ hrs. or a car that travels 135 miles in 1¾ hrs.

During reflection, the teacher initially pointed out that S11 left the solution incomplete (refer Figure 2). The researcher suggested the teacher to solve the



problem in the way student has tried. The teacher was amazed by the complexity of the solution and while discussing student's thinking shared “I never thought she could think like this”. Instances like these where the teacher and researcher discussed each response and the related (sub-)concepts helped in challenging teacher's assumptions about the students' thinking, marking a step towards developing sensitivity to students' thinking and discovering its logic and complexity.

## CONCLUSIONS AND DISCUSSIONS

This study is an attempt to discover the nature of teacher knowledge about students' mathematical thinking and its manifestation in classroom. The in-depth analysis of a concept and two classrooms was insightful in characterising teaching-learning practices in terms of goals of the teacher, sources of knowledge about students, and decisions while teaching. Different ways in which students approached the proportion problems enriched the reflections with the teacher and served as an authentic context to challenge teacher's notions about students' thinking. The study suggests the need to engage teachers in the process of articulating their knowledge and problematising their assumptions. The affordances arising from knowing about students' thinking and while using it for teaching would be an interesting extension to the work. Evidences from a teacher's engagement in a classroom-based task, which involved thinking about students before and after the lesson, anticipation of and reflection on students' thinking and learning, comes as powerful experience to use such tasks as potential sources of teacher learning and in gaining *knowledge in practice*. Further, consistent efforts with teachers to unpack students' mathematical thinking can help in sustaining this process. The larger aim is to empower teacher communities to engage in the process of discussing students' responses, and sharing knowledge generated from their diverse classrooms.

The researchers plan to extend this study by designing *classroom-based tasks* aimed to empower teachers to collectively build an understanding of students' mathematics in a mathematical domain through sharing of experiences, critically evaluating their practice and engaging in conversations around students' work. The teacher-researcher relation visualised is that of scaffolding and conducting an inquiry where we attempt to understand the mathematics in students' responses and ways in which it can challenged and scaffolded. It is hoped that such a model would help in learning from the richness of teachers' practices and also serve as a potential exemplar of teacher learning and teacher education.

## References

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing Mathematics for Teaching – Who knows Mathematics Well Enough to Teach Third Grade, and How We Can Decide? *American Educator*, pp. 14-46.

Carpenter, T. P., Fenemma, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using Knowledge of Children’s Mathematics Thinking in Classroom Teaching: An Experimental Study. *American Educational Research* Journal, 26 (winter), 499-531

Cochran-Smith & Lytle (2000). A Focus on Student and Teacher Learning. In S.J.Finley (ed.), *Instructional Coherence: The Changing Role of the teacher*, SouthWest Educational Development Laboratory.

Even, R., & Tirosh, D. (2008). Teacher Knowledge and Understanding of Students' Mathematical Learning and Thinking. In Lynn D. English (eds.), *Handbook of Research in Mathematics Education,* Second Edition. Lawrence Erlbaum Associates. pp 202-222.

Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics Teaching and Classroom Practice, In F. Lester (eds.) *Second Handbook of Research on Mathematics Teaching and Learning,* National Council of Teachers of Mathematics.

Goldin, G. (2000). A Scientific Perspective on Structured, Task- Based Interviews in Mathematics Education Research. In A.E. Kelly & R.A. Lesh (eds.) *Handbook of Research Design in Mathematics and Science Education*, Lawrence Erlbaum Publishers, London.

Hill, H., Ball, D., & Schilling, S.G. (2008). Unpacking Pedagogical Content Knowledge: Conceptualising and Measuring Teachers' Topic- Specific Knowledge of Students. *Journal of Research in Mathematics Education*, 39 (4), pp 372-400.

Lamon, S.J. (1993). Ratio and Proportion: Connecting Content and Children’s Thinking. J*ournal for Research in Mathematics Education,* 24 (1), 41-61

Lamon, S.J. (2006). *Teaching Fractions and Ratios for Understanding.* Lawrence Erlbaum Publishers, London.

NCF (2005). National Curriculum Framework 2005. *Position Paper National Focus Group Report on Teaching of Mathematics*, National Council of Educational Research and Training , New Delhi.

NCFTE (2009/10). *National Curriculum Framework for Teacher Education: Towards preparing professional and humane* teacher. New Delhi: National Council for Teacher Education.

Schön, D. (1984). *The Reflective Practitioner: How Professionals Think in Action.* Basic Books.

Shulman, L. (1986). Knowledge and Teaching: Foundations of the New Reform. *Harvard Educational Review*, 57(1), 1-22

Zaslavsky, O., & Peled, I. (2007). Professional Development of Mathematics Educators. In B. Choksi & C. Natarajan (eds.) *The epiSTEME Reviews*, Vol 2, MacMillan, India.