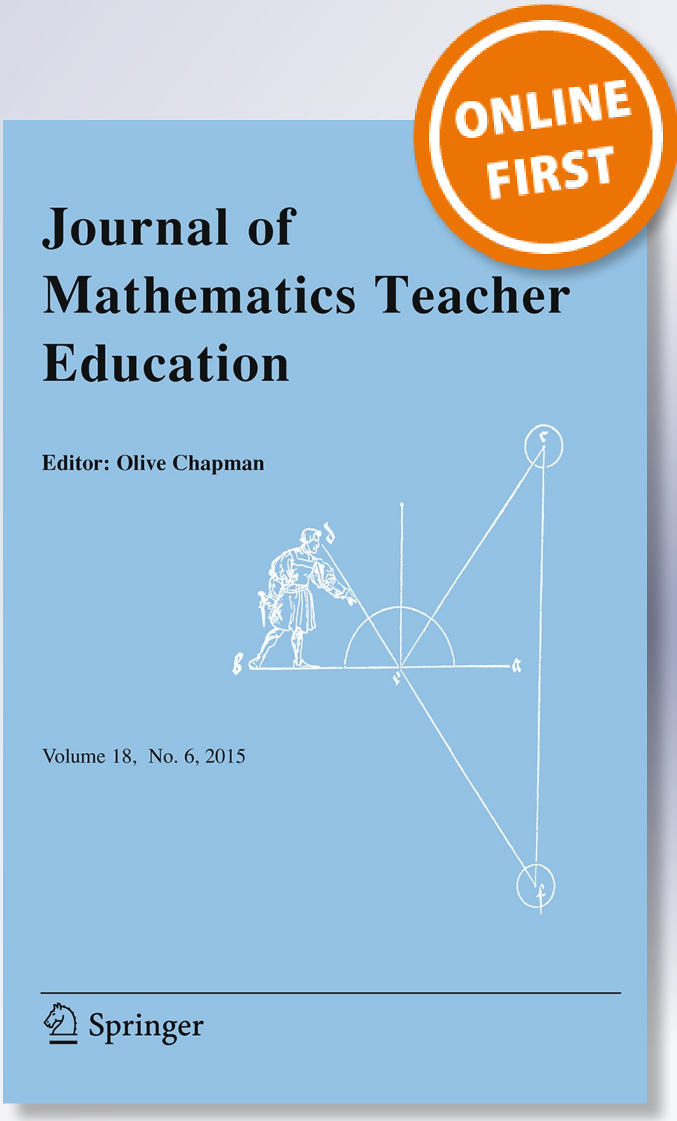
Teachers' construction of meanings of signed quantities and integer operation

Ruchi S. Kumar, K. Subramaniam & Shweta Shripad Naik

Journal of Mathematics Teacher Education

ISSN 1386-4416

J Math Teacher Educ
DOI 10.1007/s10857-015-9340-9



Springer

J Math Teacher Educ
DOI 10.1007/s10857-015-9340-9

Teachers' construction of meanings of signed quantities and integer operation

Ruchi S. Kumar[1] · K. Subramaniam[1] · Shweta Shripad Naik[1,2]

Abstract Understanding signed quantities and its arithmetic is one of the challenging topics of middle school mathematics. The *specialized content knowledge* (SCK) for teaching integers includes understanding of a variety of representations that may be used while teaching. In this study, we argue that meanings of integers and integer operations form the foundation for the construction of SCK about representations used to teach integers. We report that teachers' concerns about teaching the topic of integers implicate issues of meaning, although this may not always be explicitly acknowledged by teachers. We develop a framework of integer meanings synthesizing previous research, and describe how the framework allowed teachers to investigate a wide range of representations including contexts and thereby construct SCK in a professional development setting. Teachers constructed SCK by connecting various meanings of integers with one another and with representations including contexts. Teachers made two important shifts, from exclusively using the state meaning of integers to including the application of change meaning to representations and from exclusive use of formal models to including contexts to teach integer addition and subtraction. An implication of the study is that frameworks of meaning for key mathematical topics could be an important component of pre- and in-service teacher education.

Keywords Integer meanings · Representations · Specialized content knowledge · Teaching integers · Teacher professional development

✉ Ruchi S. Kumar
ruchi.kumar31@gmail.com; ruchi@hbcse.tifr.res.in

K. Subramaniam
subra@hbcse.tifr.res.in

Shweta Shripad Naik
shwetan@umich.edu

1 Homi Bhabha Centre for Science Education, TIFR, Mumbai, India

2 University of Michigan, Ann Arbor, MI, USA

Published online: 16 December 2015

Introduction

There is widespread recognition that specialized forms of mathematical content knowledge are important for effective mathematics teaching (Hill et al. 2008; Ernest 1999). A commonly used framework for understanding the specialized knowledge of mathematics needed for teaching was proposed by Ball et al. (2008). Ball et al.'s framework is a modification of Shulman's original (1986, 1987) framework of teacher knowledge to include a richer description of content knowledge that mathematics teachers need. Besides forms of pedagogical content knowledge (PCK) as described by Shulman, Ball et al.'s framework specifies additional components of mathematical content knowledge that teachers need to have. Among these, specialized content knowledge (SCK), understood as mathematical knowledge that is specialized for the work of teaching, has important implications for designing pre- and in-service teacher development programs.

SCK is described as "mathematical knowledge beyond that expected of any well-educated adult but not yet requiring knowledge of students or knowledge of teaching" (Ball et al. 2008, p. 402). The first part of the description contrasts SCK with "common content knowledge" or CCK, which mathematically educated adults are expected to possess. The second part of the description contrasts SCK with "pedagogical content knowledge" (PCK) as envisaged by Shulman. However, Ball et al. (2008) as well as others have pointed out that the differences between SCK and other components may be subtle and the boundaries between them fuzzy (Hill 2010; Carrillo et al. 2013).

Other criteria used to identify SCK are functional—as knowledge that *enables* teachers to carry out certain tasks involved in teaching. For Ball et al. (2008), SCK undergirds a set of tasks that are an essential part of the work of teaching, tasks such as finding an example to make a specific mathematical point, recognizing what is involved in using a particular representation, linking representations to underlying ideas and to other representations, and selecting representations for particular purposes. Knowledge that supports these tasks includes such elements as knowing why algorithms work, having a repertoire of representations of a mathematical concept, and knowing the affordances and limits of particular representations (Ball et al. 2008).

The formulation of SCK strongly suggests its topic-specific nature. While there is general acceptance of the importance of specialized mathematical knowledge for teaching, there is a need to understand in detail and topic-wise, what constitutes such knowledge in order to design professional development programs that can help in building and strengthening it. This is important in a context where SCK components are largely absent in most professional development programs (Darling-Hammond and Richardson 2009; Kumar et al. 2013). This study is focused on the specialized content knowledge related to the topic of integers. Mitchell et al. (2014) analyze examples of teacher actions in the classroom in which such knowledge is implicated. Arguing that teaching the topic of integers involves the use of multiple representations, they develop a typology of tasks that teachers carry out in the course of teaching integers using multiple representations, which include recognizing and abiding by the representation's conventions, unpacking procedures through careful use of representations, connecting different representations, and flexibly moving between representations to support student understanding. Their analysis points to the central role of understanding and using multiple representations while teaching a difficult topic like integers, and hence shows that understanding representations is a core part of the SCK needed to teach integers. There is a need, however, to move from teaching tasks to a structured account of the underlying knowledge that can form an input for designing teacher development. The bulk of the literature on topic-specific SCK or MKT is situated in the context of developing tools to measure teachers' knowledge (Hill et al. 2005; Herbst and Kosko 2014). Consequently, the focus has been on identifying elements of such knowledge, rather than on developing topic-specific frameworks. Researchers who have developed frameworks have aimed at a topic-general listing of types of SCK (e.g., see Carreño et al. 2013). Further, recognizing that teachers' knowledge is a dynamic construct (Cochran et al. 1993), a picture of how teachers construct such knowledge and the organization of ideas and concepts that facilitate such construction is necessary. In this paper, we elaborate on the key concepts related to the meaning of integers that help organize knowledge about a range of representations for teaching integers, and aim to understand how teachers construct knowledge about representations using these key concepts.

In our work, we draw on data from teachers collaboratively planning to teach a topic and engaging with tasks related to SCK designed specifically for teacher professional development, to describe key components of SCK that are critical to teach integers effectively. We claim that the construct of meaning is the key to organizing knowledge about multiple representations and to developing facility with them for purposes of teaching. Drawing on previous research on the teaching and learning of integers, we identify three interconnected layers of meaning relevant to teaching integers—of the minus sign, of integers, and of integer operations. These different layers of meaning form a framework to understand a variety of representations of integers, which allows one to connect different kinds of representations with one another, and to provide meaningful explanations and justifications of the structure and procedures based on representations.

We present our analysis in the sections below. In the next section, we sketch the framework of meanings of integers drawing on previous research. Following this, we discuss how this framework may be applied to a variety of representations involving integers to explore underlying meanings. In subsequent sections of the paper, we apply this framework to address two research questions: (1) What were the teachers' concerns about the teaching of integers and how are they related to issues of meaning of integers and (2) how did teachers construct SCK for teaching integers using the framework of integer meanings through the exploration of contexts. The third section below describes the study design and the data used to answer the research questions. In the fourth section, using the data from a teacher professional development setting, we show how teachers' concerns about teaching procedures for operations with integers, when probed, run into questions of meaning. In the fifth section, we discuss teachers' engagement with integer meanings to make sense of a variety of representations, to judge whether they are appropriate to use in the classroom, and to design their own representations. In the sixth section, we offer evidence of the teachers' take-up of the framework of meanings for use in their own teaching. In the final section, we elaborate on how the framework of integer meanings explored through various representations provides a basis for the construction of SCK, which has important implications for pre- and in-service teacher education.

Meanings of integers and integer addition and subtraction

Most mathematics teachers recognize that the topic of integers is difficult for learners. The difficulties faced by learners bear similarities with the difficulties that mathematicians in the past have had with the idea of negative numbers (Hefendehl-Hebeker 1991). The teaching and learning of integers has been a topic of research for several decades, and researchers have developed frameworks to understand student difficulties and guide teaching approaches. Some frameworks emphasize the symbolic aspect, namely the negative (and the positive) sign, while others emphasize the "meaning" of signed numbers or integers. Drawing from the literature, we briefly outline below the various meanings of the minus sign, of integers, and of integer addition and subtraction. These three layers of meaning are interconnected and provide a basis for working with representations while teaching integers. Vlassis (2004, 2008), adopting a Vygotskian perspective, emphasizes the symbolic aspect and focuses on the multiple uses of the minus sign. She lists three denotations of the minus sign: the unary, the binary, and the symmetric functions. In the unary function, the minus sign is attached to a number to form a negative number, as in "-6." Second, the minus sign is used to signify the binary operation of subtraction in arithmetic or algebra as, for example, in "5 – 3." The first two uses identified by Vlassis correspond to the distinction emphasized in other studies between the use of the minus sign to indicate a signed or directed number as opposed to the use of the minus sign to indicate the subtraction operation (Glaeser 1981). The third use of minus sign refers to the unary operation or function of taking the additive inverse of a number as, for example, in $-(-6)$. This unary function is a symmetric function. This sense is less frequently emphasized in other studies. We note that "taking the inverse" is especially important in the context of letter numbers and algebra. When an expression such as "$-x + 3$" is to be evaluated for $x = -3$, the symmetric function interpretation of the "$-$" symbol comes to the fore.

As distinct from the meaning of the minus sign, frameworks developed by other researchers emphasize the meaning of signed numbers or integers. Researchers commonly distinguish between the interpretation of a signed number as a property or characteristic of an object and as a transformation or change (Thompson and Dreyfus 1988). Vergnaud (1982) uses a three-way distinction in the meaning of a signed number, as *state*, *transformation,* and *static relationship*. As state, an integer may, for example, refer to the ambient temperature. The change or transformation in temperature from hour to hour may also be represented by an integer. We may also use integers to represent the temperature of one place relative to another to indicate how much hotter or colder it is—a static relation. A further layer of meaning is of the operations of integer addition and subtraction, which may represent contexts of *combine*, *change,* or *compare* (Fuson 1992). We may combine positive and negative scores on a test to obtain a net score. We may use the subtraction operation to find the change in temperature or to compare two temperatures. We also note that a change or relation may be represented both by an operation and by an integer. For example, if the temperature fell from 30 to 16 °C in a few hours, then the change may be represented by the operation "$16 - 30$" or by the result of the operation, "-14 °C." Thus, while state may be represented by integers, change and relation may be represented by both integers and the addition–subtraction operations. The three layers of meaning—of the minus sign, of signed numbers, and of the addition and subtraction operations—are, as we shall see, interconnected and important in grasping the mathematical ideas underlying a variety of representations used to teach integers.

Integer representations and underlying meanings

The variety of representations used to teach integers are of broadly three kinds: symbols, models, and contexts, along a spectrum from abstract to concrete. While symbolic representations involve the use of numerals and the "+" and "$-$" signs, contexts refer to situations that may be real or realistic (sufficiently real to the students), which involve the use of signed numbers and operations. These may, for example, be about profit and loss, assets and debts, height above and below sea level, or people entering and leaving a bus (for a review, see Schwarz et al. 1994). In contrast to realistic contexts, models are more formal in nature, and are thought of as "a way to support students organizing their thinking that can be modeled/inscribed in the form of physical tools and symbols" (Stephan and Akyuz 2012, p. 431). Based on their review of several studies, Stephan and Akyuz categorize the models used to represent integers as neutralization or as number line models. In the neutralization model, there are positive and negative quantities and cancellation is a salient operation. Contexts to which the neutralization model applies are positive and negative electric charges, or assets and debts. The number line model makes the order aspect more salient. Contexts such as height above and below sea level, floors in a building, are examples where the number line model applies. One of the characteristics of the number line model is that it does not readily make sense to add two states that is two points on the number line (such as two floor numbers in a building), while subtraction can be readily interpreted (as, e.g., the directed distance between two floors).

A deeper examination of a context may reveal the relevance of multiple models. For example, debts and assets seem to be best described by the neutralization model since a debt and an asset of equal value cancel one another. However, combining assets and debts makes sense only in relation to the notion of "net worth," which is the sum of the assets and debts taken with their proper sign. "Net worth" is a state variable and fits more closely with a number line model, where each distinct state represents a point on the number line. It does not make sense to add two points on the number line (two net worths) unless one changes the context to one where two entities with different net worths are merged. Analogously, in the context of electric charges, while equal positive and negative charges cancel one another (neutralization model), combining charges only makes sense in relation to a notion of "net charge," which is closer to a number line model.

The various contexts that have been explored by researchers or have appeared in instructional materials have varying instructional possibilities and potential (Schwarz et al. 1994). It needs to be emphasized however that a context typically packs in more mathematical possibilities than may appear at first sight. In other words, it may allow multiple interpretations of signed quantities and the application of both, the number line or the neutralization model. We have mentioned the example of temperatures at different times of the day, where temperature as well as change in temperature can be represented using integers. Here, the salient quantity, namely ambient temperature, is always positive under tropical conditions. But the context allows one to speak of a "derived" quantity, namely change in temperature, which may be positive or negative. Allowing for the possibility of defining such derived quantities makes the contexts richer in mathematical meaning, and integers play a role in expressing and computing with such derived quantities.

The framework outlined above consisting of the meaning of the minus sign, integers, and integer operations, together with the broad types of representations—symbols, the formal models of number line and neutralization, and contexts is summarized in Table 1. We claim that the framework points to critical constituents of the specialized content knowledge needed to teach the topic of integers. We expect to show through the analysis of the interaction among teachers in our study that the framework is useful in supporting teachers' construction and exploration of contexts. The framework allows teachers to identify derived quantities in situations and to deepen the mathematical meaning embodied in them. We aim to show that as teachers explore contexts and models, they build the repertoire of representations that is accessible to them while teaching the topic of integers.

Table 1 SCK framework for teaching integers

Meaning of the negative sign	Meaning of integers	Meaning of addition–subtraction of integers
Unary function	State	Combine
Binary function	Change	Change
Symmetric function	Relation	Relation
Models: number line models/neutralization models		
Contexts: eliciting salient quantities and derived quantities		

Study design

The study discussed in this paper is part of a larger qualitative study designed as a professional development intervention with teachers. Participants in the larger study were mathematics teachers teaching primary and middle grades in a nation-wide Government school system and were nominated by their principals as "effective teachers." The study covered a span of two academic years (see Fig. 1). In the first year, a 10-day workshop during the summer vacation was held for 13 teachers. The goals of the workshop were strengthening teachers' knowledge relevant to teaching, providing opportunities to articulate and reflect on beliefs and developing a sense of community among teachers, teacher educators, and researchers participating in the study (for details, see Kumar et al. 2013). The workshop tasks included observing and reflecting on non-traditional teaching, learning through solving and analyzing problems, anticipating and reflecting on student responses, discussing math education research literature, analyzing textbooks, and articulating beliefs about teaching, students, and mathematics. While the tasks included a range of topics and concepts in school mathematics such as whole numbers and operations, fractions, ratio and proportion, and algebra, the topic of integers was not addressed in any of the workshop sessions. The workshop was followed by visits by the first author to the classrooms of two teachers—one primary and one middle school teacher (about 1 month each). During these visits, the first author frequently reflected on the lessons together with the teacher and discussed plans for future lessons. The visits revealed that teachers needed to develop knowledge and resources for specific topics which would facilitate the change in teachers' practice toward developing student understanding. This was addressed through the collaborative lesson-planning workshops held in the second year.

In the second year, six one-day workshops were held for collaborative lesson planning (CLP) spread over a period of 5 months, while the teachers were teaching in their schools. The specific topic of integers was chosen as the focus of the workshops by the four middle school teachers, all of whom had attended the professional development workshop in the first year of the study. Out of the four teachers, the first author visited the classrooms of two teachers, while they were teaching integers. At the end of the second year, the teachers' group conducted a 2-hour workshop session on teaching integers for peer teachers from the same school system.

In this paper, we discuss the work of the middle school group of teachers in the CLP workshops. Table 2 provides selected background information of the group of teachers. A resource team of five members consisting of researchers (who also played the role of teacher educators) and graduate students planned and facilitated the meetings of the CLP workshop. Usually, two to three resource team members were present for each workshop meeting.

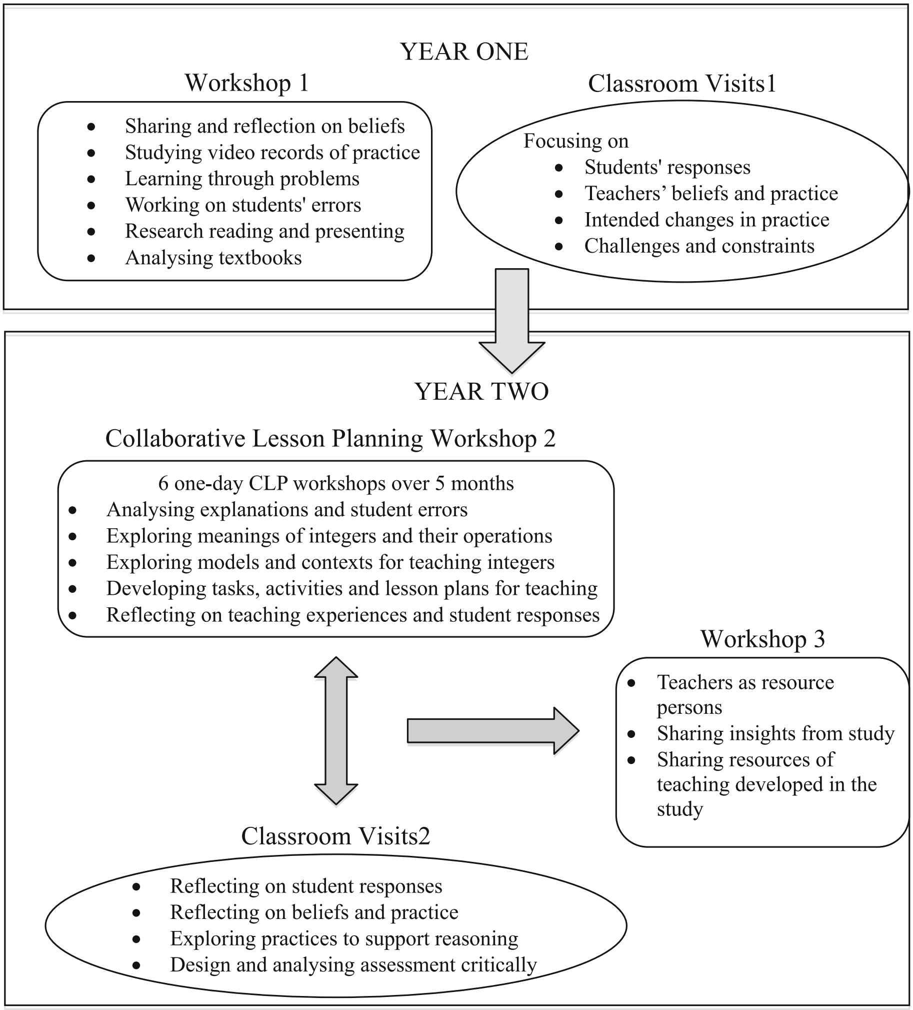


Fig. 1 Study design showing professional development efforts in year 1 and 2 of the study

The aim of the CLP workshops was to focus on a topic that was challenging to teach, to develop a deeper understanding of the topic, and to plan for teaching. The teacher educators, who were also researchers, refrained from conveying to the teachers that they needed to make specific shifts in their teaching practice, for e.g., from teaching rules to teaching for understanding and reasoning. The focus was more on collaboratively developing mathematical knowledge for teaching and resources, which was expected to facilitate teacher learning and influence teachers' decisions in selecting and designing tasks. The middle school teachers' group chose the topic of integers to be taught in Grade 6 as the topic for the CLP workshops. Meetings were held over 6-day spread over a period of 18 weeks. The discussion in the workshops can be broadly divided into four phases: (1) initial discussion of issues related to the teaching of integers (Day 1 and Day 2), (2) engagement with contexts in which integers can be applied meaningfully (Day 2 and Day

Table 2 Background information of the participant teachers

Teacher names (pseudonames)	Age and gender	Qualification	Teaching experience (primary + middle/secondary)	Avg. no. of students in class (last 3 years)
Swati	42, F	M. Sc. Maths, B. Ed.	10 + 7	45
Anita	47, F	B. Sc. Maths, B. Ed.	20 + 3	40
Rajni	53, F	M. Sc. Maths, B. Ed.	0 + 23	45
Ajay	54, M	B. Sc. Maths, B. Ed.	0 + 22	40

3), (3) planning for teaching (Day 4), and (4) reflection on teaching and preparing a workshop session for other teachers (Day 5 and Day 6). The phases are convenient divisions with overlaps and elements of each phase present in the other phases.

The teacher educators' role was to initially elicit from the teachers the approaches that they used in the classroom and the challenges that they faced. On the second day, a worksheet (see "Appendix 1") designed by the teacher educators based on Vergnaud's (1982) framework of integer meanings was worked on by the teachers. This led to an extended discussion on contexts where integers were used and the integer meanings associated with the contexts. On subsequent days, the teacher educators supported teachers in examining the learning outcomes addressed by the textbook chapter, in designing instruction and in preparing for a workshop session for peer teachers. Such support consisted in helping individual teachers in identifying and preparing learning resources that they wished to use in their classrooms, and sometimes in designing student tasks around a context chosen by the teachers.

The data that we analyze in this article consist mainly of transcripts of audio recordings of the CLP workshop. The audio recordings included interactions between teachers and teacher educators in the workshop as well as teachers' reports and reflections on using resources developed in the workshop for teaching. Additional data included teachers' individual lesson plans and presentations made by teachers to their peers in the last meeting of the workshop. We focus on the participants' exploration of representations involving the use of integers, including contexts and models, and their developing understanding of the meaning of integers and of addition and subtraction of integers in relation to a variety of representations. The discussion in all the four phases of the workshop was fully transcribed. Each utterance by a participant (i.e., each turn) was coded to identify aspects related to the topic of integers using the following categories: speaker, mathematical purpose, pedagogical purpose, integer meaning, operation meaning, type of representation, and specific model/context discussed. The category "mathematical purpose" coded for mathematical content, and included the codes "integer meaning," "integer order," or "addition–subtraction." The category "pedagogical purpose" described the pedagogical concern reflected in the talk and included the codes "student thinking" and "evaluating accessibility." It also included codes that captured teachers' engagement with the mathematical content without explicit reference to the teaching context such as, "explaining a mathematical point" and "evaluating mathematical consistency." The codes for integer meaning and operation meaning were obtained from the framework described in the previous section. Types of representation were "symbol," "context," and "formal model."

The first and second authors independently coded the transcripts. After initial coding, codes were merged and simplified to remove ambiguities. Differences between the two coders in coding were resolved through discussion; when they could not be resolved, both codes were marked together for the particular turn. The codes were used to collect together utterances related to a common theme, which indicated the broad features of the discussion. The codes were also used as filters to focus on specific aspects of interest and to validate the claims made. In the discussion of the results below, we support our argument using extracts from the transcript guided by the coding scheme.

As mentioned earlier, we draw on data from the study to answer two questions: (1) What were the teachers' concerns about the teaching of integers and how are they related to issues of meaning of integers and (2) how did teachers construct SCK for teaching integers using the framework of integer meanings through the exploration of contexts. The first question is explored in the next section where the discussion aims to show that the concerns that the teachers expressed were related to issues of meaning, although this was not apparent to the teachers at first. The subsequent section deals with how teachers developed understanding of representations for integers, their meaning, and interconnections. This is followed by a description of the teachers' use of ideas from the workshop in their own teaching. In the final section, we discuss implications for formulating topic-specific SCK and its role in teacher development.

Teaching concerns and issues of meaning

The discussion in the collaborative lesson-planning workshop took place in the backdrop of impending teaching of the topic of integers by the teachers. The textbook chapter on integers for Grade 6 (National Council of Education Research and Training [NCERT] 2005) formed a reference point for the discussion. The topics covered in the chapter were need for integers, location of integers on the number line, comparison of integers, and addition and subtraction of integers. Three of the four teachers (Swati, Anita, and Rajni—all pseudonyms) were teaching in Grade 6 and hence related the discussions of the first 4 days (Phases 1–3)to their plans for teaching in that grade. Ajay (pseudonym), who was teaching in Grade 7 that year, said that the discussions were useful for him too, since teaching the integers topic in Grade 7 involved revisiting what they had done in Grade 6.

In the initial phase (Phase 1), the teachers identified issues and concerns that they faced in teaching integers and discussed representations used by them to teach integers. In the discussion, the teacher educator's prompts included asking teachers about student difficulties and common errors, asking them for the students' thinking underlying these, and how the teachers addressed them in their teaching. Further inputs and discussions evolved on the basis of what teachers said. In the initial discussions, teachers discussed several student errors and underlying causes that were important to address. Most of the students' difficulties that teachers identified had to do with the integer operations of addition and subtraction using symbolic expressions (multiplication and division operations were not discussed since they were not included in the Grade 6 curriculum).

The teachers did not always explicitly connect students' difficulties with difficulties about the meaning of integers or integer operations. Initially, when the teacher educators suggested that students' difficulty with operations could be because they did not understand the meaning of integers, the teachers responded that students did not have a problem with the "meaning of integers." However, as we point out below, several issues concerning the learning of integer addition and subtraction led to an exploration of underlying issues about the meaning of integers. We discuss excerpts from the CLP meetings, where teachers raised issues centered around the teaching of procedures for integer operations using symbolic representations or formal models. Teachers' concerns were about addressing student errors and developing meaningful explanations using representations. Through an interpretation of excerpts from the discussion, we show how issues of meaning underlie teachers' concerns. Further, we show that teachers' SCK about integers was limited in terms of awareness of meanings associated with integers and operations, awareness of the distinction between minus as sign of integer and that of the subtraction operation, challenges faced in giving meaningful explanations for procedures on representations, and lack of knowledge of conventions of representations. In the subsections below, we use the framework of meanings outlined earlier to specifically focus on the distinct meanings of the minus sign, the meaning associated with the subtraction operation, and the conflict between focusing on rules and focusing on meaning.

The following excerpt records an exchange about a student error that occurred at the very beginning of the discussion on Day 1, and is indicative of the kinds of concerns that teachers had about the teaching of integers. The numerical code in the bracket indicates day and session number followed by the serial number of the utterance (turn). Thus, "1.1; 17" indicates that the utterance is from Day 1, Session 1 and is number 17 in the sequence of turns in the discussion.

Excerpt 1

Swati: If we write $7 - 6$ they will say 1. If we say $-7 + 6$ then they will make 13, they may put negative sign… (1.1; 17)

Anita: For subtraction they have to first convert it into addition, which children forget to do… Using buttons – that [subtraction] is not there, only addition is possible not subtraction… when subtraction problem comes they make this error (1.1; 18)

Swati: Concept of subtraction-addition becomes confusing because both places they have this negative sign. In addition also… negative sign is there for integers. (1.1; 19)

Rajni: They do *ulta* (reverse). (1.1; 20)

Anita: They should remember, no? Though they know it very well…. But when they have to do it they are in a hurry and they forget and make error. (1.1; 21)

In the five consecutive turns quoted above, teachers are citing examples one after another, rather than responding to one another. Each of the three teachers Swati, Anita, and Rajni identifies an error or offers an explanation for errors made by students. Swati identifies the error known as the detachment of the minus sign ($-7 + 6 = -13$) (Linchevski and Livneh 1999), and follows it up with her explanation of the underlying reason why students make this error in 1.1; 19. Anita identifies the subtraction operation as difficult because students forget to convert it into addition. She follows this up with an explanation—students respond well to the two-color button (neutralization) model, but she thinks that the model applies only to addition and not to subtraction. Rajni identifies the problem of students' incorrectly reversing the order in subtraction as a problem that teachers need to address. Each of the error patterns identified in this brief excerpt connects to discussion threads that were about issues of meaning. These discussion threads were specifically about the meaning of the minus sign and the meaning of the subtraction operation, which we discuss below.

Distinct meanings of the minus sign

The discussion thread related to the distinct meanings of the minus sign shows a growing realization among teachers of the importance of the distinction between the sign for a negative integer and the sign for the subtraction operation. This realization is part of a general movement toward sensitivity to issues of meaning. The exchange in Excerpt 1 suggests that teachers were aware of the common errors that students generally make while computing with integers. They described student errors making reference only to the procedures that students need to use and did not think that the errors were due to the kinds of meaning that students made of the expressions. In Turns 1.1; 18 and 1.1; 21, Anita attributes students' errors to their forgetting rules or procedures. This was a frequently invoked explanation of students' mistakes. For example, Rajni, discussing the error cited by Swati in Turn 1.1; 17 above, remarked, "How much ever example we give… they should keep in mind -7 and $+6$, that minus they will forget" (1.1; 64).

Swati, however, offers a different explanation for students' difficulty with integer operations in Turn 1.1; 19. She is pointing to the fact that students may be confused at seeing the minus sign in addition problems, which is not the case for whole numbers. She appears to suggest that the minus sign has a different meaning in the middle grades that students need to internalize, while in earlier grades they are only accustomed to interpreting the minus sign as indicating the subtraction operation. This indicated awareness of the challenge faced by students in distinguishing sign of minus as operation and integer but was articulated through an example rather than as a general difficulty faced by students.

The teachers' initial explanations for procedures associated with specific representations that they used in the classroom did not exhibit a distinction between minus as sign of operation and as integer, leading to possible misinterpretations. In session 1.2 during a discussion about modeling subtraction of integers on the number line, a teacher suggested that while moving on the number line, one must reverse the direction of movement on encountering a minus sign. This would ensure that for $3 - (-4)$, one starts from 3 and moves correctly toward the right. This was challenged with the example of $-3 - 4$, where moving from zero to -3 and then reversing the direction would lead to an incorrect answer. The teacher educator then made a suggestion that he thought might help.

> TE1[1]: You have to make a distinction between minus as the operation sign and minus as negative sign. (1.2; 68)

This distinction was readily appreciated by the teachers, and Ajay applied it to explain the difference between the minus sign in "$4 - 2$" and "$-4 + 4$," as seen in the following excerpt.

Excerpt 2
Ajay: Here the meaning of -2 is different…. The first minus [i.e., in "$4 - 2$"] is operation and the second [i.e., in "$-4 + 4$"] is number (1.2; 93)
Rajni: Second is quantity.
Ajay: -4 is one number…. The first is a sign of operation, but the second is not operation.
Swati: Sir, this is clear to us, but for a 6th Grade child, this *operation sign* – he will not understand [the distinction?].
Ajay: This is the point that we have to explain in a simple way.
Rajni: We have to differentiate the two…
Swati: They don't know…
Rajni: Yes, they don't know that.
TE2: For them, probably all the signs mean operation.
Swati: It is the same for them, all [the signs] are addition, subtraction. (1.2; 102)

[1] TE indicates teacher educator and 1 indicates which teacher educator it is out of the three.

The teachers thus acknowledged that students may not understand the distinction between minus as sign of operation and sign of integer and that hence this is an important issue to be dealt with in teaching. The discussion shows teachers thinking about the meanings held by students of the minus sign vis-a-vis the meanings invoked by teachers while explaining the procedures for integer operations using representations. However, at this point, the discussion of meanings was centered around symbolic expressions and teachers attempted to identify the features of an expression that can indicate whether the sign denotes operation or integer. Teachers attempted to articulate criteria using which they themselves identify the sign as indicating integer or operation. One suggestion was that "+" sign always meant operation, since the "+" sign is generally implicit for the positive integers. Another criterion suggested was that whenever two signs appear in succession, the first is the sign of the operation. The issue was not satisfactorily resolved at this point.

In the discussion above, we see the teachers moving from an implicit recognition by one of them (Swati) of the distinct meanings of the "−" sign to explicit acknowledgment of the distinction and its importance, and to attempts to articulate criteria for identifying the distinct meanings in a symbolic expression. There is however a flipside of the distinction, which is the question, "Why is the same sign used for two distinct meanings, rather than two distinct signs?" This question surfaced when the teachers were discussing tasks for children around a shopping mall with floors above and below the ground. One of the tasks was to ask students to number the floors in the mall, and it was expected that students would number floors below the ground starting with "−1." However, Ajay raised a question—why would students accept using the minus sign here, which they understand as the sign for subtraction. This led to an interesting discussion on the advantages and disadvantages of numbering basement floors with the letter "B" as opposed to the minus sign. A satisfactory resolution of this question calls for an explanation of why the "−" sign is used to denote both a "negative" quantity and the subtraction operation. This explanation emerged in a subsequent discussion in a specific context, which we describe later. We note, at this point however, that the teachers now thought of the meaning of the sign together with the meaning of the integer.

Meaning of the subtraction operation

The meaning most commonly applied to subtraction of whole numbers is "taking away" a smaller quantity from a larger quantity. However, the "take away" meaning has to be re-contextualized for understanding subtraction of integers, which may involve adding zero pairs ($+1-1$) to preserve the value of minuend, before "taking away" the subtrahend. Other meanings of subtraction like difference or comparison can be used consistently across whole numbers and integers. The discussion on student errors led to teachers realizing that the issue of the meaning of the subtraction operation is important for the teaching of integers.

Returning to Excerpt 1, we note that in Turn 1.1; 20 Rajni is pointing to the order error in subtraction made by students, which she elaborated later: "What should be subtracted from what… that is the main mistake they [students] are making. Subtract 7 from 3, they will do $7-3$…. That order only we have to teach… so many examples we should give" (2.1; 1). In Rajni's view, students did $7-3$ instead of $3-7$ because they had difficulty interpreting the English sentence "Subtract 7 from 3." She thought that the error of forgetting to reverse the order needed to be addressed while teaching and she mentioned it several times in the course of the discussion. A discussion of this error took place in Session 1 of Day 2, which was a combined session with primary and middle school teachers. Teacher educator 1 (TE1) suggested a different explanation for the error by asking whether students would make the order error for "subtract 2 from 5." Most teachers agreed that they would not do this because they were familiar with $5 - 2$, which is taking away 2 from 5. However, they have a problem with taking away 7 from 3 or $3 - 7$. As a primary teacher said, "In the primary classes, we fill it in [their] mind that you cannot take away 5 from 2. How can you take away? You can never take away 5 from 2." (2.1; 28) Teachers thus confronted the need for students to understand what "taking away" a bigger number from a smaller number actually means.

The "take away" meaning was extended to integer subtraction while discussing how the neutralization model of two-colored buttons could be extended to subtraction. While teachers appreciated the usefulness of this model for subtraction, they also anticipated difficulties that may arise while attempting to apply meaning considerations consistently.

In Excerpt 1, Turn 1.1; 18, Anita refers to the use of the two-color button neutralization model for the addition of integers. This model is discussed in the textbook and also taught by the teachers, but only for the addition operation and not for the subtraction operation. Anita apparently believed that the two-color button model does not work for subtraction. She thought that this might account for why students found the subtraction of integers especially difficult. TE1 took the opportunity to model the subtraction operation using two-color buttons, using the meaning of subtraction as "take away." Problems such as $7 - 3$ and $-6 - (-3)$ are relatively straightforward since one can take away three positive buttons in the first case and three negative buttons in the second. Problematic cases are handled by introducing "zero pairs" that is a pair of buttons with opposite colors. Thus, $6 - (-3)$ could be modeled as follows: introduce 3 "zero pairs" (i.e., 3 positive and 3 negative buttons) to the 6 positive buttons already laid out on the board. This does not change the value of the set of buttons on the board. Now take away 3 negative buttons, which leaves 9 positive buttons. This explanation was new to the teachers and led them to believe that the two-color button model could be used for subtraction.

Although the teachers thought that the two-color button (or card) model was useful, they did raise issues about the meaningfulness of specific actions carried out using this model. Thus, while discussing how addition of integers works with the two-color button model, Swati raised the question of why a red card and a black card cancel each other: "We say it is zero … it is $+1$ and -1 we know. [But] they start counting all of them… Here they can see $+1$ and -1 [but] they make it 2, they don't consider it zero." (1.1; 132, 134)

Rajni had a different way of using the two-color buttons to model subtraction, which she explained in Session 2.1. She started with the rule that subtraction problems can be changed into addition problems by replacing the subtrahend with its additive inverse (or "opposite"). Thus, $3-4$ could be rewritten as $3 + (-4)$, and could be solved by adding four black cards to 3 red cards giving the result -1. Similarly for the problem $3-(-4)$, one could rewrite it as $3 + 4$ and add 4 red cards to three red cards to get 7. However, both Swati and Anita objected saying that the problem was being changed. In the quote below, Anita expresses her dissatisfaction with suggested ways of modeling subtraction, both that of changing the color of the buttons and that of adding zero pairs.

> Anita: But in this method I feel you are manipulating the question… change the color… for zero you add 2 buttons… so you have changed the question… (2.1; 79)

Rajni had used the meaning of integer as opposite to explain the subtraction using two-color buttons, but it was not convincing to the other teachers. The discomfort seems to stem from modeling the procedure using two colors to get a correct answer without delving into explanation of why subtraction of an integer is equivalent to addition of its inverse. Also, the procedure for subtraction in the case of integers is not consistent with subtraction of whole numbers needing an explanation for why the procedure needs to be modified. The group however did not go deeper into discussion about meanings connected with this representation.

Teaching rules versus teaching with representations

Teachers shared the representations that they used for teaching integers in Phase 1. Teachers' talk about representations made explicit their beliefs about teaching integers, their preferences for use of specific kinds of representations, the meanings used implicitly to identify features that can be represented by integers and the perceived role of rules in the teaching of integers. Besides the use of the formal two-colored button model to teach addition, the teachers used the formal number line model to teach integer addition and subtraction. In this case, addition and subtraction were typically interpreted in terms of movement on the number line, at times involving seemingly arbitrary rules about reversing the direction of the movement when encountering a minus sign. Teachers and teacher educators expressed the view that the rules seemed arbitrary and did not seem meaningful. Despite these reservations, teachers almost exclusively used only formal models, and did not use contexts, to teach the operations of integer addition and subtraction.

Teachers mentioned contexts involving the use of integers while teaching, but only while introducing integers. The teachers made a distinction, between a subtopic that they referred to as "need for integers" and the subtopic of "operations with integers." They believed that contexts were useful to introduce students to the "need for integers," but were not very useful in teaching operations with integers. While introducing integers (also referred to as "giving the concept of integers"), teachers explained the meaning of positive and negative numbers by referring to contexts involving opposite quantities such as increase–decrease, depth-height, and above-below. However, they generally did not draw on meanings or contexts while discussing integer operations. Teachers also explained the meaning of integers by referring to the number line. Rajni, for example, associated meaning of the symbols + and − with directions on the number line.

Although the teachers used the two formal models of colored buttons and the number line to teach integer operations, the models were thought to be useful only initially, when the numbers dealt with were "small numbers." The students needed to eventually learn to operate with big numbers, and teachers felt that for this they needed to know the rules. All the teachers acknowledged that they explicitly teach the rules for operations with integers, but they differed in the importance they gave to rules. For Ajay, the main aim of the chapter was to know the "laws of integers" which are needed to solve problems with bigger numbers efficiently. However, Swati thought that students find it difficult to remember all the rules. Anita felt that rather than teaching rules explicitly, one should let students construct rules as they worked with models such as the two-colored button neutralization model. The excerpt below reflects these tensions.

Excerpt 3

Ajay: I think that when we give the concept of button, same time we can give concept of rule. (1.1; 83)

Anita: No, why don't we make rules through buttons – if white color is more then it is positive… this rule can be constructed. (1.1; 85)

Rajni: We are calling them for buttons, ultimately we should tell them rules otherwise big numbers they will face problems. Here only they are making mistakes. (1.1; 86)
Ajay: The main thing that you have to tell in integer is rules. If they understand rules then they can do everything. (1.1; 87)
Swati: They are not able to remember [rules] that you know. Every time that is the problem. (1.1; 88)

As we see from the excerpt above, talk about remembering or forgetting rules stands in contrast to and in tension with talk about understanding the meaning of symbols and operations. Teachers' talk about use of representations indicated that they preferred rules and symbolic representations over using contexts and models for teaching integers. This is perhaps because of their belief that representations like contexts and models are only useful for small numbers and for introducing students to integers but are not useful in developing fluency in computation using integers.

Teachers' talk in Phase 1 reflected their concerns about teaching integers, but indicated gaps in their SCK in terms of a limited repertoire of representations, limited knowledge of meanings which can be attributed to integers, signs and operations, lack of distinction between these meanings in their discourse, and inadequate explanations of procedures using representations. These knowledge gaps constrained teachers' understanding of student errors and the conceptual shifts needed in moving from whole numbers to integers. However, even in Phase 1, the discussions around student errors and the models used for teaching integers extended teachers' SCK in important ways. Two elements of the teachers' construction of SCK were (1) awareness of the distinction between the use of the minus sign for integer and for operation and of the importance of this distinction and (2) awareness of the meaning of the addition and subtraction operations as applied to the neutralization model and a striving for consistency of these meanings. These constructions occurred in the context of identifying the challenges faced by students in working with symbolic expressions or with the neutralization model, and evaluating the representations that the teachers had been using.

Teachers' engagement with contextual meaning of integers

In the second phase (Days 2 and 3), the teachers systematically engaged with the meaning of integers through tasks that called for proposing contexts or interpreting contexts using the framework of meanings of integers and integer operations. The teachers were exposed to the integer meanings of state, change, and relation through worksheet tasks (see "Appendix 1"). A large number of contexts were discussed, integer meanings explored, and judgment was made about their pedagogical usefulness. Table 3 presents contexts, which were discussed over 10 or more turns. Prompts by the teacher educators included questions about which integer meaning applied to a given context, whether the use of integers was appropriate, and which contexts were useful for the classroom. Of the large variety of contexts discussed, the teachers found the following contexts to be useful for the classroom: integer mall, change in day temperature, scores on tests, and change in a baby's weight. Discussion of the many contexts helped deepen teachers' SCK for teaching integers. We support this claim with two kinds of evidence. One kind of evidence is derived from analysis of teachers' talk, which reveals the aspects of SCK that were extended and built upon during the course of the CLP workshop discussions, using their initial talk as a

Table 3 Turns of teacher and teacher educators' talk while discussing contexts for teaching integers

Contexts discussed	Total number of turns	Turns of teachers' talk	Turns of teacher educators' talk
Integer mall—floors in building and movement of lift	211	89	122
Temperature	96	51	45
Marks/score	53	32	21
Baby's weight	51	26	25
Profit/loss	44	24	20
Mixing water at different temperature	30	16	14
Ticket reservation	29	16	13
Loan taking/giving	28	17	11
Family size	26	16	10
Queue	26	12	14
Length of shadow	24	16	8
Water level	22	9	13
Journey by train	20	12	8
Steps	19	11	9
Altitude—heights of different vehicles	15	7	8
Speed of car	11	7	4

frame of reference to chart their growth. For this, we describe examples of teachers' selection, interpretation, design, and evaluation of contexts and highlight discussions of the meaning of integers and of integer addition and subtraction. The second kind of evidence is the teachers' self-report of what they have learnt, and the changes in their teaching as a result of participation in the study, which is discussed in the next section.

The first subsection below describes teachers' engagement with the use of integers to represent change and relation, which marks the important shift from the use of integers to represent only state. The second subsection describes their attempts to interpret the addition and subtraction operations in contexts, which marks the next important shift from the exclusive use of formal models to also using context-based representations to teach integer addition and subtraction.

Integers as representing change and relation

Most of the initial examples given by the teachers of contexts for teaching integers were of state, which became apparent to the teachers when they became explicitly aware of the alternative senses of change and relation that integers may denote. The teachers used integers to represent states in contexts that might be familiar to students, such as temperature, profit and loss, height above and below ground level, or position of floors in a building above the ground or in the basement. In these examples, the positive and negative integers were associated with opposite states like above-below, profit–loss, and increase–decrease. However, there were occasions where they used integers to inappropriately represent "opposites" like number of boys and girls, number of children sitting and standing, without offering an interpretation of what canceling positive and negative quantities might mean in these contexts. Over the course of the workshops, the variability in the meanings represented with integers increased along with the awareness of different meanings. Teachers gave examples of contexts for representing integers; they evaluated and challenged examples, and proposed and designed activities for teaching based on the contexts that they had discussed.

Using integers to represent change

It was challenging for the teachers initially to identify features of representations that correspond to the change meaning of integers. The meaning of change was introduced when, during the initial discussion, Swati asked how one might explain that $+1$ and -1 cancel to give zero (1.1; 127). TE1 suggested that one may interpret $+1$ as an increase of 1 and -1 as a decrease of 1, so they together cancel each other resulting in no change. However, initially teachers preferred to represent change only with operation rather than also as an integer. An engagement with this issue is reflected in Rajni's attempts to incorporate the change meaning into her explanation of addition and subtraction on the number line as movement. She now interpreted movement to the right as increase and to the left as decrease. However, her association of the "increase–decrease" was still with the operations of addition or subtraction rather than with the positive and negative integers.

> Rajni: We are telling them that add means right and subtract means left. Then they will ask $3 + (-4)$…. So we will go where? $3 + (-4)$… it is equal to left only actually but it is increasing. Plus means it is increasing but -4, so negative weight is increasing (1.2; 21, 25)

We note that Rajni is still connecting increase to the operation of addition, while the negative integer represents a state ("negative weight"). Teachers' resistance to the use of unary integer to represent change signals a limitation in underlying beliefs about what features of representation were appropriate to represent with integers.

In contrast to Rajni's seeming resistance to using integers to represent change, Anita designed a context to show addition of integers, where the integers represented change. She proposed a context where stones are added or removed from a bowl containing an unknown number of stones. The number of stones added and removed was represented using integers.

> Anita:…already there is a collection of stones and you add some thing and by taking away these stones there will be decrease in the number of stones. But together there will be increase of 2 stones. Like that similarly increase 2 and decrease 4, what is the result – like that. We will ask and get the number and convert these activities into mathematical form. We will see how they are going to write. There you can introduce minus 3. (2.1; 116)

Here Anita has explicitly used unary integers to represent a transformation. Moreover, the context and the framing of the task make the change meaning salient, and the representation of a decrease using a negative integer meaningful. Similar contexts, where change is salient and is represented using integers, have been used by other researchers to support integer learning, for e.g., persons entering or leaving a discotheque (Linchevski and Williams 1999). An explanation that Anita offered indicated how she understood integers as representing transformation in contrast to integers as representing state. She described her new idea as a recognition that "-2 is not always from zero." That is "-2" does not always mean 2 less than zero, but could mean 2 less than some arbitrary or unknown number. This prompted Swati to describe the unknown number as a "reference point" (2.1; 131). The idea of the *reference point* emerged as a key construct, which was useful in later discussions in distinguishing the meaning attributed to integer in different contexts. We mark this as an important moment in the teachers' construction of SCK through an exploration of meanings and contexts.

Teachers appreciated the need to explicitly distinguish between the state and change senses of integers as indicated in the Swati's and Anita's comments below. Swati discusses how the meaning of state and change can be distinguished in the context of integer mall as negative integers representing lower floors indicate state, while the change is represented by integers depicting the movement of the elevator. Anita points out how in the context of representing change in water level in a tank the reference point will keep on changing since the initial level will be the reference point for determining the change denoted by the final level of the water in tank. She is able to reiterate her earlier argument that reference for negative integers is not always from zero by using the meaning of change in a context.

> Swati: Child has to understand the difference between the movement and the state because when we are writing numbers here −3 so he will say it is here [as basement floor], here when −3 is coming it is telling you that we [have moved] 3 floors down (4.2; 235)
>
> Anita:…Initial level can be the reference point always…let that be the reference point…initially 10 litre and then increase [represented by] + sign and decrease 10 litre [so] how much? (3.2; 335)

This notion of reference point was initially proposed and taken up by Swati and Anita; however, later Rajni and Ajay too understood the importance of this construct and used it in their discourse to distinguish different meanings of integers. In a session with peer teachers (to be discussed later), Rajni explained the significance of reference point to clarify how one determines the height (state) of an object using the sea level as a reference point.

> Rajni :…here in this question we have referred sea level for knowing the position of the bird and position of the diver… [if] simply we are telling 25 m above a bird is sitting on tree… so you don't know from where… from the ground from the depth of the sea or whatever we don't know that.. so for referring this bird, sea level is the reference point… That is called as state..

Swati, who had proposed the notion of a reference point, went on to use it in insightful ways. Here she uses the notion to distinguish the contexts where integers represent change from those where they represent state.

> Swati: [State is] where the reference point is not changing…. When we talk about change, reference point is changing every time. [In the context of change in profit from day to day]…. We are comparing today with tomorrow and that day with the next day. So reference point is always changing. (2.2; 407)

Using the meaning of integers as change, the teachers were also able to identify additional features in contexts that could be represented by integers. For example, an initial context constructed by the teachers to represent change using integers was that of test scores. Ajay had suggested the inclusion of negative points for a wrong answer as an example of using integers—one has to combine (add) the positive and negative points to get a total score. In a later discussion, Anita suggested modifying the task to record only the change in the test score of a student over a series of tests. The change would be positive or negative taking the score on the previous test as the reference point and the change can be combined (combine–change) to show the student's progress over time. Similarly, teachers discussed the contexts of weekly change in a baby's weight and hourly change in ambient temperature, for which they designed tasks of representing change using integers (note that in tropical conditions, there is no possibility of day temperatures falling below 0 °C.)

Excerpt 3
(Following a discussion about weekly change in baby's weight.)
TE1: Similarly [Any] other situation… you can think of which will be interesting to students?
Swati: Change in temperature from morning till night.
TE1: That's very nice.
Anita: Different time interval… 9 am [temperature increases from then on] then again in evening it decreases.
TE1: So we can give… rather than temperature you can give the change in temperature. (2.2; 167-171)

In the task for students developed by teachers involving representing weekly change in weight of a newborn baby, they asked students to give reasons for an unexpected event (such as a negative change in the baby's weight) as it may help in meaning making.

TE1 (*reading the task written by teachers*): Represent the above data (i.e., change in baby's weight) as integers… What could be the reason for 200 gm. decrease in first week? What could be the reason for 500 gm. increase in 6th week? How much does the baby weigh in the second week?…. In which week did the child gain maximum weight? What is the total weight gain or loss in the first month? (3.3; 254)

The teachers were able to adopt the change meaning for identifying features of representation to be represented by integer. This was facilitated by the explicit distinction between the meanings of state and change and through considering the possibility of representing by a unary integer what was usually represented by an operation. Thus, a distinction was made at two levels—between operation and integer and between two different meanings that can be attributed to an integer. However, as Swati pointed out in the remark quoted earlier, the two meanings of state and change are also related in that both signify being less than or more than a reference state. In the state meaning, the reference point is fixed and taken to be the zero point. In the change meaning, the reference is to a previous state that might be designated by an integer different from zero. Swati and Anita used the distinction between state and change meaning in exploring new contexts. In the process, they extended their SCK by extending the range of meanings accessed for thinking about integers, which in turn led to a greater variety of contexts and context features being represented by integers.

Using integers to represent relation

As in the case of representing change, teachers initially preferred to represent relations using the subtraction operation rather than using unary integers. The worksheet presented two situations to prompt thinking about the use of integers as relations: "Me and my sister are standing in a queue to buy ice-cream. How far is my sister from me?" The second situation referred to two persons standing on different floors of a building. Teachers represented these situations using the operation of subtraction (or incorrectly, using addition).

Anita: Me and my sister standing in the queue. How far is the sister away from me, there we have to do subtraction…. In the second case we have to add… to find the distance between [floors].
TE1: In the first case, suppose I say how far is the sister from me.
Anita: Yes, then it is subtraction (2.2; 13-15)

Another situation described by TE1 during the discussion led to the teachers accepting the use of unary integers to represent a relation. He described an airplane in the air with its instruments displaying the relative altitude of other planes nearby—heights above the plane indicated with a positive sign, and below with a negative sign. This made the relation salient and meaningful. Other contexts were discussed to illustrate the use of integers to represent relations—depth in water, temperature, and counting years in different eras in different cultures, relative position of runners in a race, relative differences in test scores.

Swati, who had earlier connected the meaning of state and change through the construct of reference point, made a similar insightful comment about representing relations using integers:

Meera is standing 3 position[s] ahead [of me] and Radha 7 position[s] behind me so Radha here, me here, and Meera here. I want to know where I am standing from the starting position. So if you don't know the starting point it is not possible to find the position from starting point. Here it is only relation you are taking…. Generally we take 0 as the reference point … but in this particular question 0 is not a reference point…. We have taken something [else] as reference with respect to it we are finding the position so it is a relation.

Hence, in the teachers' interpretation, the meanings of state and relation are similar, with an important difference of reference point. The reference point for state is fixed by convention, while the reference point in the case of relation is arbitrary. Thus, we see the construct of reference point being used to distinguish different meanings of integers.

One context, the "integer mall" with a lift (elevator) containing only buttons marked as "+" and "−", was worked on in some detail since it contained features corresponding to all the three meanings of integers, as well as the various senses of integer addition and subtraction. The integer mall (Bajaj and Kumar 2012) is illustrated in Fig. 2. Note that floor numbers correspond to a state meaning of integers, while instructions for movement of the elevator, in the form of number of presses of the "+" or "−" buttons, correspond to the change meaning. The position of any floor in relation to a given reference floor provides the relation interpretation. Thus, teachers were able to identify different meanings within the same context and differentiate them based on the generalization they made about critical features that contributed to the meaning.

Use of only state meaning to identify features to be represented by integers may lead to limited understanding of integers as exhibited by teachers in their initial discourse in workshops when they gave inadequate explanations for how and why procedures using representation work to give the correct answer. Knowledge and use of other meanings increase the variety and flexibility in use of contexts while aiding in building meaningful explanations for procedures associated with particular representations. Through exploration of contexts involving integers and thinking about the meaning of integers explicitly, teachers were able to identify the critical features of the context that can be represented by integers and thus differentiate between the two meanings of integer as state and change. This led to not only an increase in the number of contexts identified as useful for teaching integers but also increased potential of using contexts for meaning making.

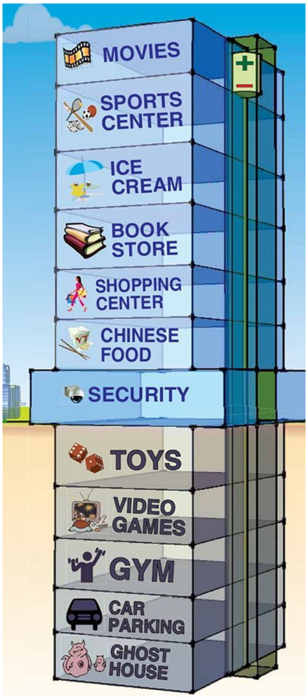


Fig. 2 Integer mall—a context improvised collaboratively by participant teachers

Addition and subtraction of integers

After the exploration of contexts using the framework of meanings of integers and signs, teachers engaged in tasks of exploring contexts where it would be meaningful to represent addition and subtraction of integers. They constructed problems where integers could be combined (added), and which involved change and relation that could be represented using addition and subtraction. This provided opportunities for teachers to connect meanings of integers with meanings of operations, identify which contexts are meaningful to represent integer addition and subtraction, and establish connection between different representations through meanings.

The discussion of a range of situations led teachers to realize that not all situations contain features that correspond to integer addition and subtraction. An interesting context that was discussed was mixing water at two different temperatures—a situation suggested by one of the teachers. Through a discussion led by the teacher educator, teachers realized that the resultant temperature is not the sum, but a weighted average of the two initial temperatures. Temperature change, on the other hand, could be represented using positive and negative integers, and could meaningfully be added. This was the case for the hourly change in ambient temperature, another context suggested by teachers.

The integer mall context (Fig. 2) contained a feature in the form of buttons on the elevator marked "+" and "−". Addition could be used to find where the elevator would stop after a certain combination of positive or negative button presses. It meant understanding that equal number of positive followed by negative button presses will take one to the same floor, i.e., no change. Teachers felt that this notion of no change should be explored both from the ground floor as well as any other positive or negative floor. This corresponds to the neutralization model, where two opposite changes neutralize resulting in no change.

Swati: We can include problem of from any floor pressing plus twice and minus twice … same number "+" times and same number "−" times so it will come back to the same floor. So those +2 have cancelled −2. So no change. (4.2; 3)

Here Swati is drawing a connection between the integer mall context and the neutralization model by identifying that equal amount of upward and downward movement cancel one another. As we had noted, she had earlier asked how one can convincingly explain why +1 and −1 sum to zero. Using the change meaning of integers and the combine meaning of addition, she is able to devise a satisfactory explanation for why equal and opposite integers cancel to yield zero, illustrating an important mathematical idea. Thus, change meaning helped the teacher in constructing this explanation by connecting it with features in the integer mall representation. The use of integers to represent opposite changes made sense, so did the cancellation of additive inverses to yield zero. Using the meaning of combining change, addition of integers was easy to grasp, and teachers felt that students will be able to complete the addition of integers without recourse to rules.

The teachers also pointed out that students will have to represent multiple presses of the "+" and "−" buttons using integers.

Swati: Two times plus coming to +2, so we are forming integers (4.2; 7)
Swati: Two plus [es] is +2 and one minus −1 (4.2; 29)
Ajay: When you press ++− you actually press +2 −1… (4.2; 30)
Ajay: This is the essence of this chapter. (4.2; 34)

In the context of the integer mall, teachers discussed how finding the directed distance between two floors could be represented using subtraction. The directed distance could be interpreted as the movement required to go to the target floor from the starting floor. Students could verify this distance easily from a visual representation of the context, and could associate it with the subtraction operation. Thus, "$a - b$" could be interpreted as the movement required to reach floor a from floor b. Here the subtraction operation corresponds to the "compare" sense and yields the distance between two floors, which could be interpreted as a required or actual change, or as a relation. The sign of the integer obtained as the result of subtraction could be further confirmed by the direction of movement. For example, if moving from −2 (basement floor) to 5th floor, the movement can be expressed as $5 - (-2)$; students could verify from the picture that the distance between the two floors is 7 and since the movement is upwards and would require pressing + button 7 times, the answer would be +7.

TE3: So how will we phrase it so that the [corresponding] mathematical expression is $3 - (-2)$? (3.3; 126)
Anita: From −2 you are moving up to 3rd floor. That means movement is upward 5 floors…(3.3; 131)
TE1: So $3 - (-2)$ is +5 (3.3; 132)
Anita: Up… because it is +5 (3.3; 133)
Ajay: Answer is 5. But if students ask why then we will have to give a reply. (3.3; 134)
Anita: Why is $3 - (-2)$ [equal to] +5? (3.3; 137)
Swati: From −2nd to 3rd floor… 5 upwards. (3.3; 140)

> TE1: See, we can think of it as 2 steps. 3−0. If we reach the zeroth floor first, then I have to go from zero to 3. So how to reach zeroth floor? It is zero minus minus 2… zero minus minus 2 is always +2 because if I have to go from −2 to zero then I have to press +2… (3.3; 141)

TE1 is offering here an explanation of why subtracting an integer is the same as adding its additive inverse, by interpreting movement in two steps with the zeroth floor as an intermediate station. Thus, connection was made between a mathematical idea—the equivalence of subtraction with addition of the inverse—and a transformation on a representation in the form of the integer mall context, illustrating an important piece of SCK. This addresses the issue raised earlier by teachers of how we can meaningfully interpret the rule of changing subtraction to addition of the inverse. Although there was no evidence of the teachers taking up this idea, it points to the possibilities contained in a rich context like the integer mall.

The teachers' engagement during professional development in exploration and design of contexts for use in teaching helped in constructing important aspects of SCK for teaching integers. These aspects include identifying contexts that can be meaningfully modeled by integers and integer operations. Within these contexts, the teachers were able to identify critical features that correspond to particular meanings of integer. In particular, using integers to represent change made it possible to include many contexts involving the addition operation, as well as to identify derived quantities that could be represented and operated with. Expanding the meaning of integers to change and relation, as well as incorporating the change and compare meanings of subtraction, made it possible to meaningfully model the addition and subtraction of integers on contextual representations like the integer mall. Thus, teachers were able to use the potential of a context to a fuller extent by identifying multiple meanings of integers within a context. Further, teachers were able to make connections between contexts and other representations like models using meanings of integer and their operation as a framework and making connections between meanings. They were able to interpret movement on the number line as increase or decrease, and to provide a more meaningful explanation of why additive inverses sum to zero. Thus, meanings helped in bringing coherence among different representations that could be used for teaching integers as well as an increase in variety of contexts for teaching.

Impact on teachers' use of representations for teaching

In this section, we examine the impact of the teachers' engagement with integer meanings and exploration of representations on their classroom teaching. Evidence of impact is obtained from the unit lesson plans that they developed individually, their self-reports on what they learnt in the CLP discussions, their reports of how they used contexts, tasks, and meanings developed in the workshops in their classroom teaching, and their reflections about changes in their teaching. The teachers' reports in the workshops were corroborated by observations by the first author of classroom teaching by Swati and Anita. What teachers chose to report in the workshops about their classroom experience indicated what they had found significant in their learning, and hence we consider this useful and important to analyze.

Impact on lesson plans

Teachers made individual written plans in Phase 3 for teaching the unit on integers consisting of about 10 lessons. All the teachers had used several contexts in their plans (see "Appendix 2" for an example of a lesson plan made by Anita). Anita, Rajni, and Swati had used the integer mall context in several lessons, for example, representing movement from any floor in either direction using integers and later modeling addition and subtraction of integers using the mall context. They also used contexts like temperature, scores on tests, milestones on roads, vertical position of vehicles in the air and in water, baby weight chart among others in their lesson plans to teach different ideas related to integers. Ajay, in contrast, planned to use contexts only for introducing integers. He strongly believed that students needed to be told the "laws of integers of adding integers of same sign and of different signs." In the case of the other three teachers' lesson plans, the tasks included a balance of questions which called for representing features or actions in a context as integers and integer operations, as well as questions on evaluating symbolic expressions containing integers. Anita included questions for eliciting student meanings for the minus sign like "Give examples of situations from your daily life where you have seen use of minus sign." These three teachers also acknowledged change in their approach from telling rules in the beginning to exploring contexts first with students and then either introducing or generalizing rules.

Teachers' self-report of their learning from the CLP workshops

The teachers, while reflecting on their teaching, explicitly acknowledged that SCK elements such as knowing about contexts and their connections with meanings of integers and operation were a powerful resource that can support classroom teaching. Besides such explicit self-reports in the workshop citing their learning, they also indicated what they had learned through the choices they made while leading a workshop session for peer teachers on the teaching of integers.

Following their engagement in the CLP workshop, they felt they had more "understanding" of teaching, "more resources," and felt "more confident" after the workshop. Swati said,

> Actually we did it in so much detail here, so I could… I was more aware. I realized that the student needs clearer understanding of integer. Otherwise we would clearly say 'no, not like this. Do like this'. This is how we used to deal … so that is the change in us I could observe.

Rajni said "I thought [of] integers always as numbers and never connected it with any situation… after coming here I thought like this." Anita added that the textbook problems mostly involved the sense of integer as state and only some of change through movement, while now they were able to develop different contexts to represent change and relation through integers.

The teachers acknowledged change in their teaching as a result of the engagement with contexts. Anita said, "Usually we follow the textbook method… This was entirely different as I used this button activity and lift (elevator) context wherever required." The teachers expressed a desire to do such an exercise for other chapters in the textbook.

After their experience of changing their own teaching, the four teachers led a session for other teachers where Anita explained, "First it is important for us to understand… These 3 senses [of integer] make us clearer… This is what happened with us, we learnt it here only and it made our concept of integer more clear so it is better in teaching."

All the four teachers jointly prepared presentations for their peer teachers. They decided the themes of presentations based on what they felt was useful for learning about teaching integers through reflecting on their experience in the research project. The first theme was discussion on the different meanings of integers like state, change and relation, which they discussed using several contexts, asking participating teachers to think of examples of contexts and the associated meanings of integers. The second theme was about how contextual and open ended tasks beyond those given in the textbook help students think and reason about integers rather than solve problems mechanically. Discussion on this theme also involved issues that arise when such an approach is taken in the classroom such as students' beliefs and resistance, and communication with parents and administrators. The third theme was about how specific representations and tasks developed collaboratively in workshop helped in eliciting and developing students' understanding. It also featured examples of how teachers interpreted student responses using contexts and responded to students' ideas. The fourth theme was about presenting and discussing several models (like neutralization and number line models) and explaining the procedures associated with such models using meanings of integers. These presentation themes selected by teachers indicated the value teachers had started attaching to the framework of meanings of integers, use of contexts in the teaching of integers, students' thinking and exploring different representations for the teaching of mathematics. It also indicated that teachers found the SCK aspects focused in workshop like integer meanings, representations like contexts, and models and associated explanations useful in their classroom teaching.

Impact on teaching practice (reported)

In Phase 4 of the workshops, the teachers shared their experience of teaching integers and reflected on how their participation in professional development workshop had impacted their teaching. The teachers' reports and reflection indicated that four aspects of their practice have been influenced, namely their use of context in teaching, focus on reasoning, responses to students, and their understanding of students' thinking behind errors.

The teachers reported how they had used different contexts to engage students in meaning making. The integer mall was especially mentioned by the three teachers who used it as being helpful to initiate discussion on integers. Anita appreciated how students in her class could give explanations using contexts about why certain quantity should be labeled as positive or negative.

The teachers also discussed which contexts they found useful for comparing integers while teaching. Swati felt that the borrowing–lending context is not that useful for comparing integers as the amount borrowed is represented by negative integers, but student tends to reason that borrowing Rs. 3 is more than borrowing Rs. 2, and thus, −3 is more than −2. Rajni and Ajay suggested modifying the question by asking "who is more rich?" rather than comparing who borrowed more. Ajay pointed out a similar problem using the temperature context when students know that −5 degrees is colder than −2 degrees and thus find it problematic to believe that −5 is less than −2. Anita shared how in the context of a quiz, positive and negative points, respectively, for correct and wrong answers was useful for comparing integers as students were able to reason that the person getting −3 has a lower score than the person getting −2 as the former has made more mistakes. In this type of discussion, teachers realized how associating a contextual parameter with negative or positive sign plays a role in comparing quantities.

Anita reported how students connected the representation of integer mall to the vertical number line when discussing a question which involved a floor number beyond what was visible on the integer mall. Students proposed that one can imagine more floors, and the teacher proposed imagining a vertical number line which extended infinitely on both sides. This indicated that students and teachers were able to use the integer mall context to develop a generalized image of number line thus establishing a connection between a powerful context and a widely used model.

The teachers shared how knowledge of meanings of integers helped them in developing student understanding and responding to students in the classroom. Anita had selected a variety of contexts for discussion in classroom, which catered to state, change as well as relation meaning. For example, she used the temperature context and positions of floors on the building to discuss state and relation and movement from one floor to another and increase and decrease in temperature to discuss change. Teachers found that some students readily accepted minus sign as part of integer for labeling basement floors since they had already seen this in malls and other buildings. Some teachers used this to discuss how opposite aspects of a context like height and depth could be represented by positive and negative integers. However, the students could be using integers more as a label than a signed quantity. This became evident when teachers shared that while doing a worksheet for labeling the basement floors, students started labeling lowest floor as -1 and -2 and so on. While discussing this problem, teachers arrived at an understanding that students do this as they use their knowledge of natural numbers thinking -1 to be smaller than -2. Anita shared how she had used the idea of reference point in integer mall context to justify that the floor below zeroth floor should be -1 since one moves *one floor down* to reach that floor, while one moves *one floor up* to reach $+1$ floor and also that $+1$ and -1 are both at one unit distance from zero and in opposite directions, thus having different sign. She pointed out how the relation meaning was useful in convincing students that positive and negative integers are opposite to each other. This observation addressed the question raised by Ajay and discussed earlier, about why the minus sign is appropriate to designate basement floors rather than some other sign. The deeper underlying issue here is of why the same sign is used for the subtraction operation and the negative integer. This discussion helped other teachers too to reflect on their teaching. After this conversation, Swati revisited this idea in her class when doing comparison of integers using number line. She pointed to students how integers can be compared on number line based on the number of units from zero. While in earlier lessons she had accepted reasons like a number being "to the left of zero" as being smaller to the number on the right of zero (which resembled a rule), post this workshop conversation Swati explained and accepted students' answers for comparison based on using zero as a quantity for comparison rather than a label. She focused students' reasoning on how many units is a number larger or smaller than zero. In doing this, zero is being used as a reference quantity for comparison of a signed quantity rather than as a placeholder.

Students in Anita's class also came up with interesting examples of integer use: buttons on a TV remote to increase or decrease the volume, numbers used by doctors for prescribing spectacles. Anita shared her excitement over eliciting these examples from students, which indicated different meanings of integers. Anita asked students to write their observations thus giving them an opportunity to express their own ideas instead of telling them what to write in notebooks. She appreciated how as a result she was able to know what students think and understand and how they obtained answers. She said "they are able to comprehend what they have done and why the answer is this… what is left out is the green color" (for addition using the neutralization model where red and green buttons denoted integers).

In another instance reported by her, Anita had planned to discuss representing distance across floors through subtraction of integers by subtracting the destination floor from the starting floor. Thus, $4 - 3$ represented distance from 3rd floor to 4th floor. When she asked students to interpret this expression using the integer mall context, a student interpreted it as starting from 5th floor, going 4 floors up and coming 3 floors down. Exhibiting flexibility in her thinking, Anita appreciated students' idea and suggested representing it as "$5 + 4 - 3 = 6$" where in she discussed how 5 represented the starting floor and the resultant movement of moving 4 floors up and 3 floors down represented by expression $+4 - 3$ is $+1$. Here, the knowledge of different meanings of integers helped the teacher to flexibly interpret what student was saying into a mathematical expression while she figured out way to discuss the idea that she wanted to communicate to students. Anita's teaching here is responsive to students' ideas rather than making them accept the ideas authoritatively communicated by the teacher. Reflecting on this episode, Anita shared how she "changed [her] track according to the answer"

> I had to relate their answer to my questions … so that is what we should know immediately how to interpret their answers suitable to us…. This makes us also feel good in being able to interpret their answers/that also gives us some satisfaction…. That is why it is a learning experience…. Instead of saying their answer is wrong it is one way of thinking… makes us feel good…

Connecting student errors with student thinking, Swati shared how while teaching operations on polynomials in another class, she found that a student was not able to understand that subtracting an expression is same as changing the sign of the terms and adding. It lead her to think again about the issue of distinction between minus sign as indicating operation and integer sign and how this connection between them needs to be established while teaching subtraction of integers.

The SCK developed during the CLP workshop had an impact on teachers' practice although to varying degrees depending upon the strength with which teachers held their beliefs about teaching through telling rules and the extent to which they engaged and developed their SCK during collaborative planning. The impact was visible in teachers' lesson plans, their self-reports on their learning and reports of classroom experiences. However, even the classroom experience can be termed as an initial exploration of teaching using the resources developed in the workshops. We expect that it would take time and several cycles of teaching for teachers to develop deeper SCK and integrate it with other elements of knowledge for teaching.

Conclusion and implications

The response and take-up by the teachers in our study support our claim that the framework of integer meanings forms an important part of the SCK for teaching the topic of integers. Following exposure to and work with the different meanings of integers, we noted several key movements and shifts in the participating teachers' discourse. These included a movement toward relating teaching concerns with issues of meaning, a shift toward using context-based representations rather than the exclusive use of formal models to teach integer operations, developing lenses to analyze contexts in terms of the meanings of integers embedded in them, using such analyses to make judgments about the appropriateness of representations for teaching and learning, striving for consistency of meaning and designing contexts for teaching. The shift toward the use of contexts for teaching a difficult topic like integers is significant in itself. Further, we witnessed teachers constructing, for themselves, SCK for teaching integers through interconnecting different meanings associated with integers. Further evidence of the importance of the framework of meanings comes from teachers' take-up of resources and ideas from the workshop into their classrooms, and their self-reports concerning the relevance of integer meanings for teaching.

Analysis of the initial talk by the teachers in the CLP workshops regarding student errors and representations indicated gaps in the teachers' SCK in terms of their limited repertoire of representations, the explanations associated with their use of representations and making connections between representations. They spoke about student errors and their teaching concerns in a manner largely disconnected from issues of meaning and focused on procedures as conventions dissociated from meaning. They also explicitly disavowed that students' difficulties were with the meanings of integers.

In the initial discussion, the teachers indicated that they used contexts to introduce integers and the need for integers, but not for integer operations. Their initial preference was to teach operations through the use of formal models like the number line or the two-color buttons, or through rules. The teachers articulated a tension between choosing to explicitly teach rules and building on students' intuitive understanding of contexts. What explains the teachers not using contexts to teach operations? We hypothesize that their limited understanding of the meanings associated with integers constrained them not to use contexts to teach addition and subtraction, and also limited the explanations that they offered for operations performed on formal models like the two-color buttons or the number line. The teachers did not have a satisfactory explanation for why a "positive" button and a "negative" button cancel each other, or why subtraction of negative numbers on the number line involved tricky combinations of change of direction and movement.

Awareness and exploration of meanings in relation to contexts using the integer meanings framework led to an increase in the variety of situations that can be represented by integers. This contributes to increase in richness of the example space (Watson and Mason 2005) that teachers can access for generating tasks, guiding classroom interactions, and assessing learners' understanding. Appreciating the change meaning of integers allowed the teachers to use contexts where "changes" could be combined in the form of integer addition, leading one of the teachers to design a context of a bowl containing an unknown number of stones, to which stones could be added (an increase represented by a positive integer), or from which stones could be removed (a decrease represented by a negative integer). In analyzing a variety of contexts, the teachers used integers to represent derived quantities that were different from the salient quantities—for example, change in temperature as opposed to temperature, change in baby's weight as opposed to weight, and relative position in the mall as opposed to floor number. This led teachers to design and adopt such contexts where integers represent change and hence could be added meaningfully—hourly change in temperature, weekly change in a baby's weight and movement of an elevator in an "integer mall." Interpreting an integer as representing a "static relation" allowed further exploration of contexts and the possibility of modeling the subtraction operation using contexts. In the integer mall, for example, subtraction was used to find the movement required to move to a target floor from a given floor.

The teachers' movement from using integers to represent only states to representing transformation and relation is an important one, whose significance and challenge has been identified by other researchers (Thompson and Dreyfus 1988). This is related to the move from representing transformation using the subtraction operation to representing it using an integer. The teachers initially chose to represent change by means of the subtraction operation rather than using integers. Representing the *process* of change using an integer is an essential step—reifying a transformation into an object that can be represented as a number. In some accounts, this is at the heart of algebraic ways of thinking (Sfard 1991), which calls for flexibly interpreting symbols as representing both process and object. The move from representing transformations as operations to representing them as integers similarly reflects a flexible understanding of process-object duality.

Merely becoming aware of the various meanings of the minus sign, of integers and of integer addition and subtraction does not constitute SCK for teaching mathematics. Using the framework of meanings, teachers need to construct further elements of SCK by relating it on the one hand to teaching concerns and on the other to representations. We found evidence of three ways in which teachers constructed SCK for integers in the workshops. Firstly, as we have mentioned before, teachers identified features and processes associated with representations, especially contexts, that corresponded to one or the other meaning of integers. Secondly, teachers connected various meanings of integers through their insight about the key idea of a reference point. They noted that in contexts where a sequence of changes is represented by integers, the reference point is constantly shifting. They noted that to represent state using integers, they need to fix a "zero" as a reference point by convention, while to represent relations, the reference point is arbitrary. The teachers also made connections across different layers of meaning, by relating the distinction between the two meanings of the minus sign (integer and subtraction operation) to the distinction between the state and change meanings of integers. Ajay raised the question, also raised by Anita's students, as to why basement floors are marked with a minus sign. At the heart of this question is an important mathematical idea, namely that the sign used for the subtraction operation is also the sign for a negative integer. Using the insight about the connection between the state and change meanings, Anita was able to explain that the floor number was related to the amount of change needed to reach the floor from the reference point of the zeroth floor. Finally, the teachers used the framework of meanings to interpret student errors (the difficulty in extending the take away meaning of subtraction to "taking away" a negative integer), to offer explanations using representations (moving right on the number line corresponds to an increase) and finding new ways of modeling procedures for addition and subtraction using representations (subtraction using the neutralization model, or using the "integer mall").

The teachers' construction of knowledge by probing meanings associated with representations indicates the importance of understanding the distinctions and connections between the several meanings of integers. This suggests to us that a framework that distinguishes different meanings may function as a foundation on which further elements of knowledge relevant to teaching could be built. In our study, we have chosen to support this claim with a fine-grained description of teachers' construction of elements of SCK, rather than probe what individual teachers' gains in SCK using objective measures. Using measures such as a paper–pencil test would not have made it possible to capture teachers' construction in a detailed manner. However, we believe that on the basis of detailed descriptions of teachers' constructions, it may be possible to develop further measures of SCK that are detailed and specific.

We note that all our teachers were highly experienced, knowledgeable and resourceful. They had many years of teaching experience. They were aware of student errors and were familiar with the textbook and the curriculum. Given this fact, the lack of detailed attention paid to issues of meaning in the initial phase of the workshop was remarkable. It suggests to us that the knowledge encoded by the framework of integer meanings is an important part of SCK that is not gained directly through the practice of teaching alone. One reason for this might be that developing such distinctions and frameworks needs deep engagement with issues connected with both content and with learning of content. Hence SCK elements such as integer meanings may be important bridges between the knowledge acquired through mathematics education research and the knowledge that is essential for effective teaching.

The remarks above also suggest that SCK elements such as connections between meaning and representation are important for both pre-service and in-service teacher education. There is an under-emphasis on content knowledge in teacher preparation (Chazan and Ball 1999). Our analysis points to how such content could be designed, at least for certain topics in school mathematics. Exploring distinctions and connections among meanings of mathematical objects and processes, between mathematical objects and various representations, may be important to include as part of the mathematical knowledge required to prepare teachers. Such "framework of meanings" may be important not only for the topic of integers, but also for other topics such as fractions, and operations with fractions and whole numbers (Kieren 1988; Ma 1999; Fuson 1992).

Thus, the SCK elements identified in this article could be expanded to other topics, and could form the basis for work with pre- and in-service teachers, exploring ways in which teachers construct SCK using meaning frameworks as the foundation. Even within the topic of integers, the framework that we developed did not include the meanings associated with integer multiplication and division. This is work that remains to be done. Semantic interpretations of multiplication and division have been proposed by researchers (Schwartz 1988; Subramaniam 2013). Schwartz (1988) pointed out that a majority of multiplication word problems fit into the schema – intensive quantity/rate × extensive quantity = extensive quantity. For example, in the "integer mall" context, the rate may correspond to the speed of the elevator, the first extensive quantity to time duration, and the second extensive quantity to relative position. Both the speed of the elevator and the time duration may be either positive or negative. The expression $-3 \times (-2)$ may represent the position where the elevator was 2 s ago relative to its current position if it was traveling downward with a speed of 3 m/s. Contexts need to be probed for such meanings that may be associated with integer multiplication and division.

There has been extensive work by mathematics education researchers on the role played by representations in the learning of mathematics. It is increasingly acknowledged that teachers' understanding of representations associated with specific mathematical objects and processes must be deep to enable their effective use in teaching. Thus, deep knowledge of representations is recognized to be a critical component of SCK for teaching mathematics. This paper attempts to show how interventions can be designed for in-service, and by extension for pre-service teachers, that can lead to the construction of such knowledge.

Appendix 1: Worksheet 1

One of the difficulties that children face is in interpreting negative numbers. What does "-2" exactly mean? There are broad senses in which negative numbers or more generally integers (positive, negative numbers, and zero) are interpreted.

1. *As a change* Change includes increase or decrease, movement up or down (or forward and backward), or positive or negative growth (e.g., total annual sales of a company). Think of situations which involve change and can therefore be described using integers. The situations should be meaningful and interesting. Some suggested examples are given below. Think of more such examples.

   - ∞ Increase–decrease: Make a table of the weight gained by a baby every week (may be negative, what does it indicate?),
   - ∞ Movement forward/backward or up/down: Change in tennis ranking of a tennis player, change in run rate from over to over.

   Make a presentation of such data in a way that would be interesting to students.

2. *As a state* We can specify the state of something we are interested in using integers but only when it is meaningful to talk about positive and negative states.
   Think of such situations where integers represent sate. Some suggested examples are given below. Think of more such examples.

   - ∞ Position of a lift in a building which also has basement floors
   - ∞ Temperature of water in a freezer

   Again think of ways in which such situations can be presented in an interesting way to students.

3. *As relation between numbers and quantities* An important point here is that this is a directed relation. The relation makes sense if we distinguish the direction of the relationship and use positive and negative numbers to indicate it.
   Consider these two examples:

   - ∞ Me and my sister are standing in a queue to buy ice-cream. How far is my sister from me?
   - ∞ Me and my sister are on different floors of a tall building with several basement floor levels for parking. How far away is my sister from me?

   Why is it meaningful to give the answers to these questions using integers? Is there any difference between the two examples? Think of more examples where relations can be represented using integers.

Appendix 2: Class VI: integers

Daily Lesson Plan

Day 1

The chapter will be introduced using the DREAM MALL figure.

The child learns that to move upward there is a "+" button, and to move downwards "−" button is to be used. Using this idea, he can number the floors accordingly.

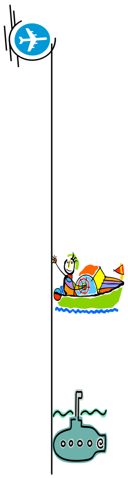
Movement problems

Suppose Sapna and Kiran are in the ice-cream parlor. Sapna wants to go to the movie hall, and Kiran wants to go for shopping. How many steps each would move?

Their attention can be brought to the point that each would move same number of steps but in different directions, they can answer using appropriate signs.

Discussion regarding the importance of using the correct sign will be done at this moment.

In the figure, the boat is on the sea level. The aeroplane is flying 2000 km above the sea level (*sic*). The submarine is at 800 km below sea level (*sic*). Express their distances from the sea level.



Have you seen numbers with "−" sign earlier?

Every day we see the weather report in a newspaper or a TV. Do you know there are places where the temperature is <0 °C?

(Refer Text Book page 154) for the list of temperatures of 5 places in India.

Integers: The first numbers to be discovered were natural numbers i.e., 1, 2, 3, 4,… If we include 0 in this collection, we get a new collection of numbers known as whole numbers. Now we find that there are numbers like −1, −2, −3, −4,… known as negative numbers. If we put the whole numbers and negative numbers together, the new collection will look 0, 1, 2, 3, 4, … −1, −2, 3, −4, … and this new collection is called integers.

References

Bajaj, R., & Kumar, R. S. (2012). A teaching learning sequence for integers based on real life context: A dream mall for children. In M. Kharatmal, A. Kanhere & K. Subramaniam (Eds.), *Proceedings of national conference on mathematics education* (pp. 86–89). Mumbai: HBCSE.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching what makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Carreño, E., Rojas, N., Montes, M. Á., & Flores, P. (2013). Mathematics teacher's specialized knowledge. Reflections based on specific descriptors of knowledge. In *Proceedings of the CERME* (Vol. 8, pp. 2976–2984). Antalya: ERME.

Carrillo, J., Climent, N., Contreras, L. C., & Muñoz-Catalán, M. C. (2013). Determining specialised knowledge for mathematics teaching. In *Proceedings of the CERME* (Vol. 8, pp. 2985–2994). Antalya: ERME.

Chazan, D., & Ball, D. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*(2), 2–10.

Cochran, K. F., DeRuiter, J. A., & King, R. A. (1993). Pedagogical content knowing: An integrative model for teacher preparation. *Journal of Teacher Education, 44*, 263–272.

Darling-Hammond, L., & Richardson, N. (2009). Research review/teacher learning: What matters. *Educational Leadership, 66*(5), 46–53.

Ernest, P. (1999). Forms of knowledge in mathematics and mathematics education: Philosophical and rhetorical perspectives. In D. Tirosh (Ed.), *Forms of mathematical knowledge: Learning and teaching with understanding* (pp. 67–83). Berlin: Springer.

Fuson, K. C. (1992). Research on learning and teaching addition and subtraction of whole numbers. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 53–187). Hilldale, NJ: Lawrence Erlbaum Associates Inc.

Glaeser, G. (1981). Epistemology relative numbers. *Research in Didactique of Mathematics, 2*(3), 303–346.

Hefendehl-Hebeker, L. (1991). Negative numbers: Obstacles in their evolution from intuitive to intellectual constructs. *For the Learning of Mathematics, 11*(1), 26–32.

Herbst, P., & Kosko, K. (2014). Mathematical knowledge for teaching and its specificity to high school geometry instruction. In J. Jane-Jane Lo, K. R. Leatham, & L. R. Van Zoest (Eds.), *Research trends in mathematics teacher education* (pp. 23–45). Berlin: Springer.

Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 41*(5), 513–545.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371–406.

Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162–181). Reston, VA: Lawrence Erlbaum Associates Inc.

Kumar, R. S., Subramaniam, K., & Naik, S. (2013). Professional development of in-service teachers in India. In B. Sriraman, J. Cai, K. H. Lee, L. Fan, Y. Shimizu, C. S. Lim & K. Subramaniam (Eds.), *Abstracts of the first sourcebook on Asian research in mathematics education* (pp. 207–211). Information Age Publishers.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40*(2), 173–196.

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics, 39*(1–3), 131–147.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates Inc.

Mitchell, R., Charalambous, C., & Hill, H. (2014). Examining the task and knowledge demands needed to teach with representations. *Journal of Mathematics Teacher Education, 17*(1), 37–60.

National Council of Educational Research and Training. (2005). *National curriculum framework*. New Delhi: NCERT.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41–52). Reston, Virginia: Lawrence Erlbaum Associates Inc.

Schwarz, B. B., Kohn, A. S., & Resnick, L. B. (1994). Positives about negatives: A case study of an intermediate model for signed numbers. *The Journal of the Learning Sciences, 3*(1), 37–92.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher, 15*(2), 4–14.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard educational review, 57*(1), 1–23.

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education, 43*(4), 428–464.

Subramaniam, K. (2013). Research on the learning of fractions and multiplicative reasoning: A review. In S. Chunawala (Ed.), *The epiSTEME reviews: Research trends in science, technology and mathematics education* (Vol. 4). New Delhi: Macmillan.

Thompson, P. W., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education, 1*(1), 115–133.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. Carpenter, J. Moser, & T. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 39–59). London: Lawrence Erlbaum Associates Inc.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction, 14*(5), 469–484.

Vlassis, J. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology, 21*(4), 555–570.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum Associates Inc.