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CONSTRUCTING THE CONCEPT OF AREA MEASUREMENT IN A CLASSROOM

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In this paper we will present an analysis of two classroom episodes of argumentation among students involving aspects of the concept of area. These episodes of argumentation occurred when students were proposing their varied solutions for a given problem. The analysis will elaborate on the process of construction of the area concept in the classroom by focusing on two major aspects of the classroom interaction. The first aspect is the structure of students' argumentation, how they defend their claim and what 'warrants' they use. The second aspect of the analysis explores various facets of the students' conceptual understanding of area and the tension they face as they move between spatial and numerical representations.

INTRODUCTION

Research in Mathematics Education has shifted its focus from looking at an individual learner to the social process of learning as a product of social interactions (Voigt, 1994). However this trend is not reflected in studies focusing on area-measurement learning (Battista, 2007). Thus, it is important that studies on the concept of area make a shift from looking at an individual student's learning to the construction of area-concept in a classroom setting. Constructing the mathematical knowledge of area measurement in a classroom requires meaning making discussions in the classroom. Forman, Larreamendy-Joerns, Stein, & Brown (1998) have emphasized the importance of mathematical argumentation among students in order to enhance the understanding of a particular mathematical concept. Some previous studies have used Toulmin's argumentation structure to analyse mathematics classroom interaction (Forman et. al., 1998; Krummheuer, 2007).

In this study, we investigate students' construction of the area concept in a classroom by a detailed analysis of students' argumentation in the classroom. We are using an adapted version of Toulmin's structure of argumentation to analyse a few episodes from our classroom (Toulmin, 2003; Toulmin, Rieke, & Janik, 1979). Toulmin's structure of argumentation consists of three main components: one, *claim* whose truth is to be established, two, *ground* consisting of facts which provide the foundation for the claim and three, *warrant*, which provides the basis to arrive at the claim from the ground. The credential of the warrant comes from the *backing*. Backing is usually field dependent and likewise warrant also varies with different fields of argumentation. The basic structure of Toulmin's argumentation layout is as shown in Figure 1.

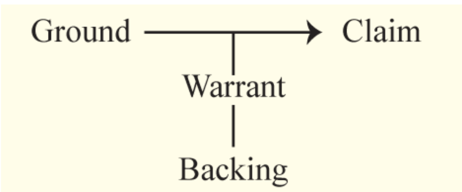

Figure 1: Toulmin's Layout of Argumentation

Reid & Knipping (2010), in their review of research on proof in mathematics education have indicated the presence of good theoretical groundwork in the field of argumentation but they have pointed to the need for empirical work in this field. Thus, the focus of our study is the analysis of empirical episodes of argumentation in the classroom. Several authors have noted that argumentation in the classroom is different from proof. In fact, argumentation is important in the classroom because mathematical proof does not convince students of the validity of mathematical results (Carrascal, 2015).

Design of the Study

The data for this paper is based on a teaching design experiment, which was part of a larger study on students' learning of the area concept. Teaching experiments have been adopted as a research methodology for various purposes, one among which is the development of ideas in a classroom environment (Kelly & Lesh, 2000). Teaching design experiments are based on our knowledge of existing research and theory and seek "to trace the evolution of learning in complex, messy classrooms and schools, test and build theories of teaching and learning (Shavelson, Phillips, Towne & Feuer, 2003, p. 25). In our study, tasks were designed for instruction based on the research literature on area learning, in particular to aid in distinguishing numerical and spatial solution strategies used by students (Battista, 2007), and to study the interaction between the two. The data was analysed by focusing on the interaction among students and between the students and the teacher during episodes that called for reasoning about areas of figures. The argumentation framework prompted us to focus on those episodes where varying claims are put forth, are challenged and justified.

From the main episodes analysed in the paper, we can see that the episode of argumentation emerged as a result of keeping our teaching methodology in line with the variation theory (Holmqvist, Gustavsson, & Wernberg, 2008). It is reported that, variation theory allows one to have different learning outcomes by making small changes based on reflections within the class. Variation theory is focused more on what the student experiences than what the teacher wants the students to experience (p. 128). The variation theory distinguishes between the intended and enacted object of learning and the pattern of variation used by the teacher to achieve that. Thus this approach allowed us to have newer insights by accommodating variations in our classroom setting.

The teaching was done by the researcher (i.e., first author) and her colleague over 12 days with approximately 120 minutes every day. The topic of instruction was area measurement and, each day began with a warm-up game or activity for 20-30 minutes followed by the tasks based on area-measurement. In all, 30 students participated in the study which included both sixth and seventh grade students.

Data collection happened through video recording of each lesson, with a fellow researcher writing the lesson log each day. Each day's lesson was followed by a debriefing session with fellow researchers about the conduct and planning of the day's and the next day's lesson.

In this study, we are focusing on two different episodes in detail, which involve instances of argumentation in the classroom. The two major aspects of our analysis are: (1) to look at the structure of argumentation in students discussion in the classroom and (2) to look at the conceptual underpinnings in these discussions. Pseudonyms are used to protect the identity of the students.

Episode-1

In one of the tasks, the students were asked to make different possible rectangles for a given size on a graph paper and then write the numerical facts. One of the sizes given was 15 units. For this size students came up with various facts like 3×5=15, 2×7.5=15, 1×15=15. Sajaad came up with 30×1/2=15. He came to the black board and made a 6×5 rectangle and divided it vertically into two halves to show that there are 15 units in each half. The teacher then asked the students to come up with more ways to divide a 6×5 rectangle into two equal parts. Many students suggested horizontal division and Sajaad suggested diagonal division as well. Most of the students agreed that the rectangle can be divided vertically, horizontally or diagonally into halves. But when Sajaad tried to divide his 6×5 rectangle diagonally into two halves, he was stuck as he was not able to identify 15 units as contained in each triangular part. So he thought that the diagonal division was not giving half the area. In an attempt to convince Sajaad that a diagonal division too will produce halves, the teacher prompted Sajaad to check if the diagonal divisions are congruent. The teacher gave him a pair of scissors to check whether the two pieces are equal without actually counting the units in each part. But Sajaad was not convinced that the two pieces are equal, as indicated in the following transcript.

> Teacher: … *Ye ek dusre ke baraber ho raha hai? Ho raha hai na?* [… Are they becoming equal to each other? They are becoming equal, right?]

> Sajaad: *Nahi* . [No.]

After cutting the rectangle diagonally into two halves, he was unable to superimpose the congruent halves without the teacher’s help. Even after the teacher demonstrated that the halves are indeed congruent, he reacted as:

> Sajaad: *Lekin ye aa kyun nahin raha hain pandrah.* [But why we are not getting 15 for this.]

Using Toulmin’s argument structure, we identify the claim as “When a 6×5 rectangle is diagonally divided into two equal halves, the area of each triangle is 15 units”. The data for this claim consists in the two halves being congruent to one another. However, the inference from this data to the claim is mediated by other assertions, which can be categorized as “warrant” following Toulmin’s scheme. The argument structure for the student shows that even when two parts of a whole seems spatially or geometrically congruent (equal), there is a doubt about the numerical value of the area being exactly half that of the whole. We interpret this as a gap between the spatial understanding and the numerical understanding considering the fact that Sajaad recognizes congruence of the two parts which is clear in his response to another student as follows.

> Merajuddin: *Lambayi aur chaurayi mein fark hai, isiliye aadha nahi katega.* [Length and breadth are different, so it won't get cut into two (equal) halves.]

> Sajaad: *Aadha katega lekin ginti mein pura nahi hoga.* [It will get cut into two (equal) halves, but we will not get the total when we count.]

The teacher tried to convince the students that even if we cannot count 15 units in each of the triangular halves, since the two triangular halves of the 6×5 rectangle are equal halves it must be half of 30. After this the teacher moved on to discuss other number fact problems. However, the students did not appear to be fully convinced as evidenced by Raziya’s subsequent intervention. Raziya intervened to bring the focus back on the area of the triangular half. She said that she could make the 15 units and she showed the teacher how this was possible on her graph paper. The teacher then asked her to demonstrate this to the class on a bigger graph paper. Thus two different kinds of warrants emerge in the episode. The argument structure for the teacher was different from the argument structure for the student as indicated in Figures 2 and 3.

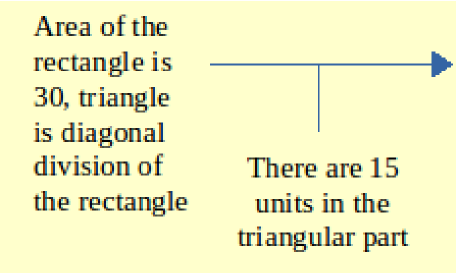

Figure 2: Student's argument structure

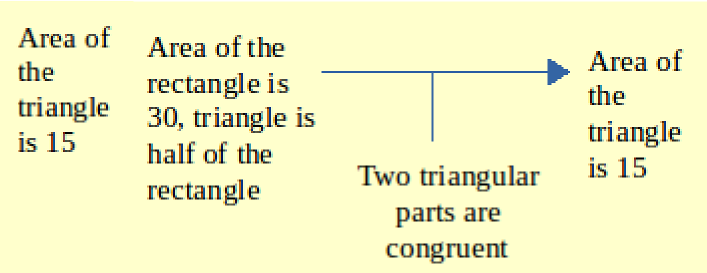

Figure 3: Teacher's argument structure

Later, Raziya came to the board and pointed out that although the teacher had said that we couldn't count the units in the diagonal division of a rectangle, it can actually be counted. She showed her work on a bigger graph paper to explain how diagonally dividing a 6×5 rectangle gives 15 units in one half. As can be seen in Figure 4, Raziya is using the strategy of moving parts to complete the units along the diagonal in one triangular half. Raziya supported Sajaad's argument by providing the same warrant that he and other students were looking for.

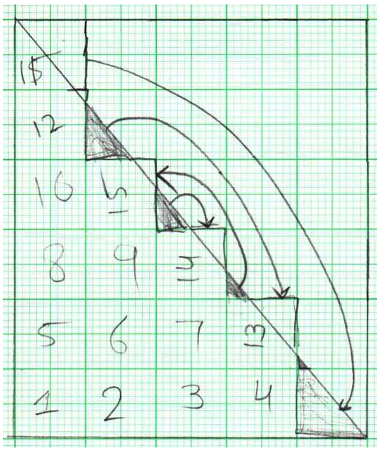

Figure 4: Raziya showing that the triangular part contains 15 units

This episode demonstrates that a few students including Raziya were still engaged with the problem of finding 15 units in the diagonal division of the rectangle. The students were seeking a warrant to support the claim through identifying units in the figure, which was different from the teacher’s warrant for the claim.

Episode-2

In contrast to the previous episode, this one involved a more complex task. Work on the task was split over two days lasting for more than an hour in all and was accompanied by very rich discussion. This episode is spread across the fourth and fifth day of the teaching sequence. In this episode, students were asked to find the size of six given shapes outlined on an inch-graph paper in terms of the inch square unit. The last of the shapes elicited multiple answers from students, some of which are shown in Figure 5.

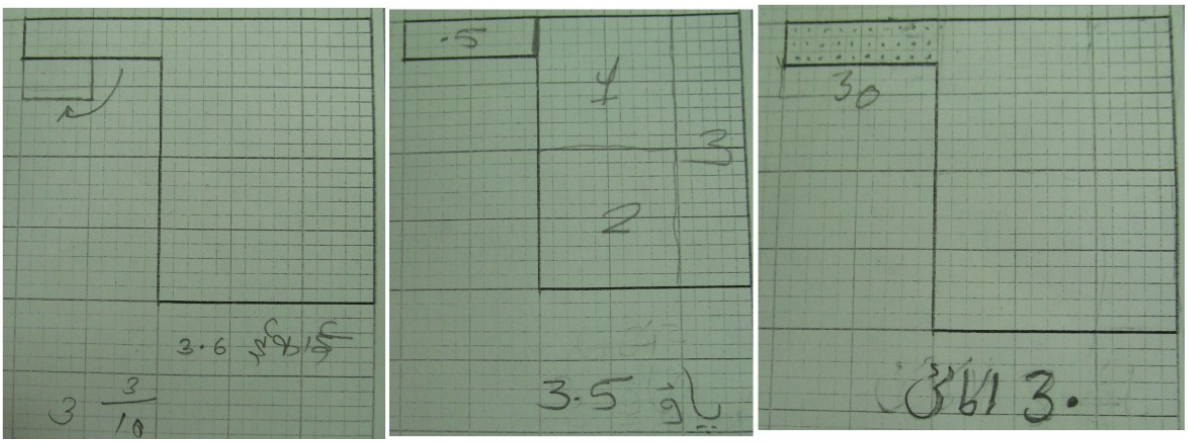

Figure 5: An instance of multiple solutions

Thus, in contrast to the previous episode where the student focussed only on completing the unit to get their solution, here they had to find out the value of a small part of the unit. Most students could identify the three complete units, however many of them struggled with the remaining part that extended to the left of the rectangle. There were several discussions among small groups within the class about to how to represent the remaining part.

By the end of the lesson, the teacher asked all the students to share their answers. There were multiple answers suggested by different students, which included "3 quarter, 3.3, 3.6, 3.60, 3.5 half, 3 3/10, 3.30, 3.1". The teacher wrote all the answers on the board by the end of the fourth day, and announced that they would be discussed the next day. On the fifth day, the teacher asked each student to defend his or her solution in front of the class. This episode has four different parts where different students were using different units as a backing to support their solution.

Part-1

Suhana explained the solution for 3.3 given by a boy the previous day, (who happened to be absent on Day 5), which is reproduced in the excerpt below:

> Suhana: *Teen box hai na chote chote wahi ginke likha usne* [There are these three small boxes, he have counted them]

Thus, in this solution the student has identified the extended part on the left as consisting of three rectangular strips calling them three small boxes.

Part-2

Aliza claimed 3.6 as the value for the given space and justified her claim as below:

> Aliza: *Teacher agar ye, ek box rehta ye wala, isko hum teen asariya paanch mante, aur isme ek box wo jo ek chota wala tha na wo ek zyada hai, isiliye teen asariya panch mante na, toh usme ek aur box aa gya toh teen cheh manenge na usko* [Teacher if there was one box, this one, then we would have considered that three point five, here there is one more box, so it will be considered three six]

The teacher drew the same shape on the blackboard in an enlarged version for everyone to see. Aliza explained her solution to the class by working on the figure made on the blackboard. She erased one-half of the remaining extended part and moving it to the bottom of the other half (similar to the first example in Figure 5). While in the previous case Suhana referred to three strips, in this case Aliza refers to six strips made after moving one half the remaining part. So in the previous case, the three rectangular strips were recognised as .3 in 3.3. However, in Aliza's case she identified six rectangular strips, which corresponded to .6 in 3.6. Aliza further said if there were five such strips, they would have been recognised as .5 in 3.5. But her claim was countered by Raziya as below:

Aliza: *Kyu? teen asariya panch, agar ye ek khana nahi rehta toh teen asariya panch bolte na* [Why? If this one space was not there, then we will call this three point five right]

Raziya: *Nahi bolte* [We won't call that]

Aliza: *Kyu nahi bolte?* [Why not?]

Raziya: *Teen asariya panch yani aadha, wo toh adha nahi hai pau hai* [Three point five means half, that is not half, its quarter]

Thus, Aliza misidentified the quarter part made after moving the smaller parts as point five. But Raziya's comment was aimed at making Aliza notice that the part she is referring to is not half but quarter of the full unit.

Part-3

Raziya and Afia came to defend 3 3/10. Raziya referred to the earlier lessons on how to represent fractions and took the example of *Roti* (Indian bread). Afia drew a circle on the board to represent a *Roti*, and made ten division on it.

Raziya: Dus tukre kiye, aur isme se maine teen hisse kha liye, toh phir kaise likhenge [Made ten divisions, of which I have eaten up three parts, then how will we write that]

Aliza: Teen batte dus [Three by ten, i.e., 3/10]

Using that context as base, Raziya justified her claim to the whole class as below:

Raziya: *Teen batte dus na, toh waise hi ye line mein agar humlog aise aare mein lete hai, toh usme teen line thi, aisi teen line thi, dus line hai aur teen line, toh kya hua, dus batte teen hua na, toh teen box aur dus batte teen* [Three by ten right, so similarly if we take this line horizontally, then there are three lines, ten lines are there, and three lines, then what will be the value, ten by three, so three boxes and ten by three]

Aliza: *Dus batte teen nahi cheh batte teen, cheh batte dus hoyega na* [Not ten by three, six by three, six by ten will be the value]

Raziya: *Cheh batte tab hoga, tumne khali aari line gini hai* [Six by will be when you have only counted horizontal line]

Afia: *Aisi line aisi, niche nahi hai, aise hi* [This kind of line, its not going down]

Raziya: *Cheh batte agar bolenge na toh apne ko aari aur khari dono leni padhegi, isme aari bhi dus hai, khari bhi dus hai, toh agar hum cheh batte lete hai, toh cheh batte bees lenge, aur agar usko katenge toh phir, do daham dus, do tiya cheh, teen batte dus aaya, toh apka answer aayega teen sahi teen batte dus* [If we say six by, then we have to take both horizontal and vertical lines, here there is ten horizontal and ten vertical, so if we are taking six by, then we have to take six by twenty, and then if we cancel them out, two tens are ten, two threes are six, three by ten will come, so your answer will come as 3 3/10]

Thus, Aliza was consistently looking at the remaining part as six divisions, So Raziya tried to fit her reasoning in Aliza's argumentation structure by referring that in the case of taking six divisions also, there will be twenty divisions in all. So in that case also Aliza's solution will come out to be same as three by ten.

Part-4

Sajaad came to defend 3.30 as the answer for the task. He

Sajaad: *teen hissa ye bhi hai, teen hissa ye bhi hai, toh ek line main ek khane main ja rha hai panch, panch line, idher teen hain na idher se doh lenge, isko ek khana banaenge pau kerenge, aur idher ka ek line bach gya, toh ye pau ka pachees hota hai, toh pachees ka ye, aur panch ye ek line ko manenge toh panch aur pachees, tees ho gya na, teen point thirty* [this has also three parts and this has also three parts, so in one line (or part) there is five, five line, here there is three (lines), so we will give two, we will make it one quarter, and here one line remained, so here quarter will be 25, and if we consider one line as 5 then 5 and 25 will be 30, 3.30]

Sajaad mostly uses the context of money to justify his reasoning. First he recognised the remaining part as consisting of six rectangular strips as was done by Aliza. He then moved two of these strips at the bottom of the remaining three, thus made a quarter of the full unit. He recognized the quarter as 25 paise and the remaining one strip as 5 paise. Thus recognised 30 paise for the remaining part and 3.30 as the value for the given shape. His use of the context of money was known by most of his classmates.

Thus, in this episode, even with the same primary data, students came up with different claims by following different solution (or argumentation) structure.

Discussion

Some broad insights that can be drawn from these episode are:

- Disconnect between the geometrical and numerical understanding

In the first episode, students argumentation was based on unit-structuring as they were more comfortable relying on counting the full units rather getting convinced by the congruent halves shown by the teacher. Even though the two triangular halves were spatially or geometrically equivalent, students' assurance came from the numerical value of 15 units. This indicates a disconnect between students' spatial and numerical understanding of area-measurement. This also supports Sarama & Clements' (2009) claim that the problems in the learning of area-measurement could be due to difficulty in connecting the spatial and numerical aspects. The basis of students' reasoning was more aligned to the additive thinking of counting units rather than the multiplicative thinking of looking at half units.

Battista (2007) have emphasized that students must be able to extend their reasoning to different forms of units. The task used in the second episode is based on fractional unit. In the second episode it was found that different students can identify and consider different fractional part as their fractional unit, but while representing that fractional unit they have to take care of the total number of that fractional unit in the full unit.

- Different argumentation structure of the actors in the episode

In the first episode, there was a difference between the argumentation structure of the student and the teacher. In the first episode, despite realizing that the two triangular parts found by diagonally dividing a rectangle gives congruent halves, there was a resistance in accepting that the each triangular half has half the number of units as in the rectangle. From the argumentation point of view, the basis of warrant for the student and the teacher were different. Students' warrant came from the unit structure, specifically with the number of full units that can be made in one triangular part. However, the teacher gave a different warrant which is that the two triangular halves are congruent, which did not convince the students since they were looking for a different warrant.

In the second episode, we saw that different students were working with different fractional units. The backing for their argumentation structure was based on which fractional unit they were using as a reference. From the argumentation point of view, there was a difference between the argumentation structure of the two student. In part-3 of the second episode, Raziya could convince Aliza, only by using Aliza's unit to prove that her claim holds true even in that case.

References

Battista, M. T. (2007). The Development of Geometric and Spatial Thinking. In Frank K. Lester, Jr. (Ed.). Second Handbook of Research on Mathematics Teaching and Learning, 843-908.

Carrascal, B. (2015). Proofs, Mathematical Practice and Argumentation. *Argumentation*, *29*(3), 305-324.

Forman, E. A., Larreamendy-Joerns, J., Stein, M. K., & Brown, C. A. (1998). "You're going to want to find out which and prove it": Collective argumentation in a mathematics classroom. Learning and instruction, 8(6), 527-548.

Holmqvist, M., Gustavsson, L., & Wernberg, A. (2008). Variation theory: An organizing principle to guide design research in education. *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*, 111-130.

Kelly, A. E. and Lesh, R. (2000) Teaching experiments. In A. E. Kelly and R. A. Lesh (Eds.), *Research Design in Mathematics and Science Education* (pp. 192-195). NJ: Lawrence Erlbaum Associates.

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom: Two episodes and related theoretical abductions. *The Journal of Mathematical Behavior*, *26*(1), 60-82.

Reid, D. A., & Knipping, C. (2010). Proof in mathematics education. *Research, learning and teaching. Rotterdam: Sense Publisher*.

Sarama & Clements, (2009). Early Childhood Mathematics Education Research: Learning Trajectories for Young Children. (pp. 271-315). Routledge.

Shavelson, R., Phillips, D. C., Towne, L., & Feuer, M. (2003) On the science of education design studies. *Educational Researcher*, *32*(1), pp. 25-28.

Toulmin, S. E. (2003). The uses of argument. Cambridge University Press.

Toulmin, S. E., Rieke, R. D., & Janik, A. (1979). An introduction to reasoning. New York: Macmillan.

Voigt, J. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational studies in mathematics*, *26*(2-3), 275-298.