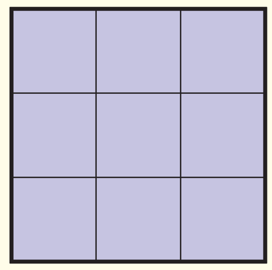
Non-integer Dimensions through Self-similarity

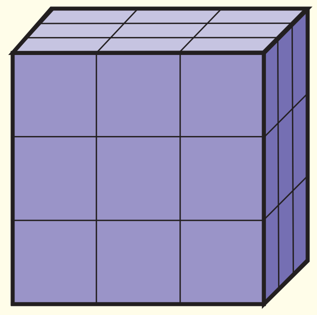
We usually think of space as 1, 2 or 3 dimensional. Geometric objects can also be 1-dimensional (e.g. Line), or two dimensional (e.g. Triangle) or three dimensional (e.g. Cube). But how can an object be 1.5 dimensional?

If a line segment is scaled up by 3 times, we see that $3^1 = 3$ copies of the original can fit into the new segment.

If a square is scaled up by 3 times, $3^2 = 9$ copies of the original can fit into the new square.

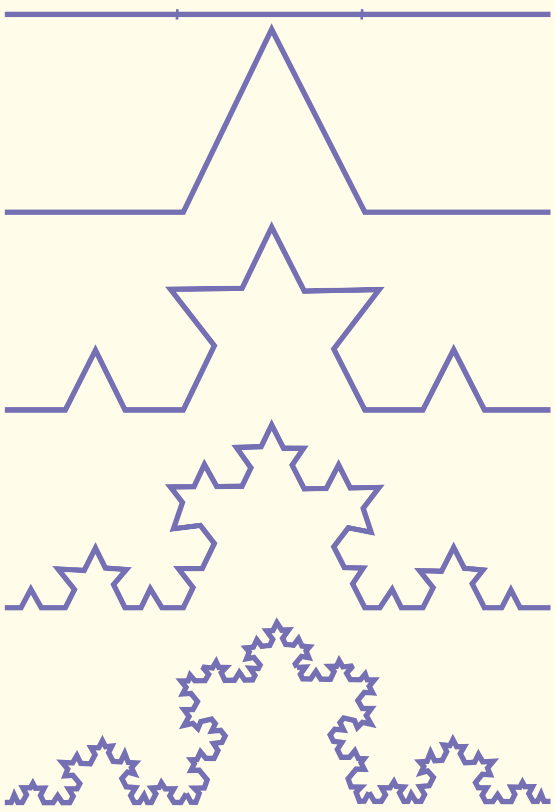


If a cube is scaled up by 3 times, $3^3 = 27$ copies of the original can fit into the new cube.



Generalizing, we can write

$$\text{Scaling factor}^{\text{dimension}} = \text{No. of copies}$$



Koch Curve

Divide a line segment into 3 equal parts. Replace the middle segment by two sides of an equilateral triangle with the same length. You get Shape 2 which is made of 4 equal line segments. Repeat the steps for each line segment and iterate this process infinitely. The shape we get is a Koch Curve.

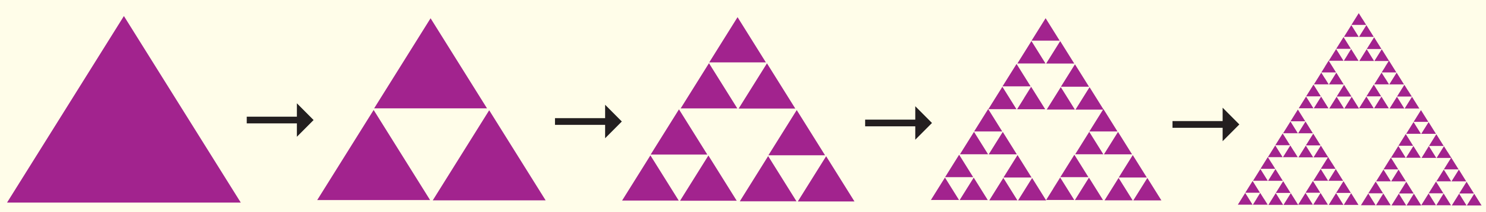
Note that if we scale up the Koch curve by a factor of 3 we get a similar shape. Also 4 copies of the original fit into the new shape.
So,

$$\mathbf{3^D = 4}$$

Simplifying, $D = \log_3 4 = \mathbf{1.2618595}$

The Koch Curve has dimension 1.262 (approx.) !

Sierpinski Triangle



We can get Triangle 2 from equilateral Triangle 1, by punching out a hole in the center in the shape of an equilateral triangle whose side is half the length of a side of Triangle 1. Now iterate the process to get the Sierpinski Triangle.
We see that if Triangle 5 is scaled by a factor of 2, then 3 copies of the original can fit into the new triangle.

Hence $\mathbf{2^D = 3}$

Simplifying, $D = \log_2 3 = \mathbf{1.585}$

The Sierpinski Triangle has dimension 1.585 (approx.) !

The Koch curve or the Sierpinski triangle both can be decomposed into smaller copies of themselves. This calculation of Dimension(D) depends on this property. Shapes with this property are called as ‘Fractals’. Fractals can have non-integer dimensions.