56

COMPENDIUM OF ERRORS
IN MIDDLE SCHOOL MATHEMATICS

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PREFACE

We are pleased to bring out this compendium of student errors in mathematics which prevail at the middle school level in India. This compendium is mainly an outcome of the so called Mathematics Teachers' Quality Circle which Homi Bhabha Centre for Science Education (HBCSE) organized during 1993-94. This was an informal group of about 25 teachers which met once in two months for a day long workshops devoted to issues regarding improvement in the quality of mathematics teaching at the school level, especially at the middle school level (Std. V, VI, VII in the state of Maharashtra).

In one of the workshops the teachers worked on the common errors made by their students. They brought forth a large number of errors, some of them previously not so well known. HBCSE scientists in their field work verified a number of these and added a few more based on their analysis of students' answers in examinations. We found that our collection of errors in algebra and arithmetic had become large and significant enough to be put it into a book form. It would be of interest to teachers and parents alike who wish to know about the common errors their students / children make. We realized during our endeavour, that most of their errors are *methodic*, that there are some understandable patterns in them. Knowing these patterns and methods, we believe will be of considerable help in remediation of these errors. Such remediation is an important step towards making learning of mathematics a pleasant experience for the students.

Of course, by no means this collection can be claimed to be exhaustive. There may be a large number of errors not found here. Yet we humbly trust that this compendium will serve a useful purpose in instruction. We would like our readers to go through the material presented here critically. We shall welcome their comments and suggestions.

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Contents

Section Zero

INTRODUCTION

We begin with a simple, rather obvious example. Suppose a student is asked to find out the highest common factor (h.c.f.) of 12 and 16 by the method of factors. How does he go about solving this problem ? He factorizes 12 and 16. From the factors reads the h.c.f. He knows what the h.c.f. is, what factors are, how to get factors of 12 and 16 and the sequence of the steps to find the solution. Thus the student follows a procedure. For carrying out the procedure he has to know certain concepts, definitions (h.c.f., factors) and rules (e.g., how to read off the h.c.f. from the factors) and the procedure as a whole (the flow of its steps, appropriateness).

Such a systematic step by step procedure is known as *algorithm*. We find that most of the arithmetic and algebra at the primary and middle school level is algorithmic in nature.

When students are asked to solve a problem, they try to bring to their mind the procedure, the algorithm, that they have learnt in their class. In doing so, i.e., in reconstructing the algorithm there often arise gaps. These gaps may arise with regards to

1. concepts and definitions
2. rules for carrying out the steps, or
3. the flow of and relations between the steps leading to the procedure as a whole.

Often such gaps are filled by the students in their own alternate ways. For example, when asked if 15 is a prime, many students answer in the affirmative. They equate prime with odd. Odd is a simpler concept to remember and the students replace the more complex concept of prime with this simpler alternative concept.

It is now known that most of the errors children in their arithmetic at the primary and middle school level make are in the form of filling up the gaps in the required algorithms. Borrowing the terminology from computers further, one calls these errors *bugs*. Some of the bugs are easy to correct because they do not have a deeper, alternative conception behind them. For example, students often interchange h.c.f. and l.c.m. This is a bug which can be removed rather easily. On the other hand there are other bugs which are very hard to correct because they are rooted in alternative conceptions that are difficult to dislodge. For example, students take the h.c.f. of numbers which are relative prime to each other, such as 8 and 9, to be 0 and not 1. This is because when they write factors of 8 (= 2 x 2 x 2) and 9 (= 3 x 3), they do not find any common factor. The absence of what one is looking for is shown by the zero for the student. Thus there is a *naturally attractive*, strong alternative reasoning behind the students taking 0 as the h.c.f. and this may completely overwhelm the correct reasoning. We refer to such sticky, persistent bugs as *stereotypes*. In the following pages we have

56

identified some of the common bugs and stereotypes encountered in middle school arithmetic. The topics covered are

1. Divisibility
2. Fractions
3. Decimal fractions
4. Indices
5. Elementary algebra.
6. Signed numbers

Each error is given in the form of an argument made by a student. Occasionally, the question to which the argument comes as a response is also given. For each error we discuss below the student argument the nature of the error including its possible origin and remedy for correction. In section 1 on Divisibility the discussion is more systematic. We give there below the student argument the exact error and discuss separately the origin of the error and the remedy. We also give the *type* of the error depending on its origin. First it could be either a bug or a stereotype as mentioned above. Secondly, since an error is mainly the result of a gap in the

required algorithm, we have tried to identify the nature of the gap, whether it is

1. due to conceptional inadequacy,

2. due to a rule being improperly applied, or

3. at the level of the entire procedure.

Fourthly, often, the error may be triggered due to inadequacy of language used in the definitions, rules or procedure names. An attempt is made to classify the errors in divisibility according to these four causes. We believe that such a classification helps in devising a strategy to guide the students to correct their errors. The classification is not given for errors described in the sections following Section 1, but the readers may themselves profitably use the scheme given in this section.

A bug may be removed by a strategy of explanation relevant to its cause. Thus we may explain to the student either

1. the meaning of a given label, or
2. the definition or concept involved, or
3. the proper use of rules, or
4. the procedure as a whole.

Stereotypes, however, cannot be removed in this soft way. In their case, first we must confront the students with the consequences of their errors to make them realise the seriousness of the errors and only then provide explanation as in the case of bugs. (An example of this is found in the first error included in the Section 1, on Divisibility, where the student equates *odd* with *prime*, e.g., 15 is prime for him. He should be confronted by showing to him that odd numbers like 9, 15, 21 are composite and not prime. Many other examples calling for this strategy may be found in the following pages.)

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Section One

DIVISIBILITY

1. Question

Give examples of prime numbers.

Answer

7, 9, 13, 15 etc.

Error

The student takes 9, 15 etc. (odd composite numbers) to be prime.

Discussion

The student has most likely forgotten that a prime number is not divisible by any other number than 1 and itself. He, therefore, fails to notice that 9 is divisible by 3 and 15 is divisible by 3 and 5. He then replaces the concept of *prime* by a relatively simpler, easier to remember concept of *odd*. In this process he is helped by the observation that all primes except 2 are odd. The student as yet does not possess the critical ability to see that the converse, viz., all odd numbers are prime, is not true. In the absence of such critical ability he ends up with the stereotype, prime = odd.

Type

The error here is found to be a recurrent one and quite common at the level of middle school. It may be termed as a stereotype arising out of conceptional inadequacy (which here corresponds to the replacement of a more complex by a simpler concept).

Remedy

First the student should be confronted showing that 9, 15, 21 are odd, but not prime. He should then be reminded that a systematic way to conclude about the primeness of a natural number is by checking patiently whether the number is divisible by all primes less than it.

There is another dimension to this error. Many students find it very hard to handle the subtle two parameter situation here. They fail to comprehend that a number may be classified in two different ways; one by a parameter of odd / evenness and the other by a parameter of prime / compositeness. An odd number, therefore, can be either a prime or a composite number. In order to correct the students' misconception, the following two way tabular exercise may be helpful. The students may be asked to enter numbers beginning with 2 in appropriate positions as indicated :

	Odd	Even
Prime	3, 5, 7, 11, 13, 17, 19, 23, ...	2
Composite	9, 15, 21, ...	4, 6, 8, 10, 12, 16, 18, 20 ...

They then realise that 4 is both even and composite, but composite $\neq$ even. Similarly 5 is both odd and prime, but odd $\neq$ prime.

2. Question

State the divisors of 24.

Answer

1, 3, 4, 6, 8, 12, 24.

Error

The student omits from the list of divisors only one number, namely, 2.

Discussion

Here, the student seems to be aware of what a divisor is and that one should check whether the given number (dividend) is wholly divisible by its likely divisor. We found that the students who were making the above error of omission were indeed following a procedure to check divisibility. They were, however, following a simpler procedure instead of actual division. They were using multiplication tables. The multiplication tables they use in our part of India go from 1 upto 10. As a result when the students recite the tables, 24 occurs as a product in their tables of 3 ($3 \times 8 = 24$, $8 \leq 10$), 4 ($4 \times 6 = 24$), 6, 8, 12 and 24, but not in the table of 2, since $2 \times 12 = 24$. (It is interesting to note that the students do not use commutativity. Also,

they include 1 as a divisor because they are explicitly told by the teachers that any number is divisible by 1). We expected and have verified that the same set of students omits 3 from the list of divisors of 36.
(Note : 3 x 12 = 36)

Type

Clearly we have here a stereotype arising out of inadequacy at the level of a procedure (which here corresponds to an entire procedure being replaced by a simpler alternative).

Remedy

The students should first be confronted with the question why 2 was omitted. Is 24 not divisible by 2 ? How would you show that 24 is divisible by 2 ? We should carry out for this the actual division and then tell the student that use of tables is only a shorter alternate route to actual division. Further

this alternate route is applicable only in a limited number of cases. For example, it is not very useful to find divisors of 96, which are 1, 2, 3, 4, 6, 8, 12, 16, 24, 48, 96. The number 96 is *accessible* in tables of 12, 16 and 24 only. (students do tables upto 30, not beyond).

3. Student argument

12 is wholly divisible by 6. Therefore 6 is a prime factor of 12.

Error

6 is a factor of 12, but not a prime factor of 12.

Discussion

The student clearly mistakes a factor for a prime factor. He is replacing the more complex concept of a *prime factor* by the simpler, easier to remember concept of a *factor*.

Type

The error here may be said to be a bug arising due to concept inadequacy. We do not feel that the bug is so well entrenched as to be called a *stereotype*.

Remedy

It is necessary to explain to the students that factors of a number can be classified either as prime or composite. For example, 1, 2, 3, 4, 6, 12 are all factors of 12, but only 2 and 3 are prime. (Also, it should be mentioned that 1 is not taken to be prime.) Similarly, 2 and 5 are the only prime factors of 20 and so on.

4. Student argument

1, 2, 3, 6, are factors of 6.
$\therefore$ 6 = 1 x 2 x 3 x 6

Error

Instead of writing 6 = 2 x 3, the student takes it as a product of all of its divisors.

Discussion

A number can be expressed as a product of two or more of its factors. For example, 10 = 2 x 5, 20 = 2 x 2 x 5 etc. The student is extending this rule to cover *all* the factors of a given number. This clearly is happening in the absence of a simple monitoring check on the part of the student that

1 x 2 x 3 x 6 = 36 and not 6.

It is found that the students often know that their conclusions like 1 x 2 x 3 x 6 = 6 are wrong, but they do not have the necessary conviction to correct the situation and prefer to *live with* a known contradiction.

56

Type

Here is a bug arising on account of inadequacy in the application of a rule. The case does not warrant to be called a stereotype.

Remedy

Clearly, the student should be first shown the obvious contradiction that 1 x 2 x 3 x 6 = 36 and not 6. The student very well knows 6 = 2 x 3. From this what is to be generalised has to be made clear to him. In this product 6 does not occur; 1 may be said to occur trivially. Similarly, if we write 12 = 2 x 2 x 3, the expression does not contain factors 4, 6 and 12. Further it is worth mentioning that this way of expressing a number as a product of its factors is not unique, e.g. 24 = 3 x 8 or 4 x 6 and so on.

5. Student argument

2 and 3 are prime factors of 12.
$\therefore$ 12 = 2 x 3

Error

12 = 2 x 2 x 3 , and not 2 x 3.

Discussion

Here, too, as in the case of the earlier error (argument 4), the student is generalising from the fact that a natural number can be expressed as a product of two or more of its factors, e.g., 12 = 2 x 2 x 3, 42 = 2 x 3 x 7, 20 = 2 x 2 x 5 etc., but he seems to restrict himself only to cases like 10 = 2 x 5, 6 = 2 x 3, where the product involves prime factors only and each of the prime factors occurs only once in the product. He then concludes that any number is the product of its prime factors with each factor occurring only once in the product and applies this rule to any number like 12 not withstanding the contradiction that 2 x 3 = 6 and not 12. As pointed earlier, the student does not have the conviction to use a monitoring check, and continues to live with the contradiction.

Type

We have here a bug arising out of inadequacy in the application of a rule.

Remedy

To begin with, it should be pointed to the student that 2 x 3 = 6 and not 12. On the other hand, we should remind the student that 12 = 2 x 2 x 3. Thus when one writes a number as a product of prime factors, a prime factor may occur more than once in the product. It is necessary to say that the students' basic idea that a number is expressible as a product of its prime factors is right, but his conception that every prime

factor occurs only once covers only certain cases like 6 = 2 x 3 or 10 = 2 x 5 and has to be corrected to include cases like 4 = 2 x 2 or 24 = 2 x 2 x 2 x 3 etc.

6. Student argument

8 = 2 x 2 x 2

9 = 3 x 3

∴ the h.c.f. of 8 and 9 is 0.

Error

The h.c.f. of two numbers which are relatively prime is 1 and not 0.

Discussion

Clearly, the student seems to know the steps of the procedure by which a common factor of two numbers is worked out. He, therefore, looks for a common divisor of 8 and 9 from the factors in the expressions which he has used and does not find any. There is thus a gap in his procedure of finding the h.c.f. and uncomfortable with this, he fills this gap with his own hypothesis that the h.c.f. is zero. Most probably zero symbolises for him absence of what he is expecting, namely, the common factor. This may be an expression of a deeper stereotype in the mind of the student that absence (of a desired result) = 0.

Type

This is an example of a relatively lasting, recurring error (stereotype) arising from a gap, felt in the application of a procedure. The gap is filled using a deeper stereotype.

Remedy

Here first we should confront the student by reminding him that zero cannot be a factor of a natural number. Thus the question of getting a common factor 0 does not arise. Next we should ask if 1 is a factor of 8 and 9 and show that 1 is the common factor, in fact,

the h.c.f. We should rewrite ,

$8 = 2 \times 2 \times 2 = 1 \times 2 \times 2 \times 2$ and

$9 = 3 \times 3 = 1 \times 3 \times 3$

This makes explicit that 1 is the h.c.f. of 8 and 9.

7. Student argument

(a) $20 = 2 \times 2 \times 5$

$24 = 2 \times 2 \times 2 \times 3$

$\therefore$ h.c.f. = $2 \times 2 \times 2 \times 3 \times 5 = 120$

$\therefore$ l.c.m. = $2 \times 2 = 4$

(b) $4 = 2 \times 2$

$12 = 2 \times 2 \times 3$

$\therefore$ h.c.f. = 2

(c) $84 = 2 \times 2 \times 3 \times 7$

$70 = 2 \times 5 \times 7$

$\therefore$ h.c.f. = $2 \times 7 = 14$

and l.c.m. = $2 \times 7 \times 3 \times 5 = 210$

56

Errors

In (a), the student interchanges h.c.f. and l.c.m.

In (b), the student gets the h.c.f. wrong, he gets 2 instead of 2 x 2.

In (c), his l.c.m. is 2 x 7 x 3 x 5 (= 210), instead of 2 x 7 x 2 x 3 x 5 (= 420)

Discussion

The first is a case of wrong labelling, triggered by the words highest and lowest. These words are *misleading* for the student, who thinks the *highest* common factor of 20 and 24 should be higher (greater) than both these numbers. Similarly the *lowest* common multiple should be lower (smaller) than both of these.

In case (b), the student rightly takes a common factor, but in doing so he includes

the prime factor 2 only once. He probably thinks that in the h.c.f. a prime factor occurs only once and hence concludes that the h.c.f. = 2. The student does not stop to check if 2 is indeed the h.c.f.

In case (c), the student follows the right procedure to get the l.c.m. He takes first the h.c.f. and multiplies it by the non-common factors. While doing so, he omits the factor 2, because he thinks this factor to be already *included* in the h.c.f.

Type

The first bug (a), may be seen as a bug triggered by improper interpretation of concept labels (language difficulty). Incidentally, the language difficulty seems to be more severe in the case of students of *Marathi* medium. The (b) and (c) may be seen as bugs resulting from improper application of a rule or step in a procedure.

56

Remedy

In case (a), the student should be first told that he has interchanged the h.c.f. and the l.c.m. He may have most likely done this due to *misinterpretation* of the words *highest* and *lowest*. He should be told that these words apply to common factors and common multiples respectively. In fact, the h.c.f. is less than or equal to the smaller of the given numbers and l.c.m. is greater than or equal to the larger of the given numbers.

In case (b), we should ask the student to verify that 4 is a common factor of 4 and 12 and a number smaller than 4 cannot be the h.c.f. Two points are important here :

1. The h.c.f. need not be smaller than either of the given numbers. It could be equal to the smaller of the given numbers, as for 4, 12 or 9, 18 and so on.

2. **The h.c.f. includes all common factors. The h.c.f., therefore, may have certain factors repeated.**

For example,

8 = 2 x 2 x 2
12 = 2 x 2 x 3
∴ h.c.f. of 8 and 12 is 2 x 2 = 4.

In case (c), first the student should be told that the l.c.m. cannot be 210, since 210 is not a multiple of 84. The correct l.c.m. is 2 x 7 x 2 x 3 x 5 = 420, which contains besides the factors of the h.c.f., all the non-common factors, no matter whether they are already *included* in the h.c.f.

8. Question

Find the h.c.f. of 12 and 8 by the method of division.

Answer

$12 \div 8$ and $8 \div 4 \quad \therefore$ h.c.f. = 2

$$\begin{array}{r} 1 \\ 8\,\overline{)\,12} \\ -\ 8 \\ \hline 4 \end{array} \qquad\qquad \begin{array}{r} 2 \\ 4\,\overline{)\,8} \\ -\ 8 \\ \hline 0 \end{array}$$

$\therefore$ h.c.f. = 2

Error

The h.c.f. is 4, not 2; 2 is the quotient and 4 is the divisor (factor).

Discussion

In the long division algorithm, the answer is given by the quotient read from above the dividend. The student who is familiar with this rule from earlier classes is using the same rule for reading out the h.c.f. by the

method of division. The word *division* has most likely induced the error.

Type

The error is not probably very serious and can be corrected easily. We may classify it as a bug induced by language (i.e., here by an *unfortunate* word clue).

Remedy

Since the student's procedure is right, he should be told from where he should read off his answer. He should be encouraged to check whether his answer is correct.

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56

Section Two

FRACTIONS

1. Student argument

(a)

$$\frac{2}{3} + \frac{1}{4} = \frac{3}{7}$$

(b)

$$\frac{5}{12} - \frac{11}{12} = -\frac{6}{0} = -6$$

(c)

$$\frac{7}{12} + \frac{7}{8} = \frac{7}{20}$$

(d)

$$\frac{23}{39} = \frac{20}{30} + \left(\frac{1}{9} + \frac{1}{9} + \frac{1}{9}\right)$$

Discussion

(a) shown here is the most common error found in problems on fractions. Clearly, in this case the student is not able to recollect the procedure of addition of fractions. He then replaces this procedure with the familiar procedure of addition of integers, treating in the process the numerator and the denominator separately. The student takes the addition sign to mean addition of what is on the left and what is on the right of the sign. Realizing that there are no simple numbers on the left and right, but some

56

complicated objects (fractions), each of which has two component numbers (numerator and denominator), he handles each component number separately. Thus for him the numerator and the denominator are disjoint compartments, *an ordered pair*. This device enables him to use the known procedure of addition of two numbers. He, therefore, argues

$$\frac{2}{3} + \frac{1}{4} = \frac{2+1}{3+4} = \frac{3}{7}$$

The error is obviously sticky, since the students' prescription is a *natural* extension of the operation of addition familiar to him. It is indeed a stereotype.

(b) The error here is of the same kind as in (a) except that it corresponds to subtraction. Further, since the denominators are equal, their difference is zero. The student uncomfortable at getting a zero in the

denominator simply ignores it and writes only the numerator as the answer.

(C) Here the student is essentially following the same procedure in (a). The difference is that the numerator of the two fractions being the same, the student refrains from adding the numerators, but proceeds to add the denominators. Alternately, he may partially recollect the procedure of adding fractions with equal denominators, where the numerators are added and the common denominator is retained. As a result of the partial recollection, he may turn the procedure upside down, and add the denominators, since in the present case the numerators are equal, and denominators are different.

(d) This is of the same kind as (a), (b) and (c); the only difference being that it is addition of fractions reversed, i.e., the student breaks up a given fraction :

$$\frac{23}{39} = \frac{(20+3)}{(30+9)}$$

$$= \frac{20}{30} + \frac{3}{9} = \frac{20}{30} + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)$$

It is to be noted that the student breaks up the fractions twice, the second time as

$$\frac{3}{9} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

2. Student argument

(a)

$$\frac{2}{8} + \frac{3}{8} = 2 \times 8 + 3 \times 8 = 16 + 24 = 40$$

(b)

$$\frac{4}{15} + \frac{2}{10} = \frac{(4 \times 3) + (2 \times 2)}{30} = \frac{16}{30}$$

(c)

$$\frac{5}{9} + \frac{5}{12} = \frac{(5 \times 4) + (5 \times 3)}{9 \times 12} = \frac{35}{108}$$

Discussion

These errors are essentially due to the fact that the student does not remember the complete procedure to add the fractions and there are gaps in his reconstruction which he leaves uncovered or fills up wrongly.

In (a) he forgets that the denominator should be 8 x 8. What is surprising is the absence of criticality. The student gets 40 for the answer and does not realize that even its order of magnitude is wrong. He clearly has

no feel for the magnitude of fractions.

In (b) he follows up the procedure of adding fractions with unequal denominators properly; in fact he properly takes the l.c.m. of the denominators. The only error he makes is that he interchanges the factors that the numerator should be multiplied with.

In (c) he again follows the procedure of adding fractions properly; he also rightly multiplies the denominators with the respective factors; the only error he makes here is that in the denominator he takes the product 9 x 12 instead of the l.c.m. 36. He is aware that the l.c.m. is used here. That is why his numerators are multiplied by appropriate factors. He probably confuses the situation with that in which the denominators are relative primes, and their l.c.m. is their product itself.

e.g.,

$$\frac{5}{9} + \frac{3}{7} = \frac{5 \times 7 + 9 \times 3}{9 \times 7}$$

All the three errors can be said to be bugs arising due to inadequacy in the application of rules. They probably are not so sticky as to be termed stereotypes.

3. Student argument

$$\frac{7}{8} + \frac{3}{5} = \frac{(7 \times 3)}{(8 \times 5)}$$

Discussion

Here the student replaces the operation of addition by multiplication. His reasoning may not, however, be that simple. When fractions with unequal denominators which are prime relative to each other are added, we take the product of these in the denominator of the resulting fraction. The student applies this rule also to the numerator. Viewed this way, the error here may be classified as a stereotype arising out of improper application of a rule.

4. Student argument

(a)

$$\frac{2}{5} \times 10 = \frac{(2 \times 10)}{(5 \times 10)} = \frac{20}{50} = \frac{2}{5}$$

(b)

$$\frac{2}{7} \times 9 = \frac{2}{7 \times 9}$$

(c)

$$\frac{3}{8} \times 12 = \frac{\overset{1}{\cancel{3}}}{8} \times \overset{4}{\cancel{12}} = \frac{4}{8} = \frac{1}{2}$$

(d)

$$\frac{3}{8} \times 12 = \frac{\overset{1}{\cancel{3}}}{8} \times \overset{4}{\cancel{12}} = \frac{1}{8 \times 4} = \frac{1}{32}$$

Discussion

All the above errors arise because the student does not know how to handle an integer as a fraction. In (a) and (b), the student remembers that 10 is a multiplicative factor. In (a), he multiplies both the numerator and the denominator by it. In (b), he multiplies the denominator and not the numerator.

In (c), the student wrongly cancels a factor of 3 from the numerator and the integer multiplier. He then goes on further to cancel, this time rightly, so, a common factor from the numerator and the denominator. In (d), the student cancels a factor of 3 as in (c), and then puts the integer multiplier in the denominator, as was the case in (b). The errors here may be termed as stereotypes arising out of inadequacy at the conceptual level. (The student does not know how to express an integer in the form of a fraction.) In (c) and (d), this error is compounded by another bug arising because the student does not know the rules of cancellation of a common factor.

5. Student argument

(a)

$$\frac{5}{9} \times \frac{2}{9} = \frac{(5+2)}{9} = \frac{7}{9}$$

(b)

$$\frac{7}{27} \times \frac{9}{14} = \frac{(7 \times 14) + (9 \times 27)}{(27 \times 14)}$$
$$= \frac{(98 + 243)}{(27 \times 14)} = \frac{341}{378}$$

Discussion

It is clear that in both the cases above; the student follows the addition recipe for multiplication. He knows his addition procedure well. What is worth noting here is the students' absence of criticality; his *mechanicalness*. On probing we find that students often expect that the problems on

fractions to be complicated, and therefore feel that a relatively simple computation like,

$$\frac{5}{9} \times \frac{2}{9} = \frac{10}{81}$$

cannot be *true*. This feeling *fortifies* his substitution of the addition recipe for multiplication.

6. Student argument

$$\frac{3}{25} + \frac{5}{18} = \frac{\overset{1}{\cancel{3}}}{\underset{5}{\cancel{25}}} + \frac{\overset{1}{\cancel{5}}}{\underset{6}{\cancel{18}}} = \frac{1}{5} + \frac{1}{6}$$
$$= \frac{6+5}{5 \times 6} = \frac{11}{30}$$

Discu ssion

It is very interesting to note that the student is procedurally very systematic. Though he uses the cancellation rule for multiplication wrongly for addition, he handles the later part of the problem correctly. It is quite likely that the student, not being clear as to when cancellations can be made, has developed a stereotype : *Cancel when you see common factors between the numerator and the denominator*. Whether the numerator and the denominator in the cancellation correspond to the same fraction or not does not matter to him.

7. Student argument

(a)

$$18 \div \frac{1}{2} = 18 \times 2 = \frac{36}{1} = \frac{1}{36}$$

(b)

$$6 \div \frac{1}{2} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}\ ;$$

$$\frac{4}{15} \div \frac{1}{5} = \frac{15}{4} \times \frac{1}{5} = \frac{3}{4}$$

Discussion

In (a), the student knows that division by a fraction is multiplication by the reciprocal of the fraction. But he is unsure and therefore at the end takes again a reciprocal and puts the final answer as

$$18 \div \frac{1}{2} = \frac{1}{36}$$

clearly, he is mechanical and does not have a feel for what he is doing.

In (b), the student takes the reciprocal of the initial number instead of the subsequent number. He knows that he has to take a reciprocal, but forgets which one. Such inversions are quite a common occurrence in the students' errors.

56

8. Student argument

(a)

$$\frac{5}{9} \times \frac{4}{7} = \frac{5}{9} \times \frac{7}{4} = \frac{35}{36}$$

(b)

$$\frac{5}{9} \times \frac{4}{9} = \frac{5}{9} \times \frac{9}{4} = \frac{5}{4}$$

Discussion

Here the student has used the procedure for division also for multiplication. This has happened after the student has been introduced to division by fractions. He has learnt to take the reciprocal and then follow the procedure properly. But he has not realized that the procedure is relevant only for division.

9. Student argument

(a)

$$\frac{4}{5} > \frac{3}{4}, \quad \because 4 > 3,\ 5 > 4$$

(b)

$$\frac{4}{7} > \frac{3}{4}, \quad \because 4 > 3,\ 7 > 4$$

Discussion

In case (a), the result is fortunately right, since 16 > 15; the student's reasoning, however, is wrong. This reasoning may lead to a wrong conclusion as in (b). Clearly the student has found an alternative test of deciding the smaller or larger fraction and this test is easier and more *natural* than the test of cross multiplication. The error here may be termed as a stereotype with a wrong rule being followed.

56

10. Student argument

$$\frac{4}{5} < \frac{3}{4} \qquad \because 12 < 20$$

Discussion

Clearly, the student multiplies the numerators and denominators instead of cross multiplying. He therefore arrives at a wrong conclusion.

11. Student argument

(a)

$$4\frac{3}{5} = 4 \times \frac{3}{5}$$

(b)

$$4\left(\frac{3}{5}\right) = 4 + \frac{3}{5}$$

Discussion

The student is not familiar with the mixed fraction notation. In the first case he takes $4\frac{3}{5}$ as a product of 4 and 3/5. In the second case he considers $4\left(\frac{3}{5}\right) = 4 \times \frac{3}{5}$ not to be different from $4\frac{3}{5}$.

He does not realize that the brackets imply product. Probably, according to him the brackets contain only one term and may, therefore, be removed.

i.e., $4\left(\frac{3}{5}\right) = 4\frac{3}{5} = 4 + \frac{3}{5}$

12. Student argument

$$\frac{23}{43} = \frac{2\not{3}}{4\not{3}} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{36}{96} = \frac{3\not{6}}{9\not{6}} = \frac{3}{9} = \frac{1}{3}$$

Discussion

The student seems to generalize from examples of cancellation like,

$$\frac{21}{35} = \frac{3 \times 7}{5 \times 7} = \frac{3}{5}$$

His idea is probably to remove the common factor from the denominator well as numerator irrespective of the way it occurs. Formation of this stereotype probably gets reinforced by an example from algebra

$$\frac{3x}{5x} = \frac{3 \times x}{5 \times x} = \frac{3}{5}$$

13. Student argument

(a)

$$\frac{12 \times 5 + 7}{5} = 12 + 7 = 19\ ,$$

$$\frac{2a + 5}{a} = 2 + 5 = 7$$

(b)

$$\frac{a^2 + 5}{a} = a + 5$$

(c)

$$\frac{a + 5}{5} = a$$

(d)

$$\frac{a}{a + 5} = \frac{1}{5}$$

Discussion

As in 12, it seems that the idea of cancellation of a common factor from the numerator and the denominator is firmly rooted in the mind of the student. Obviously he is uncritical about it. He does not separate either from the numerator or the denominator, as the case may be, the common factor before cancelling it. It is important to give contrasting cases, e.g.,

(a)

$$\frac{12 \times 5 + 7 \times 5}{5} = \frac{(12 + 7) \times 5}{5}$$
$$= 12 + 7 = 19$$

(b)

$$\frac{a^2 + 5a}{a} = \frac{a(a + 5)}{a} = a + 5$$

(c)

$$\frac{5a}{5} = a$$

For (c), one may give various values to a and show that

$$\frac{a + 5}{a} \neq a$$

For (d) also the same technique as for (c) could be used.

14. Student argument

$$\frac{306}{3} = 12 \quad \text{rather than } 102$$

Discussion

The student tries to simplify the fraction by division. He ignores the zero in the long division and comes with a wrong answer. This reflects a lacuna the student most likely has been carrying on for a long time.

15. Student argument

(a)

$$-\frac{3}{4} \times 5 = \frac{-3 \times 5}{-4} = \frac{15}{4}$$

(b)

$$-\frac{3}{4} \times -5 = \frac{-3 \times (-5)}{-4} = \frac{3 \times 5}{-4} = -\frac{15}{4}$$

Discussion

Obviously, the students' treatment of the signs of fractions is wrong. Many students take , i.e. they take the minus sign in front of a fraction to be distributive with respect to the numerator and the denominator resulting in errors of the above type. If the distributivity of the negative sign is firmly entrenched in the mind of the student, the error may be sticky enough to be referred to as a stereotype.

16. Student argument

$$\frac{\frac{2}{8}}{\frac{3}{4}} = \frac{2}{\frac{\left(\frac{8}{3}\right)}{4}} = \frac{2 \times 4}{\frac{8}{3}} = \frac{2 \times 4 \times 3}{8} = 3$$

whereas the expected correct answer is

$$\frac{\frac{2}{8}}{\frac{3}{4}} = \frac{2}{8} \times \frac{4}{3} = \frac{1}{3}$$

Discussion

The error arises here because the student does not realize that what is asked is a division of one fraction (2/8) by another (3/4). The confusion is in the position of the bar indicating the division.

17. Student argument

$$\frac{2}{3} + \frac{1}{3} \times \frac{4}{7} = 1 \times \frac{4}{7} = \frac{4}{7}$$

Discussion

The order in which students perform operations in an expression is often wrong. In the absence of brackets they proceed from left to right irrespective of the standard order of operation (multiplication / division, then addition / subtraction). If the expression naturally follows this order as in

$$\frac{2}{3} \times \frac{1}{3} + \frac{1}{9},$$

no error arises; but if it does not, as in the above example, an error does arise.

****

Section Three

DECIMAL FRACTIONS

1. Student argument

½ = 1.2 ; ¾ = 3.4

Discussion

This is a stereotype found in earlier years with students who have not even remotely understood the meaning of decimals. They seem to merely put the decimal point where there should be a bar separating the numerator and denominator in a fraction.

2. Student argument

1 Rs. and 50 paise = Rs. 1.50
1 Rs. and 5 paise = Rs. 1.5

Discussion

This is a common bug found in earlier years when the students are introduced to the decimal notation for money.

3. Student Argument

(a) Place value of 5 in 23.456 is 5 (or 5/10)
(b) Place value of 5 in 23.465 is 5 (or 5/10)

Discussion

This common stereotype again found mainly in earlier years corresponds to a student's belief that every digit to the right of the decimal point has a fixed place value (either equal to the digit or the digit / 10). Powers of

(1/10) are irrelevant. (That is why the student does not give in the first case 5/100 and in the second case 5/1000 as the answers.)

4. Student argument

7. = 7.0 , 7.0 = 7.00 etc.
∴ 0.003 = 0.0003 = .00003
and 5.13 = 5.013 = 5.103

Discussion

Zeros at the end of a number written in the decimal notation are not significant and therefore may be ignored. Generalizing from here the student ignores all zeros occurring on the right of the decimal point. What is necessary here is to show to the student how the zero alters the place value of the digit following it. This is a common stereotype found even in later years.

5. Student argument

(a) 0.6 + 0.5 = 0.11
(b) 2.6 + 3.5 = 5.11

Discussion

This is a serotype in which the student thinks that the two parts, one on the right and one on the left of the decimal point, are separate compartments. So he adds 5 and 6 to get 11 and 0 and 0 to get 0 in (a) and writes the answer 0.11. In (b) he adds 2 + 3 = 5, and writes the answer 5.11. It is necessary to convince him that 0.6 = 6/10, 0.5 = 5/10 and hence 0.6 + 0.5 = 6/10 + 5/10 = 11/10 = 1.1

6. Student argument

```
(a)   9.9        (b)  4.392
   + 12.45           - 2. 49
   --------          ---------
     21.54            2.343
```



```
(c)   9.9        (d)   4
   + 12.95            - 1.03
   ---------          ---------
     21.104            3.03
```

Discussion

Here, too, like in Argument (5), the student is taking the decimal point as a separator. Once he takes the numbers on the left and right of the decimal point as separate he arranges them, as in ordinary non-decimal, addition/subtraction, in units, tens and hundred's places systematically and carries on. He has not understood the concept of place value of a digit to the right of the decimal point, and therefore does not know that 9.9 = 9.90, 2.490 = 2.49. In (a) above he adds this way, whereas in (b) he subtracts. In (c) also he adds, when he gets a carry which should affect the digits to the left of the decimal point. But this is not so for him, since he considers the two sides as separate compartments. In (d) the same hypothesis of

separate compartments persists. The student also knows that there is a decimal point to the right of 4, although it is not explicitly indicated. He does the subtraction to the left of the decimal point correctly. However, he does not know how to handle the absence of any digits to the right of a decimal point. As a result, he prefers to retain .03 from 1.03 in the subtraction.

7. Student argument

(a) 2.303 x 100 = 0.02303
(b) 2.303 ÷ 100 = 230.3

Discussion

Here there is a bug in which the student interchanges the rules for multiplication and division. Except for the interchange he makes no error. It is important that the student should have some feel for magnitudes, so that he is able to reason that

2.3 x 100 > 200 and 0.02303 is too small compared to this expected result.

8. Student argument

```
   12.45
 +   9.9
 -------
   13.44
```

Discussion

This is another stereotype in which the student first ignores the decimal point, arranges the numbers to be added properly for vertical addition (1245 + 99 = 1344) and carries out the addition. He then puts the decimal point in the sum as in the multiplication. He may have been *guided* for the whole procedure by what he is told about multiplication, i.e. to first ignore the decimal, carry out the multiplication and then worry about the decimal. A major practical

suggestion for the students will be that they should first place the decimal points of the two numbers exactly one below the other and then match the digits to the left and right systematically.

9. Student argument

(a) 15.2	(b) 15.2	(c) 15.2
x .4	x 0.4	x 0.04
--------	---------	---------
60.8	60.8	60.8

Discussion

For the student all three products are the same, because he follows only part of the procedure correctly. He ignores the decimals and carries out the multiplication. He determines the location of the decimal point in the product from only the multiplicand.

10. Student argument

```
  1.07
x 4.08
------
   856
  4280
------
  5116
```

Answer = 5.116

Discussion

Here the error does not arise so much from students' inadequacy in decimal numbers as from his inadequacy with basic multiplication. He ignores the zero in the multiplier 4.08 and gets the wrong answer 5.116 in place of the correct answer 4.3656.

11. Student argument

7/8 = ?

```
      1.142...
    ----------
 7 )  8
    - 7
      -----
      10
     -7
      ------
       30
     - 28
      -------
        20
```

Discussion

Here the procedure is right, but the first step itself is wrong, since the student is dividing 8 by 7 rather than 7 by 8. He is not *apparently* worried that his answer is > 1. It is

important that the meaning of a fraction as a division is clear to all the students.

12. Student argument

```
      125
    ---------
 2 )25
   - 2
    -------
     05
    - 4
     -----
      10
     -10
     -------
      00
```

Conclusion : 2 is a factor of 25

Discussion

This is an example of the conflict of *new knowledge* with the existing. Students learning decimal numbers newly are likely to exhibit this error. In decimal division one goes on with the division often endlessly or until one gets a zero remainder. This mixes with the student's concept of whole number division. While dividing 25 by 2 the student goes on to get a zero remainder without realizing that after 12 in the quotient his division is no longer a whole number division. Since he gets a *zero* remainder, he concludes that 25 is divisible by 2. What is interesting is that the student seems to ignore the fact that he probably knows well, i.e., an odd number is not divisible by 2.

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Section Four

INDICES

1. Student argument

$3^2 = 3 \times 2 = 6$
$4^3 = 4 \times 3 = 12$

Discussion

The error is self-explanatory. The student probably does not remember the meaning of an index, and takes the expression to mean a product of two numbers, i.e., $3^2 = 3 \times 2$.

2. Student argument

$3^2 = 2 \times 2 \times 2$
$4^3 = 3 \times 3 \times 3 \times 3$

Discussion

Here the base (the given number) and the index are interchanged.

3. Student argument

(a) $3^0 = 0$
(b) $3^1 = 1$, $4^1 = 1$
(c) $3^{-1} = -3$, $3^{-2} = -9$

Discussion

The above errors are commonly found and are *natural* to many students. In (a), the student transfers the zero from the index to the answer. He is not critical enough to check if $3^2 / 3^2 = 1 = 3^{2-2} = 3^0$. In (b) he probably

remembers that a number raised to some index gives 1. He *naturally* thinks this index to be 1. In (c), again for lack of criticality he transfers the negative sign of the index to the answer. In doing so he may properly recognize the power, e.g., for 3^{-2} he gets -9.

4. Student argument

(a) $3^2 \times 3^4 = 3^8$
(b) $3^2 \times 3^4 = 9^8$
(c) $3^2 \times 3^4 = 9^6$

Discussion

Here in (a) and (b), the student multiplies the indices instead of adding them. In (b) he also multiplies the bases. In (c) the student adds the indices, but multiplies the bases. The students' likely reasoning (alternative conception) is that the multiplication sign indicates multiplication between what is to the left of the sign and what is on the right.

Since in (a) the base is common, he feels a difference and refrains from multiplying the bases, too. In (c) he partially remembers the rule of adding indices, but gives in to his alternative conception.

5. Student argument

(a) $2^3 \times 4^2 = 8^5$

(b) $4^5 \div 2^3 = 2^2$

(c) $b^4 \times c^3 = (bc)^7$

Discussion

Here the student knows that indices have to be added in multiplication and subtracted in division. But he does not know what to do with the bases, and multiplies them as in (a) or divides them as in (b). The misconception referred to in the discussion on Error (4) prevails. In (c) the same error as in (a) is found in an algebraic form.

6. Student argument

(a) $3^2 + 3^3 = 3^5$ or 6^5

(b) $x^2 + x^3 = x^5$ or $(2x)^5$

(c) $3^4 - 3^2 = 3^2$

(d) $4^3 - 2^2 = 2^1$

Discussion

In (a) the student applies to addition the rule for multiplication. Alternatively, it could also be reasoned that the student simply adds the bases and indices. In fact, that would naturally explain $3^2 + 3^3 = 6^5$. He may, however, desist from adding the bases, if he finds a common base.

In (b), the same error as in (a) is put in an algebraic form.

In (c) we find a version of (a) appropriate for subtraction. In subtraction, when the base is common, subtraction of the bases would give a zero raised to some power. Clearly, this would look odd and engenders some inhibition in the mind of the student. He, therefore, ignores the option, and prefers to retain the common base in writing the result.

In (d), the bases being different, the inhibition does not prevail and the student subtracts both the indices as well as the bases.

7. Student argument

$(-3)^2 = -(3)^2 = -9$

Discussion

The student removes the negative sign out of the bracket without regard to its power. As a result, his final answer contains an undesirable negative sign.

8. Student argument

$a^5 \times b^5 = ab^5$

Discussion

The student's likely reasoning is $a^5 \times b^5 = (ab)^5$. He fails to distinguish between $(ab)^5$ and ab^5.

9. Student argument

$(a^5)^2 = a^7$, $((-1)^5)^3 = (-1)^8 = 1$

Discussion

The student influenced by the rule of adding indices when powers of the same base are multiplied applies it here, too. For him, $(a^5)^2 = a^{5+2} = a^7$. When the result is applied to the special case of a = -1, we get a wrong sign in the answer.

10. Student argument

$\sqrt{7x^2} = 7x$, $\sqrt{25x^2} = 25x$

Discussion

While taking the square root, the student forgets to take the root of the numerical coefficient, although he takes the square root of the literal part properly.

11. Student argument

$\sqrt{9 + 16} = \sqrt{9} + \sqrt{16}$

Discussion

The student may be generalizing from $\sqrt{9 \times 16} = \sqrt{9} \times \sqrt{16}$,i.e. he may be applying to addition a procedure that is applicable to multiplication in taking the square root. Alternately, he may be following the linearity

mode of reasoning :

$\sqrt{a^2 + b^2} = \sqrt{a^2} + \sqrt{b^2} = a + b.$

12. Student argument

$\sqrt{4} = 2$, $\sqrt{8} = 4$, $\sqrt{16} = 8$

Discussion

The student equates taking the square root of a number with halving the number. This is more commonly found when the number under the square root sign is even.

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Section Five

ELEMENTS OF ALGEBRA

1. Student argument

$3x + x + 5x = 8x$

Discussion

Here the numerical coefficient 1 is disregarded. The absence of an explicit coefficient is taken to be zero.

2. Student argument

(a) $3x + 5x = 8x^2$
(b) $5x + 3y = 8xy$

Discussion

The student here wonders what he should do with the letters. In (a), 3 and 5 being coefficients he adds them. Since he feels that something should be done with x, he squares it noting that it occurs twice. The same student would get $3x + 5x + 7x = 15x^3$

In (b) the student wrongly adds the coefficients. Knowing that they are added in case of similar terms, he applies them also to dissimilar terms. Since he does not know exactly how to handle x and y in the addition, he prefers to multiply them as in (a).

3. Student argument

(a) $3(x - 4) = 3x - 4$
(b) $(2x)^2 = 2x^2$

Discussion

These are common bugs in which the student errs while removing the bracket. In (a) he does not multiply 4 by the common factor 3. In (b) instead of 2^2 he takes just 2.

4. Student argument

(a) $(2a - 3)(a - 2) = 2a^2 - 6$

(b) $(2x - 3y)^2 = 4x^2 - 3y^2$

Discussion

In (a), the student multiplies 2a from the first bracket by a from the second bracket. He then multiplies 3 by 2, keeping the multiplication of numbers and letters separate. He does not therefore get cross terms. Further he takes the sign of 6 to be -ve. In (b), too, he keeps x and y terms separate and therefore has no cross terms. Also, he takes $(-3y)^2 = -3y^2$

5. Student argument

$(x + 8)^2 = x^2 + 64$

Discussion

This is the linearity stereotype $(a + b)^2 = a^2 + b^2$.

(Another example of this kind encountered earlier was $\sqrt{a^2 + b^2} = a + b$) Any suitable value of x ($\neq 0$) will show that the result is wrong. The student should be first convinced that the result is wrong.

6. Student argument

(a) $\frac{a^2}{a^2} = 0$ (b) $\frac{a + b}{a + c} = \frac{b}{c}$

Discussion

The student goes by the recipe of looking for common factors in the numerator and denominator. He does not stop to find whether the common factors can be legitimately separated before cancellation. In (a) he knows that the numerator and denominator being identical should cancel out. He is puzzled at the prospect both *a^2's* cancelling each other and leaving *nothing*. This nothing is represented by the zero.

7a. Student argument

Evaluate 5a for a = 2 and -3 ;

5a = 52 for a = 2

5a = 5 - 3 = 2 for a = -3

Discussion

Clearly the student does not consider 5a = 5 x a. He merely feels 5 and a to be physically adjacent as in 52. Thus here

5a = 5 x 10 + a. In the other case substituting -3 for a, adjacent to 5, as 5 - 3, leads him to the result 2.

7b. Student argument

x = 3, y = 4, xy = 34

Discussion

Here the student as in (a) does not take xy = x x y . Instead, he considers them to be two digits juxtaposed to form the decimal number 10x + y .

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Section Six

SIGNED NUMBERS (Integers)

1. **Student argument**

(a) -21 + 27 = 48
(b) -21 + 27 = -6

Discussion

These errors are a clear indication of the fact that the student has not yet realized that the signs ± are used for showing positive / negative numbers as well as for showing addition / subtraction. Unable to give meaning to the initial negative sign, the student may proceed in two ways :

(1) He may ignore the initial negative sign as in (a) and carry out the addition of two numbers 21 and 27 to get 48.

(2) He may take the negative sign to indicate subtraction as in (b) and therefore subtract the larger number from the smaller, but give the result a negative sign, since that is the initial sign.

2. **Student argument**

(a) -21 + (-27) = 47
(b) -21 + (-27) = 6
(c) -21 + (-27) = -6

Discussion

Here, too, as in **(1)**, the student is not able to handle signed numbers. In **(a)**, he simply ignores the signs in front of the numbers and adds. In **(b)** and **(c)**, seeing the negative sign he subtracts the larger integer from the smaller. He simply puts the difference as the answer in (b). In (c) he gives the difference an additional negative sign, probably because, the initial sign is negative.

3. Student argument

13 - 17 = 4

Discussion

This error is commonly found in the beginning of the study of signed numbers. Unable to handle negative numbers, the student simply subtracts the larger number from the smaller number and puts the difference as the answer.

4. Student argument

(a) 13 + (-17) = 30
(b) 13 + (-17) = -30
(c) 13 + (-17) = 4

Discussion

In (a) the student simply ignores the sign in front of 17 and adds the two numbers. In (b) he essentially does the same, except that the negative sign in front of 17 suggests to him to put a negative sign in front of the result also. In (c), spurred by the negative sign, he subtracts the larger integer from the smaller one.

5. Student argument

(a) 21 - (-23) = 2
(b) 21 - (-23) = -2

Discussion

Here, too, the student ignores the sign in front of the number. He then carries out the subtraction of the smaller number from the larger to get +2 as in (a).

In (b), the negative sign in front of 23 probably suggests to him to put a negative sign in front of the answer, too.

6. Student argument

-21 - (-23) = -2

Discussion

Again the student ignores the negative signs in front of the numbers, reduces the problem to 21 - 23 for which he gets the answer -2. He may in this process put only the difference between 23 (larger number) and 21 (smaller number) as the answer, and thus fortunately get the right answer.

7. Student argument

-21 - 23 = 44

Discussion

The recipe, *when the signs are the same, ignore the signs and add the terms*, is followed by the student. He, however, forgets that the result bears the common sign. It is likely that the student may be generalizing from the statement *two negatives make a positive* applicable to multiplication.

8. Student argument

(a) 21 - 24 + 10 = 13
(b) 21 + (-24) + 10 = 13
(c) -21 - 24 + 10 = 55
(d) 21 - 24 - 10 = 7
(e) -21 - 24 - 10 = ± 35
(f) -21 - 24 - 10 = 55

Discussion

Errors (a) to (e) have a common stereotype which occurs when there are more than 2 terms in an expression. The student always carries out the operations from the left to the right. In the fist step he takes the first two terms. He stores the result of the operation in the step and uses it with the third term. In doing so, he invariably takes the intermediate result to be positive.

In (a), he gets 21 - 24 + 10 = 3 + 10 = 13. He simply takes the difference between 24 and 21.

In (b), he essentially does the same, 21 + (-24) + 10 = 3 + 10 = 13.

In (c), he gets -21 -24 + 10 = 45 + 10 = 55. Here he may be taking as in Error (7) above -21 - 24 = 45.

In (d), he gets 21 - 24 - 10 = 3 - 10 = 7. In the first step he takes the difference. In the next step also he does the same.

In (e), he gets -21 - 24 - 10 = 45 - 10 = 35 or -35. He gets -21 - 24 = 45 in the first step probably as in Error (7) above. He then subtracts 10 from 45. At the end he may put a negative sign to the whole expression since there is a common negative sign in the original expression.

In (f), he follows the recipe, *add when the signs in front of the numbers are the same.* He either ignores putting a negative sign at the end or may think that repeated negatives make a positive.

9. Student argument

(a) (-7) x (-8) = -56
(b) -16 ÷ (-2) = -8
(c) -3 + (-4)(-5) = -23

(d) -3 (4 - 5) = -3

(e) -3 x (-4) x (-5) = -60

(f) 3 x (-4) x (-5) = -60

(g) -3 x (-15) ÷ (-5) = -9

(h) 3 x (-15) ÷ (-5) = -9

(i) (-4) x (-9) + 5 x 3 = -21

Discussion

In all these errors there is a common stereotype (-ve) = (-ve) x (-ve). The student distributes the negative sign to every number in the product. This is clear in **(a)**.

In **(b)**, the distributivity is carried to division.

In **(c)**, the student reasons this way : -3 + (-4)(-5) = -3 - 20 = -23.

In **(d)**, he proceeds -3(4 - 5) = -3(-1) = -3.

In **(e)**, he takes -3 x (-4) x (-5) = -3 x (-20) = -60. In this case he gets the correct answer fortuitously.

However, in **(f)**, he does not get the correct answer. In **(f)**, his procedure is 3 x (-4) x (-5) = 3 x (-20) = -60.

In **(g)** and **(h)**, he follows for division what he did respectively in **(e)** and **(f)** for multiplication.

In **(i)**, his intermediate step is (-4) x (-9) + 5 x 3 = -36 + 15 = -21.

10. Student argument

(a) -3 + 4(-5) = 23

(b) 3 + 4(-5) = 17

(c) 3 + 4(-5) = 2

Discussion

In **(a)** and **(b)** here the student gets 4(-5) = -20, but then proceeds to make one of the errors referred to above. In **(a)**, his steps are -3 - 20 = 23. (*Two negatives make a* *positive*). In **(b)** finding the negative number to be larger, he writes for the answer only the difference between the two numbers, 3 - 20 = 17.

In **(c)** there is rare, but peculiar, error. The student has a problem of interpreting the brackets. He is not aware of the implicit multiplication involved. He simply disregards the brackets, puts 4 and 5 together with a negative sign between them, and carries on :

3 + 4(-5) = 3 + 4 - 5 = 7 - 5 = 2.

If the initial 3 is replaced by -3, the student will most probably end up with some additional errors.

11. Student argument

(a) -3 + (-7)(-3 - 5) = -59

(b) -3 + (-7)(-3 - 5) = 59

Discussion

In (a), the student makes the error referred to in error (9) above,

-3 + (-7)(-3-5) = -3 + (-7)(-8) = -3 - 56 = -59.

For him the negative sign is distributed over each number for multiplication.

In (b), he takes double negative to mean positive for addition :

-3 + (-7)(-3 - 5) = -3 + (-7)(8) = -3 - 56 = 59

12. Student argument

(a) 3 x 15 - 5 = 20

(b) 15 ÷ 3 - 2 = 15

Discussion

In case of fractions (Error 17 there) it was seen how students take the natural order of operations to be from the left to the right. In these examples, however, the order of operations coincides with the *natural* order of going from left to right. Yet the student errs probably because, on seeing 15 - 5 = 10 and 3 - 2 = 1, which are effectively simple numbers, he is inveigled into carrying on the subtraction first. (It is possible that he always gives subtraction the highest priority.) Thus, for him,

3 x 15 - 5 = 3 x 10 = 30 and

15 ÷ 3 - 2 = 15 ÷ 1 = 15.

13. Student argument

- 98 > - 69

Discussion

This is an error found in the beginning of the study of signed numbers. The student thinks of only the absolute magnitude of the numbers in determining their order. The sign is *irrelevant* for him.

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