MATHEMATICAL EXPLORATIONS ENCOURAGING MATHEMATICAL PROCESSES IN A CLASSROOM

Harita Raval, Aaloka Kanhere and Jayasree Subramanian
Homi Bhabha Centre for Science Education, TIFR, Mumbai
aaloka@hbcse.tifr.res.in, harita@hbcse.tifr.res.in, jayasree@hbcse.tifr.res.in

In this paper we examine how open mathematical explorations encourage mathematical processes in a classroom. For this we look at two classrooms that were a part of a 9-day talent nurture camp. whose purpose was to give students a flavour of doing science and mathematics. We choose one activity that was implemented in the camp and examine how it fits into the notion of an open exploration. We then look at the implementation of this activity in two different classrooms by two different teachers and examine how far these implementations encouraged mathematical processes. We choose to focus on the processes of visualisation, making conjectures and proving. The preliminary analysis of the sessions establishes that such open explorations have a huge potential in encouraging mathematical processes in the classroom.

Keywords: *Mathematics Education, Pattern, Mathematical processes*

INTRODUCTION

Mathematical processes play a very important part in understanding and doing mathematics. The National Focus Group Position paper on Teaching of Mathematics strongly recommends giving precedence to mathematical processes over content, “Giving importance to these processes constitutes the difference between doing mathematics and swallowing mathematics” (NCF 2006, Teaching of Mathematics). The document identifies processes like formal problem solving, use of heuristics, estimation, approximation, optimization, use of patterns and visualization, representation, reasoning and proof, making connections, mathematical communication. (NCF 2006, ‘Teaching of mathematics’, p iv). Emphasis on mathematical processes helps in reducing the fear of mathematics in children’s minds and in strengthening students’ capacity to ‘do’ mathematics. By mathematical processes, we mean stages that mathematicians go through while doing mathematics. Mathematics education literature abounds in characterisation of these processes. One of the first attempts at studying the nature of mathematical processes and how it is related to content can be seen in Bell (1976), where he identifies symbolization, modelling, generalization, abstraction, and proving as the basic processes of mathematics. Mason, Burton & Stacey (2010) identify conjecturing and convincing, imagining and expressing, specializing and generalizing, extending and restricting, classifying and characterizing, as the core mathematical processes. For the purpose of this paper we choose to focus on three of these processes, namely visualisation, making conjectures and proving.

In order to provide students with opportunities to engage in these processes, teachers need to provide mathematically rich tasks/activities and classroom environment so that students are able to engage actively in mathematical discussion and discourse.

In this paper, we look at one such activity which was conducted in two different classrooms.

We examine the ‘openness of the task’ in the light of Yeo’s framework to characterise the openness of tasks (Yeo, 2015) and move on to analyze the classroom videos and elicit instances where children’s engagement in mathematical processes was apparent.

THE OBJECTIVE OF THE CAMP

The classrooms were a part of a larger talent nurture programme called Vigyan Pratibha of the Homi Bhabha Centre for Science Education (HBCSE), which is aimed at supporting high quality and well-rounded science and mathematics education. These classrooms aimed at exploring students’ thinking when exposed to an open exploration through patterns.

METHODOLOGY

The data was collected from two classrooms where the same mathematical exploration was being conducted. These classrooms were a part of a summer school held for students from 7 different English medium schools around HBCSE. All the students were Class 10 students (entering). The admission to the summer school was completely voluntary and there was no selection process. The activities were conducted by two different teachers, who both are authors of this paper. One class had 22 students (B – 12 and G – 10) and the other class had 25 students (B – 14 and G – 11). Data sources include classroom observations and classroom videos.

The objective of the activity was to encourage different mathematical processes in the classroom. In the present activity, students explored patterns of squares of natural numbers.

ABOUT THE ACTIVITY

The activity comprised of two different but connected tasks. In the first task, the students were given the table shown in Figure 1 and were asked to observe patterns in the table.

Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Square	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400

Figure 1

In the second task, the natural numbers up to 400 were arranged in a 8-column table as shown in Figure 2 and the first few square numbers highlighted. They were expected to shade in the remaining squares and look for patterns.

I	II	III	IV	V	VI	VII	VIII	I	II	III	IV	V	VI	VII	VIII
1	2	3	4	5	6	7	8	209	210	211	212	213	214	215	216
9	10	11	12	13	14	15	16	217	218	219	220	221	222	223	224
17	18	19	20	21	22	23	24	225	226	227	228	229	230	231	232
25	26	27	28	29	30	31	32	233	234	235	236	237	238	239	240
33	34	35	36	37	38	39	40	241	242	243	244	245	246	247	248
41	42	43	44	45	46	47	48	249	250	251	252	253	254	255	256
49	50	51	52	53	54	55	56	257	258	259	260	261	262	263	264
57	58	59	60	61	62	63	64	265	266	267	268	269	270	271	272
65	66	67	68	69	70	71	72	273	274	275	276	277	278	279	280
73	74	75	76	77	78	79	80	281	282	283	284	285	286	287	288

Figure 2: Snapshot of the entire table

It was expected that shading in the squares would make it obvious that the square numbers occur only in the first third columns, hinting that the only possible remainder when a square number is divided by 8 is 0, 1 or 4, leading to modular forms of $8n$, $8n + 1$ or $8n + 4$. None of this was explicitly mentioned, and the students were invited to 'look for patterns' expecting to follow along whatever patterns the students came up with, creating opportunities for students to engage in mathematical processes.

Yeo (2015) includes 5 elements in his framework to characterise openness of a task, answer, method, complexity, goal and extension. These tasks are open on the parameters of answer and method, as there are multiple answers and multiple approaches possible. For these tasks, while it is possible to anticipate some of the methods and patterns that students would come up with, it is definitely not possible to come up with an exhaustive list. The task specifies a goal – namely 'find patterns' but at the same time does not specify any particular pattern and is thus open on goals. The tasks are extendable, in that one could go on to modular arithmetic, visualisation of square numbers as the sum of consecutive numbers and so on. Thus given tasks clearly fall under the category of what Yeo calls as open investigative tasks.

The openness of the task provides affordances for multiple answers and discussions around them, thus providing ample opportunity for mathematical communication. The act of looking for patterns privileges coming up with conjectures and the tables and the arrangement in columns provide visual cues to pattern findings. The natural steps after guessing a pattern is verifying it and then proving it. Depending on the 'proof schemes' (discussed later in the paper) (Balacheff, 1988), students have, they may or may not differentiate between these two processes. Thus the task privileges mathematical communication, visualisation, making conjectures and proving among other processes. The tasks also demand very little in the nature of prerequisite knowledge and hence is accessible to all students. Based on these considerations, these tasks were chosen for implementation. We highlight below instances where these process came to the fore.

ABOUT THE CLASSROOMS AND THE FINDINGS

Before presenting the instances of students' thinking and examples of mathematical processes the students engaged in, we would like to describe the classroom practices which supported students' thinking in the classroom which in turn encouraged mathematical processes.

Both the teaching sessions began by asking students to find out patterns from Figure 1 and then share it with the class. Students were given a choice of working individually or working in groups but working in groups was encouraged. They were encouraged to articulate the patterns that they found out verbally or visually and share their findings with the rest of the class. The other students were encouraged to ask counter questions and justifications. Whenever needed the teacher would also help the students in articulating the patterns they found.

At times, the teachers suggested that students use different representations which would make the patterns clearer instead of doing it themselves.

Once they listed out the patterns on the board, it was discussed whether a pattern was true or not. A separate blackboard was used to record students' patterns. There were discussions initiated by the teachers on how to figure out whether a pattern works for all the numbers or what does a statement being true mean, which was essentially driven towards generalization. We noticed a classroom culture where students would refer to each others' pattern by citing their names, pose questions when in doubt, or comment on each others' strategy to prove it.

We now move on to examine the specific processes seen. This is a preliminary data analysis of the classrooms, and the instances that have been reported in this paper are the parts of two 3 hour classes. This analysis is a part of a larger study where we plan to study how open explorations conducted in the classroom encourage mathematical processes.

VISUALIZATION

We believe that visualization plays a central role in helping to find an effective solution for such pattern problems. Kerbs (2003) found that by using a visual approach one can generalize the patterns and Rivera (2007) confirmed that generalizations were based on visualization. And in the instances mentioned below a student is able to figure out a pattern visually. In the task, there were many instances where students have figured out patterns just looking at the number-table.

Instance 1
In class, students were asked to find out patterns from Figure 1.

S1 : Ma'am the sum of the first number and the second number when added with the square of the first number it will give you the square of the second number.

T1 : You heard what he said? [looking at the whole class]

S2 : No, we couldn't hear.

T1 : No, Ok. [looking at S1] You want to come on the board? Maybe drawing is easier for this. What you said no… If you draw that thing it might be a bit easier. [S1 walks towards the board]. So, just look at the tables what he is saying [To the class].

1 (1st number)	**2** (2nd number)
1 (Square of the 1st number)	**4**

Figure 3

S1 : [Writes on the board (See Figure 3)]

T1 : So you have a table right? What he saying is, you look at this [marking what S1 has said]. Right? Now, what he is saying is that you add these three numbers, you will get this fourth number. And he is saying it is always true, [To the class]. You are saying it will hold even if you extend the table, right? [Looking at S1]

S1 : Yes.

T1 : See we all together have to prove it. We can't just write statements like that no? [Talking to the class]

Comments: The student further goes to prove what he has written by saying that, $(x+1)^2$ is nothing but the addition of $(x + x^2 + (x+1))$. This relationship was new to the teacher too.

We see that the students had made mental figures to see the way patterns were emerging. . In other instances, students had just looked at the numbers given in the table and made their own patterns which were geometric.

MAKING CONJECTURES

Polya (1954) talks of the importance of conjectures and 'plausible reasoning' used to support them in the process of creating new knowledge in mathematics. Looking back and perceiving the steps that might have gone into coming up with the Goldbach conjecture, Polya identifies noticing some similarity, a step of generalisation and formulation of a conjecture. As the first step we recognise that 3, 7, 13, 17 are primes, 10, 20, 30 are even numbers and that the equations 3 + 7 = 10, 3 + 17 = 20 and 13 + 17 = 30 - are analogous to each other. We then pass to other odd numbers and even numbers and then to the possible general relation

"even number = prime + prime".

The conjecture is a statement suggested by certain particular instances in which we find it to be true. Now we move to examining if it is true of other particular or atypical cases. For example, the number 60 is even, can it be expressed as a sum of two primes? By a process of trial and error we come to 7 + 53. This makes our conjecture more 'credible'. Our conjecture gains credibility with the number of instances for which it is verified to be true, but it is not established beyond doubt, there is still the possibility of finding an even number that cannot be expressed a sum of two primes. Hence Goldbach Conjecture remains a conjecture almost 300 years after it was formulated.

It is important that students be given an opportunity to go through the process of discovery outlined above – of coming up with a guess, verifying that it is true and trying to prove it. In the process of discovery, the stage of coming up with plausible conjecture is of prime importance. “Anything new that we learn about the world involves plausible reasoning, which is the only kind of reasoning for which we care in everyday affairs” (Polya 1954).

The tasks outlined here provide ample opportunities to engage in this kind of reasoning as can be seen from the following instances.

Instance 2
The class was asked to find patterns in Figure 2. The students were finding patterns and discussing it with their partners or groups and then sharing them with the teacher and the class.

T1: Let’s start with more patterns. Did you see any patterns? Yes, S12. Can you show there? [pointing on a board]
S12: It’s very complicated.
….
S12: If n [leaves incomplete]
T1: If n is a natural number.
S12: n raised to 4 [teacher wrote it on the board n^4], brackets [teacher made the bracket] n plus one raised to 4 [teacher wrote $(n + 1)^4$ on board] is always divisible by 5 [teacher repeated].
[on board $n^4 + (n + 1)^4$ ’! is always divisible by 5]

Some more examples

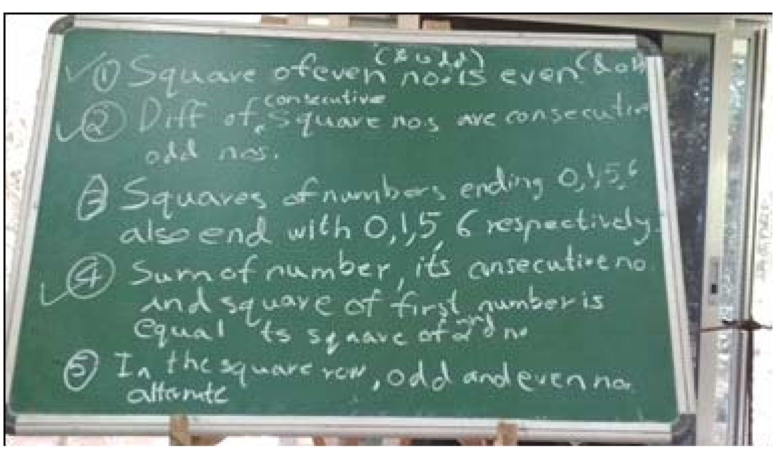

Figure 4: Some examples of students’ conjectures

There were conjectures, similar to the ones given above which were a surprise for the teachers themselves. And the teachers also had to figure out strategies to deal with these conjectures then and there. The kind of classroom environment encouraged by the teacher, gave students the confidence to make conjectures, refute them, update them and prove them and a number of conjectures came up.

We believe that, such open mathematical tasks/activities give students a taste of how mathematics is done, as they go through the process of coming up with ideas that do not work, examining and rejecting, modifying their own statements and seeing mathematics in the making. This is very different from what they do in their school mathematics. In these activities, the students were in charge and actively driving the discussion instead of passively learning definitions and theorems in the textbook. Here they come up with their own conjectures, choose the patterns they would like to investigate and the ways to prove them. In a way, this gives them the ownership of whatever that they are doing which might help in removing the fear of mathematics and the feeling of insecurity in doing mathematics.

PROVING AND PROOF SCHEMES

Students difficulties with proofs are well documented in mathematics education literature. One of the most common difficulties that students have with the concept of proofs is that they believe that a non-deductive argument, like say verifying for a few cases constitutes a proof (Weber, 2003). Balacheff (1988) differentiates between pragmatic and conceptual proofs and discusses four main types of proofs in the cognitive development of the concept of proofs. ‘Naive empiricism’ which involves asserting the truth of a result after verifying several cases is the most rudimentary but obviously inadequate proof scheme identified by Balacheff. One important aspect of understanding the concept of proof is to move from ‘it is true because it works’ scheme of the naive empiricism to establishing the truth by giving reasons. This is not an easy shift to make. However, the instances described below indicate how this happened as a matter of course in the context of these open tasks.

Instance 3

The class was asked to find out patterns from the given Figure 1.

S5 : The numbers between the square numbers are increasing by 2.

T2 : [repeated the statement] What does that mean?

S5 : Between 1 and 4, it is 2 and 3. Between 4 and 9 it is 5,6,7,8

T2 : How will I know what you are saying is correct? I take any big square number how will I know how many numbers are going be there in between?

S5 : I know!! You take the root of the first square number and then multiply it by 2 you will get to know how many numbers are there.

T2 : What you are saying now is more than what you said earlier. First, you said the numbers in between are increasing by 2, but now you said to know the number you take the square root of the smallest number and multiply it by 2 to get the numbers in between. [Discussion with the class]

T2 : I want the class to pay attention here, S6 is saying S5's pattern is proved [To the class]. Why? Can you explain to the class? It's ok go ahead explain it [talking to S6]

S6 : [Stand up at his place] His first pattern that two numbers has been added in between [looks for the exact word in the book] his pattern is been proved in Table 1.2. If we see numbers between 1 and 4, two numbers are there. Between 4 and 9, four numbers are there. Between 9 and 16, six numbers are there and so on if we see all the numbers between the two squares from 1 to 20. So, we can see that the numbers in between are 2, 4, 6, 8 and so on. [Teacher repeated by showing it on the table what S6 said]

S7 : [Immediately] Ma'am, this is not proving, this is just verifying.

Instance 4

A student has come with the pattern that if you multiply two consecutive natural numbers and then add the larger consecutive number to that product you will get the square of the larger number.

T1 : Do you think this is correct?

Class (coherently): Yes.

T1 : But, always will be correct?

Class (again coherently): Yes.

T1 : So for example, if I have 1027, 1028 and square of 1027, if I whatever multiply and I will get the square of 1028? Are you actually saying that? [and wrote on the board]

S8 : Yes ma'am.

T1 : What do you think S9?

S9 : It could.

T1 : So it might not be?

S9 : [Nods the head].

T1 : So what does one do when this happens? As S9 is saying it might work or might not? What does one do in such a situation?

S10 : Make it a theorem.

T1 : Make it a theorem. So, how do you make something a theorem? S11 how do you make something a theorem?

S11 : By proving it.

T1 : Yes, right. So you got a lot of theorems here. [pointing at the patterns students have come up with] Actually some of the theorems I have never thought about it. So, let's start proving these theorems.

Comments: In the above classroom dialogues, it is evident that students are capable of making conjectures by observing patterns and differentiating between proving a result and verifying it, which is very crucial in understanding mathematics as a discipline. In the instance mentioned above the need to prove the students' patterns came from students themselves. Both the classes went on to prove some of the conjectures they came up with as well.

We believe that the open nature of the task provided opportunities for classroom discussions as exemplified above, underscoring that verification and proof are not one and the same. Further analysis is needed to identify the features of the task or the classroom practices that enabled the move from naive empiricism to generalisation.

CONCLUSIONS

Open explorations like the one which was conducted in the reported classrooms offer opportunities for making conjectures and encourage a multiplicity of ways of thinking, ideas, approaches, and answers as compared to goal-directed problem-solving. Such explorations encourage a classroom environment which is open to discussion among students and also gives students space to makes mistakes which are an intrinsic part of the classroom process; Such open tasks shift the focus from finding the right answer or verifying and proving a given conjecture to coming up with conjectures, refuting and updating them and in general engaging in the process of making mathematics. This encouraged participation of the majority of students in both the classrooms. The openness of the activity made it possible for every student in the class to create their own mathematics. The potential of open tasks over goal-driven problem solving in encouraging mathematical processes and identification of the characteristics of tasks that aid this needs further study.

ACKNOWLEDGEMENT

We thank Adarsh Shetty and Mayuri Tawade for recording the classes. We are also thankful to the Vigyan Pratibha team at HBCSE for their help.

We acknowledge the support of the Govt. Of India, Department of Atomic Energy, under Project No. 12-R&D-TFR-6.04-0600 and the Vigyan Pratibha Project (No. R&D-TFR-0650).

REFERENCES

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London, UK: Hodder and Stoughton.

Bell, A. (1976). *The learning of general mathematical strategies: a developmental study of process attainments in mathematics, including the construction and investigation of a process-oriented curriculum for the first secondary year.* University of Nottingham.

Krebs, A.S. (2003) Middle grade students' algebraic understanding in a reform curriculum. *School Science and Mathematics, v.103, n.5,* p.233-243.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically.* Retrieved from http://books.google.com/books?id=OFEZAQAAIAAJ

National Council for education, research and training, 2006. Position paper on teaching of mathematics by the national focus group, National curriculum framework – 2005. (Retrieved from http://www.ncert.nic.in/html/focus group.htm)

Polya, G. (1954). *Mathematics and Plausible Reasoning,* Princeton University Press.

Ramanujam, R. (2010). Live mathematics and classroom processes. In R. Bhatia (Ed.), *Proceedings of the International Congress of Mathematicians, (Vol 1),* Hyderabad, 662 -679

Rivera, F. (2007). Visualizing as a mathematical way of knowing: Understanding figural generalization. *Mathematics Teacher, 101*(1), 69–75.

Weber, K. (2003). Students Difficulties with Proofs, *MAA Research Sampler 8,* Retrived from https://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof

Yeo, J. W. (2015). Developing a framework to characterise the openness of mathematical tasks, *International Journal of Science and Mathematics Education, 15 (1),* pp 175 - 191