CHAPTER 79

REPRESENTATIONS OF NUMBERS AND THE INDIAN MATHEMATICAL TRADITION OF COMBINATORIAL PROBLEMS

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ABSTRACT

This chapter provides an introduction to the mathematics associated with combinatorial problems that have their origin in music and prosody, which were studied by Indian mathematicians over the centuries starting from around the third century BC. Large parts of this mathematics are accessible without a knowledge of advanced mathematics, and there are several connections with what is learned in school or in early university education. The chapter presents expositions of such connections with, for example, binary arithmetic and Fibonacci numbers. In solving some of the problems, Indian mathematicians worked implicitly with the idea that all positive integers can be represented uniquely as sums of specific kinds of numbers such as the powers of 2, Fibonacci numbers and factorial numbers. These ideas are interesting, both in themselves and for the connections they make with aspects of culture, and hold promise for mathematics education and the popularization of mathematics.

Keywords: binary arithmetic, combinatorial problems, Fibonacci numbers, Indian mathematics, mathematics and music, mathematics and prosody

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REPRESENTATIONS OF NUMBERS AND THE INDIAN MATHEMATICAL TRADITION OF COMBINATORIAL PROBLEMS

The history of Indian mathematics has been an area of exciting new discoveries in recent decades. Fresh insights into the contributions of the Kerala mathematicians from the fourteenth to the seventeenth centuries CE are among the better known discoveries. Mathematical work from earlier periods too have been more thoroughly studied and better understood. Plofker (2009) provides a recent overview of the history of Indian mathematics. Several recent anthologies convey the excitement of current work in the field (see e.g., Emch, Srinivas, & Sridharan, 2005; Seshadri, 2010).

Our purpose in this chapter is to explore aspects of the history of Indian mathematics that may be of interest to the mathematics education community. Specifically we explore the work on combinatorial problems beginning from around the third century BC and continuing till the fourteenth century CE. The problems and the mathematical ideas developed by this tradition need only a level of mathematical knowledge available to many secondary school students. The ideas have interesting connections with Indian cultural forms, both living and historical and hence may appeal to a wider audience than those with a taste for mathematics. The material in this chapter is largely expository and draws heavily on the recent historical work and textual interpretations of among others, R. Sridharan, of whom the first author of this chapter is a collaborator (Sridharan, 2005, 2006; Sridharan, Sridharan, & Srinivas, 2010).

The development of numeral notation and forms in India provides a backdrop for the discussion of the connections of combinatorial ideas with number representations in this chapter. It is fairly well known that the decimal numeral system currently used had its origins in India and was transmitted to the West through contact with Arab culture. A decimal system of number names with Sanskrit names for the numbers from 1 to 9, and for powers of 10 up to a trillion was already developed in the second millennium BC and appears in the vedas, the oldest extant literature from India (Plofker, 2009). Large numbers were denoted by compounding names for 1 to 9 with names for powers of 10, much like in present day English. Besides these, once also finds in the vedic literature "concrete" number names, which are salient cardinalities (e.g., "moon" = 1, "eyes" = 2, "sages" = 7 from the well known *saptarṣi** or seven ancient sages). The earliest inscriptions in which a *positional* decimal numeral system is used, date to the second half of the first millennium CE. However

*First occurrences of Sanskrit or Tamil words are in italics. Subsequent occurrences are not italicized (except for names of texts) to enhance readability.

much earlier evidence for positional value exists in the form textual references. For example, the year in which a work was authored is mentioned as "*Viṣṇu* hook-sign moon." These are concrete number names where "*Viṣṇu*" (a leading diety) stands for one, "hook-sign" for nine (from the shape of the written numeral) and "moon" for one. Thus the number translates to "191" using decimal positional notation, a year measured in the Śaka era, which corresponds to the year 269 or 270 CE (Plofker, 2009). The order is actually right to left, which does not matter here since "191" is a palindrome.

It must be noted that there were other numeral systems in use through the centuries. Some of these were not based on positional value, like the alphanumeric system used by Āryabhaṭa, the author of the foundational astronomical-mathematical work *Āryabhaṭīya* written in 499 CE. In this system, consonants of the Sanskrit alphabet had specific numerical values depending on their position in the alphabet, while vowels indicated powers of 10 (Plofker, 2009). For example, the consonant "kh" had a value of 2, while the vowels "i" and "u" had respectively values of 10^2 and 10^4. Thus the syllable "khi" would mean 200, while "khu" would mean 20000. The syllable "ni" would denote 2000: "n" = 20 and "i" = 10^2. It was possible to denote the large numbers that are needed for astronomical calculations in a compact manner using Āryabhaṭa's notation, but the syllable-words that were produced were difficult to pronounce.

A more popular number system was the *positional value* based *kaṭapayādi* system in which consonants took numerical values from 0 to 9, depending on their order in the Sanskrit alphabet. The first ten consonants in the first two rows of consonants in the Sanskrit alphabet ("*k*" to "*ñ*") denote, in order, the digits "1" to "0". The next two rows also denote the same digits. So this system had redundancies—three or four consonants denoted the same digit, and vowels did not have numerical value. The redundancies allowed flexibility in the choice of a syllable combination to denote a number—often an actual Sanskrit word could be used to denote a number. Thus the word "*dhīra*" (meaning resolute or courageous) would denote 29, since "*dh*" denotes the digit "9" and "*r*" denotes "2". (Note that the order is right to left.) Of all these systems, the positional value based concrete number system described above was the most widely used in mathematical texts. For more details about the various number systems, see Plofker (2009).

COMBINATORICS IN MUSIC AND PROSODY

A rich tradition of combinatorial problems associated with the enumeration of symbol strings and mathematical techniques to solve them has existed in Indian mathematics for over two millennia. These problems have their origin not in a branch of science or technology, but in the arts—in prosody and in music. However, the mathematical ideas were pursued for their own sake as a distinct "mathematical" tradition, beyond the practical needs in poetry or music. While Sanskrit poetry largely belongs to the past or is pursued by specialized groups, Indian classical music is a living and vibrant aspect of contemporary Indian culture. (It must be noted that prosody in modern Indian languages is heavily influenced by Sanskrit prosody.) A penchant for classification and systematic organization is reflected in the classical musical traditions of India. These typically take the form of specifying an underlying basic structure or alphabet and enumerating melodic or rhythmic possibilities emerging from the basic structure. We will first look at this aspect of classical music, both with regard to melody and rhythm and then provide a brief introduction to these aspects in Sanskrit prosody. This introduction to cultural aspects provides a background for better appreciation of the mathematical discussion that follows. However, the mathematical sections can be understood independent of this background.

Combinatorics in Karnāṭak music. The two great streams of classical music in contemporary India are Karnāṭak and Hindustāni music. They share many characteristics and similarities, although the musical compositions, musicians and serious audiences are largely separate groups. We give a brief introduction to the Karnāṭak musical tradition, whose geographical center is in South India, to highlight the role of combinatorial structure in its melodic and rhythmic forms.

The melodic forms that provide the basis for both Karnāṭak and Hindustāni music are called *rāgas*, which are roughly analogous to scales in Western music. The basic specification of a rāga is in terms of the ascending and descending sequence of notes (*svaras*) in the scale. The notes are expressed using the seven "solfege" syllables of Indian classical music, which are pronounced as "Sa, Re, Ga, Ma, Pa, Dha, Ni", and commonly notated in writing using the first letter. These are short forms for the names of the notes: *Śaḍja, Riṣaba, Gāndhāra, Madhyama, Pancama, Dhaivata and* [illegible]*da*. The tonic Sa (or Śaḍja) is fixed arbitrarily, and the remaining notes have a specific tonal relation to the tonic. The notes have higher and lower tonal values, which are shown in Table 79.1. However, the seven notes correspond only to 12 distinct tone positions because of overlaps. There are four pairs of duplicate names for the same position: R2=G1, R3=G2, D2=N1, D3=N2. It must be noted however that in Karnāṭak music as it is actually performed, the tonal values of the svaras are flexible (Krishnaswamy, 2003).

Table 79.1. Notes (Svaras) and Tonal Values (*Śrutis*) in Karnāṭak Music

Note	*Tonal Values*
Sa (tonic) & Pa (fifth)	Fixed
Ma (fourth)	Higher, lower (M1, M2)
Re (second), Ga (third), Dha (sixth) & Ni (seventh)	Higher, middle, lower (R1, R2, R3, G1, G2, G3, D1, D2, D3, N1, N2, N3)

The basic rāgas of Karnāṭak music, called the "*meḷakarta*" rāgas, always have the seven notes in the correct order in the ascending and descending sequences. The meḷakarta rāgas are the "mother" rāgas, from which other rāgas are derived ("born") by omitting some notes, varying the sequence of the notes, or by interpolating notes from other rāgas. By taking different combinations of the *distinct* tonal values of the seven notes, a total of 72 meḷakarta rāgas are obtained in the following manner: 6 (number of possibilities for R-G combinations) × 6 (number of possibilities for D-N combinations) × 2 (number of possibilities for M) = 72. It is interesting to note that the enumeration of the meḷakarta rāgas has a fixed order determined by the sequence in which the tonal values are varied. For example, the well known rāga "Śankarabharanam", which is analogous to the major scale in Western music, is number 29 in the meḷakarta sequence and has the following notes: S, R2, G3, M1, P, D2, N3.

The enumeration of the meḷakarta combinations provides a convenient organization of the alphabet and vocabulary of Karnāṭak music. A useful mnemonic system exists to identify the sequence number of a meḷakarta rāga. The rāgas have formal names (sometimes different from the common names) where the first two syllables in the name gives its number in the kaṭapayādi numeral system, mentioned in the previous section. For example the formal name using the kaṭapayādi for the Śankarabharanam rāga is "Dhīra Śankarabharanam", where "dhīra" in the kaṭapayādi system denotes 29. However, it must be said that the complete list and exact order of the meḷakarta rāgas are rarely emphasized in musical training, and are present largely as background reference.

Rhythm and numbers. The rhythmic basis (*tāla*) of Karnāṭak music is similarly provided by an alphabet consisting of finger tapping, and hand clapping and waving gestures. A vocalist almost always keeps rhythm using these gestures even while performing. In the most familiar tāla system of Karnāṭak music, there are seven basic combinations of these gestures. These coupled with five forms of the tapping gesture gives a system of 35 tālas, analogous to the system of the meḷakarta rāgas. However, unlike the meḷakarta rāgas, this system has little correspondence with the actual rhythmic structure used in most compositions. Only 3 of the 35 tālas are commonly used, and two other commonly used tālas do not find a place in the table of the 35 tālas.

The striking beauty and complexity of rhythm in Indian classical music derives from exploiting combinatorial possibilities in rhythm in another sense. The player of a percussion instrument like the *tabla* in Hindustāni music and the mṛdaṇgaṃ in Karnāṭak music acquires, over time, a large stock of rhythmic phrases, which can be combined in creative ways to fit into the structure of a tāla. A striking aspect of both Hindustāni and Karnāṭak music is that complex rhythm patterns are both spoken and played on the instrument. The spoken form, called "*solkaṭṭu*" (literally "bundle of words") in Karnāṭak music consists of sets of syllables, each of which corresponds to and has a sound similar to a stroke played on the drum. Thus one may find a complex and intricate rhythm piece, several minutes long, first verbally recited in full, and then played exactly on a tabla or mṛdaṇga? Even when a percussionist trains, both forms are learned: "Throughout my training, I learned literally everything in two forms, spoken and played" (Nelson, 2008, p. 3).

The rhythm player in Indian classical music plays both solo and accompanies a vocalist or instrumentalist. It is in solo performance (often fitted into a vocal or instrumental concert) that the percussionist displays his (rarely, her) full repertoire and skill. Rhythm pieces are built up from complex phrases and sentences, which in turn are built up from a set of basic phrases and the use of rests or pauses. The basic rhythm phrases are easily recognizable to most people familiar with Indian music. For example, common four syllable phrases in Karnāṭak music are "ta ka di mi" and "ta ka jo ṇu"; a five syllable phrase might be "ta di ki ṭa tom"; a seven syllable phrase might be built as a combination of four and three—"ta ka di mi ta ki ṭa" or as a combination of two and five—"ta ka ta di ki ṭa tom".

The tāla structure provides the basic framework in which phrases are set and played. For example, the most commonly used tāla, the *ādi tāla*, consists of eight beats per cycle. Each beat is typically split into pulses, which may follow binary splits—2, 4 or 8 syllables per beat, or may follow splits based on three—3, 6, 12 syllables per beat. The percussionist designs a piece stringing together stock phrases and rests to cover several cycles of the tāla, creating contrasts, tensions and resolutions. The player often improvises on the fly while playing out a designed piece. The design and improvisation are called *kaṇakku* (literally "calculation"). Since the pulses, beats and cycles of the tāla must synchronize at crucial points during a piece, calculation and arithmetic are fundamental to the percussionists design and performance. Examples of simple and complex rhythm pieces for solo playing can be found in Nelson's *Solkaṭṭu Manual* (2008) and also in the solkaṭṭu recordings available on the web.

The classical dance traditions in India, which are also a live and vibrant aspect of the culture, share the tāla system and structure of percussion music. The syllables and spoken phrases are also a basic part of classical dance. Rhythmic compositions are often spoken out, much like in percussion music, before being performed as dance. The syllables used are similar with slight variations. Besides the classical traditions of music and dance, there are many vibrant traditions of folk music and dance spread across different regions in India. There is no doubt that the classical and the folk traditions influenced each other over the centuries. Hence it is possible that some of the aspects discussed above have corresponding features in the folk traditions. It is more than likely that research on these aspects will reveal interesting connections with numbers and mathematics.

Sanskrit prosody. The oldest extant text in Sanskrit is the *Ṛg Veda* from the second millennium BC. The four vedas, of which the *Ṛg Veda* is the oldest, are composed in specific metrical forms and have been preserved largely through an oral tradition centered around sacred ritual. The earliest authoritative discussion of these metrical forms is the work on prosody by Piṅgala from the mid-third century BC (Sridharan, 2005). The vedic metrical forms are classified on the basis of a count of the number of syllables. One of the most widely used metrical forms from the later vedic to classical periods, is the *anuṣṭubh*, a verse composed in four lines (*pādas*), each of which contains eight syllables. For example, the opening lines in anuṣṭubh verse of the Bhagavad Gita are

g g g g l g g g
dharmakshetre kurukshetre
l l g g l g l g
samaveta yuyutsavah

In the lines quoted above, each syllable is marked following the rules of Sanskrit prosody with a "*l*" or a "*g*", which corresponds to a light or a heavy syllable (*l* = *laghu* – literally "light" meaning short, *g* = *guru* – literally "heavy" meaning long). All Sanskrit poetry has the structure of the light and heavy syllables. Since there are no accents in the Sanskrit language, the meter is determined by the structure of the light and heavy syllables. The anuṣṭubh form has the number of syllables in a pāda fixed at eight, but the number of time units or "morae" is not fixed. Hence the duration taken to speak different lines of the anuṣṭubh stanza may be different. Many of the classical metrical forms have a fixed number of morae instead of fixed syllabic length, where the light syllables have a value of one mora and the heavy syllables a value of two morae.

A basic question that arises with regard to a metrical form is how many different possibilities there are of a given form. How many different combinations of laghu and guru syllables are possible when the form has a fixed syllabic length of n syllables? It is easy to see that this is 2^n since there are two possibilities (l and g) for each syllable. Similarly one can ask how many possibilities exist if the line has a fixed moraic length. As far as is known, the first text to deal with such problems is possibly Piṇgala's *Chandaḥ-sūtra*, which deals with enumerating metrical forms of a given syllabic length. Piṇgala's date is uncertain but it is possible that he lived around the time of Pāṇini in the third century BC. There was a connected (if not continuous) tradition of mathematical work on the problems related to prosody and music, that reached a mature form in the work of Nārāyaṇa Paṇḍiṭa in the fourteenth century CE.

There are several aspects of this tradition that are of potential interest to the mathematics education community. The first is that the mathematics associated with these combinatorial enumeration problems is interesting even from a contemporary perspective, and hence unexpectedly deep. At the same time, large parts of it are accessible without a knowledge of advanced mathematics, and there are several connections with what is learned in school or in early university education. The second is that numbers in the context of these problems primarily represent not quantity but serial (ordinal) position. That the mathematics associated with such representations can be interesting is a fresh and different perspective that may enrich students' experience of numbers. Finally, the methods used to solve these problems rely on uniquely representing positive integers in a variety of ways, which are vast and interesting extensions of the familiar representations of numbers in base ten or other bases. In the subsequent sections, we explore the mathematical aspects of this tradition.

THE FOUR PROBLEMS RELATED TO COMBINATORICS OF METRICAL FORMS

One of the basic questions that arise in the context of a poetic or musical form is what possibilities there are of a given type. Consider a syllable string consisting of exactly three syllables, each of which may be light or heavy. What are the syllable forms that are possible? This is the first problem discussed by Piṇgala. Piṇgala arrives at the fact that there are 2^n possibilities for a string of length n, by first enumerating the forms in a systematic manner. The systematic enumeration of forms is called "*prastāra*". Piṇgala discusses six problems associated with such forms, of which we focus on the following four problems in this chapter.

1. *Prastāra*: What are the combinations of light and heavy syllables that are possible for a given length of syllables? How do we enu-

merate all possibilities in order? What is the rule that allows one to carry out this enumeration?

2. *Saṇkhyā*: How many combinations are possible for a given syllabic length?
3. *Uddiṣṭa*: Given a string in the enumeration, how can one obtain the exact number of this string in the enumeration sequence?
4. *Naṣṭa* (converse of Uddiṣṭa): Given a number in the enumeration sequence, how can one obtain the string corresponding to this number?

We also find in Piṇgala a treatment of the *Lagakriya* problem, that is, to find the number of metres of a given length with a specified number of gurus (or equivalently, laghus). This problem, which we shall not discuss in this chapter, gives rise to the construction of what is now known as the Pascal's triangle (Sridharan, 2005).

First, we discuss the problem of enumeration or generating the prastāra for syllable strings of length n. To simplify the exposition we use the letters "a" and "b" to stand for heavy (g) and light (l) syllables respectively. Also, we have adopted a left to right convention because of the familiarity of dictionary order, which is the reverse of the convention adopted by Piṇgala. Table 79.2 gives the complete set of two, three and four letter "words" made from the letters "a" and "b". Notice that the lists are in dictionary order.

We see that each prastāra or enumeration can be obtained from the previous one by a recursive rule. To get the list of two letter words, we first prefix "a" to all the one letter words to get half of the two letter words, then prefix "b" and get the remaining half. Similarly to get the list of three letter words, we prefix "a" to the two letter words to obtain four of the three letter words, and prefix "b" to obtain the remaining four. The recursive rule actually follows from the fact that the order of enumeration is exactly the dictionary order. The rule allows us to generate the entire list from the previous one. However, it is not local enough to allow us, given a line in a particular prastāra, to generate the next line. For example, one may ask, which string comes just after "bab" in the prastāra of

Table 79.2. "Words" From a Two Letter Alphabet

One Letter Words	*Two Letter Words*	*Three Letter Words*	
a	*aa*	*aaa*	*baa*
b	*ab*	*aab*	*bab*
	ba	*aba*	*bba*
	bb	*abb*	*bbb*

three letter words? Piṇgala, as interpreted by later commentators, gives a rule to solve this problem, which amounts to the following. Going from right to left, change the first "*a*" that you encounter to a "*b*", replace all the letters to its right with a string of "*a*"s, and leave the rest of the string unchanged. The rule gives "*bba*" as the string immediately following "*bab*." It can be checked if this rule applies to all the lines in the prastāras in Table 79.2. (Note the similarity with the procedure for adding one when the numbers are expressed in the binary system with "*a*" standing for 0 and "*b*" for 1.)

We can see that the length of each prastāra is double that of the previous one, arriving at the fact that the length of a prastāra for a string of n syllables is 2^n. This is the saṅkhyā problem. We consider next the problem of uddiṣṭa by considering the following example: what is the exact position of the string "*bba*" in the prastāra for three letter words? The following line of reasoning allows us to construct a rule to solve this problem.

First, we assign numbers in the sequence starting from "0" instead of "1". Thus "*aaa*" occupies the zeroth position in the sequence of three letter words.

Since the first letter of the word *bba* is *b*, it cannot occur in the first four words of the prastāra. Its position *within* the last four is exactly the same as the position of *ba* in the prastāra for two letter words. In other words, the position of *bba* is $4+x$, where x is the position of *ba* in the two letter sequence. This gives us a recursive rule, since the position of *ba* in the two letter sequence is $2+y$, where y is the position of a in the one letter sequence, which is in fact zero. Thus we arrive at the position of *bba* as $4+2+0=6$. By substituting "1" for "*b*" and "0" for "*a*", we realize that "*bba*" is actually the binary representation of the number 6: "110", and 6 can be obtained by adding the powers of 2 with the digits as coefficients: $1\times2^2 + 1\times2^1 + 0\times2^0$. If we wish to enumerate the sequence in the natural manner from 1 to 8, then we need to increment this number by one. The position of *bba* in the sequence is then 7.

The problem of naṣṭa, which is the converse of the uddiṣṭa problem, is to obtain the string from the number that gives its position in the sequence, *given the total number of syllables*. The rule can be explained by taking the same example as above, and asking what is the string in the 7th position (using the natural numbering from 1 to 8) in the prastāra of three letter words? We arrive at this by the following rule: if the number is odd, add 1 and halve the number, write "*a*". If the number is even, halve the number and write "*b*". For the next step write the letter to the left of the previously obtained letters. So for the first step, we add $7+1=8$, halve 8 to obtain 4, and write "*a*". For the next step, 4 is even, so we halve 4 to get 2, and write "*b*" to the left of "a". Next, since 2 is even, we halve it to obtain 1 and write "*b*" to the left. Since we have obtained three letters we stop. If the string length is more than three, we continue the process till the correct number of syllables are obtained. (Note that once we obtain "1", the syllables for all the subsequent steps will be "a"s.) Reading the letters obtained, we get "*bba*".

We can also apply this rule to a number larger than 8, say, 9. Since 9 is odd, add 1, halve 10 to obtain 5 and write "*a*". In the next step, since 5 is odd, add 1 and halve to obtain 3, write "*a*" to the left. In the next step, add 1 and halve to obtain 2, write "*a*" to the left. In the next step, halve to obtain 1 and write "*b*" to the left . In this way, we obtain the string "*baaa*" as the 9th in the prastāra for four letter words. Note that in every case, we can also obtain the string by decrementing the number by 1, writing its binary representation, and substituting "*a*" for "0" and "*b*" for "1".

How does the procedure described above for solving the naṣṭa procedure work? We can understand this by considering another way to obtain the prastāra for three letter words from that of the two letter words, which is the following. Take the first word in the two letter prastāra, namely, "*aa*". We alternately append an "*a*" and a "*b*" to the *right* to get the first two words of the three letter prastāra. That is, we get "*aaa*" and "*aab*". Similarly to get the next two words we alternately append an "*a*" and a "*b*" to the *right* of the second word in the two letter prastāra, namely "*ab*". Thus we get "*aba*" and "*abb*". We can verify that these four words are the first four words in the three letter sequence. A little thought reveals that this is just another way of preserving the dictionary order as one moves from two to three letters. In general, we can get the $k+1$ letter sequence from the k letter sequence, by taking in order each row in the k letter sequence and generating two rows for the $k+1$ letter sequence by appending first an "*a*" and then a "*b*" to the right.

Hence from each word in the k letter sequence, we get two words in the $k+1$ letter sequence. More precisely, from the n^{th} word in the k letter sequence, we get the $(2n–1)^{\text{th}}$ word – which always ends in an "*a*" – and the $(2n)^{\text{th}}$ word—which always ends with a "*b*"—in the $k+1$ letter sequence.

Let us now try to understand the naṣṭa procedure for three letter words. We know that if the row number is odd, it will be of the form $2n$-1, and the string corresponding to this row number will always end with an "*a*". Now we add one and divide by 2, which gives us n. By recursion, the next step is to find the string corresponding to the row number n, in the prastāra for two letter words. If the original row number is even, it is of the form $2n$, and the string corresponding to this row number will always end with an "*b*". Now we divide by 2, which gives us n. By recursion, the next step is to find the string corresponding to the row number n, in the prastāra for two letter words.

From the solutions to the uddiṣṭa and naṭṭa problems, we can set up a one to one correspondence between numbers and strings, which Indian mathematicians were clearly aware of. From the recognition of such correspondence, it is a major leap to ask the question in what sense each word in the prastāra of three letter words in Table 79.2 represents the numbers 1 to 8. The answer clearly is that the words are just binary representations of numbers (0 to 7 in the standard binary representation instead of 1 to 8). It is not clear if Piṅgala or other mathematicians actually saw the strings as we see them now, that is, as representations for numbers. However, one wonders what really lay behind the interest in the uddiṣṭa and naṣṭa problems, which do not have any apparent practical significance.

To summarize, we note that Piṅgala provides rules for (i) obtaining the prastāra for syllabic meters, that is, meters of a fixed syllabic length in terms of strings of a binary alphabet (ii) obtaining the total number of such strings (meters) (iii) obtaining the sequence number of a given string and (iv) obtaining the string from its number in the sequence. A rule that generates a unique sequence of strings (prastāra) allows one to formulate the uddiṣṭa and naṣṭa problems. The rules obtained as solutions to the problems of uddiṣṭa and naṣṭa set up a one-to-one correspondence between strings and numbers, allowing the possibility of interpreting the strings as representations of numbers. In the next section, we consider these four problems in the context of a different type of verse—verses with fixed moraic length.

Mātrāvṛttas and Fibonacci Numbers

The metrical forms that we considered in the previous sections were those with fixed syllabic length, where the syllables could be either light or heavy. In this section we consider metrical forms called mātrāvṛttas, where the number of morae, or time units (*mātrās*) is fixed. Here too, syllables may be light (*l*) or heavy (*g*), with *l* having a value of one time unit and *g* a value of two time units. Thus a meter of the form *llg* would be 4 units long; so would a meter of the form *gg* or *llll*. The four problems discussed in the previous section (prastāra, saṅkhyā, uddiṣṭa and naṣṭa) can be posed with regard to the mātrāvṛttas. The solutions to these problems for the mātrāvṛttas are found in the work of Indian mathematicians beginning with Virahāṅka in the seventh century CE. Table 79.3 presents the prastāras for mātrāvṛttas of lengths 1 to 6.

From Table 79.3, we can see how a recursive rule allows us to generate the prastāra for a given length from the previous two prastāras. For example, to generate the prastāra of length 4, append a "*g*" at the end of each string in the prastāra of length 2, and an "*l*" at the end of each string in

Table 79.3. Prastāras for Mātrāvṛttas of Different Moraic Lengths

Total Length = 1	*Total Length = 2*	*Total Length = 3*	*Total Length = 4*	*Total Length = 5*	*Total Length = 6*	
l	*g*	*lg*	*gg*	*lgg*	*ggg*	*lggl*
	ll	*gl*	*llg*	*glg*	*llgg*	*glgl*
		lll	*lgl*	*lllg*	*lglg*	*lllgl*
			gll	*ggl*	*gllg*	*ggll*
			llll	*llgl*	*llllg*	*llgll*
				lgll		*lglll*
				glll		*gllll*
				lllll		*llllll*

the prastāra of length 3. Thus the number of strings in the prastāra of length 4 is the number of strings in the prastāra of length 3 plus the number of strings in the prastāra of length 2. Using the notation S_n for the number of strings in the prastāra of length n, we can write

$$S_4 = S_3 + S_2$$

Generalizing, we have, $S_n = S_{n\text{-}1} + S_{n\text{-}2}$

This is exactly the recursive relation for the so-called Fibonacci numbers. Virahāṅka's text may well be the first to write down the recurrence relation for the Fibonacci numbers, although it may have been known earlier. The recurrence relation gives the solution to the saṅkhyā problem of finding the number of strings in the prastāra of length n. The discussion of the mathematics associated with such prastāras was a continuing tradition. Later mathematicians, to name a few, such as Halāyudha, Kedārabhaṭṭa and Hemacandra, discussed these problems. Like in the case of the varṅa prastāras, the solutions to the problems of Uddishta and naṣṭa, are also discussed for the mātrāvṛttas. The mathematical rationale underlying the solutions is the fact that any positive integer is either a Fibonacci number or can be expressed uniquely as a sum of non-consecutive Fibonacci numbers (Sridharan, 2006). This can be easily checked by the following argument. Take any positive integer N_0. If N_0 is a Fibonacci number, we stop since the unique sum is the number itself. If N_0 is not a Fibonacci number, then there is a largest Fibonacci number S_n such that $S_n < N_0$. Now consider the number $N_1 = N_0 - S_n$. Since $S_{n+1} > N_0$, we have $S_n + S_{n\text{-}1} > N_0$ or $S_{n\text{-}1} > N_1$. We repeat the process for N_1. The construction ensures that the process will yield a unique S_n at each step and will terminate since the sequence is strictly decreasing. The number N_0 is the sum of all the Fibonacci numbers obtained in this manner. The process also ensures that we do not obtain consecutive Fibonacci numbers at any stage. Thus every positive integer can be expressed uniquely as a sum of non-consecutive Fibonacci numbers.

The naṣṭa problem can now be solved using a simple algorithm. Suppose we want to find the string corresponding to the sequence number 7 in the prastāra for mātrā length of six (see Table 79.3). We note that the prastāra for a mātrā length of six units has 13 strings. We first subtract 7 from 13, which is the total number of strings. The result 6, is to be expressed as a sum of Fibonacci numbers, which we obtain as 5 + 1. Now we write down all the Fibonacci numbers from 1 to 13 and write down an "*l*" or a "*g*" below each of them using the following rule. For all Fibonacci numbers that appear in the expression, we write down a "*g*" below this number and skip the next Fibonacci number (put a "dash" below it). Below all the remaining Fibonacci numbers write down an "*l*". So for 6=5+1, we write a "*g*" below 1 and put a dash below 2. We also write a "*g*" below 5 and put a dash below 8. We write *l* below the remaining Fibonacci numbers. As seen below, we obtain the string as: *glgl*.

1	2	3	5	8	13
g	–	*l*	*g*	–	*l*

We verify from Table 79.3 that this is the seventh string in the prastāra for 6 mātrās. We consider one more example: what is the string that is number 4 in the prastāra for 6 mātrās? First we subtract 4 from 13, this gives 9. Next we express 9 as a sum of Fibonacci numbers. We obtain 9 = 8 + 1. Now we apply the rule and write "*l*" and "*g*" below each of the Fibonacci numbers from 1 to 13 in the following manner:

1	2	3	5	8	13
g	–	*l*	*l*	*g*	–

We obtain the string "*gllg*" as the fourth string in the prastāra, which can be verified from Table 79.3. We leave it to the reader to verify the algorithm in other cases. As can be seen, the algorithm depends on the fact that each number can be expressed as a sum of Fibonacci numbers uniquely, where there are no consecutive Fibonacci numbers. The uddiṣṭa problem needs one to proceed in the converse direction. The reader is refered to the article by Sridharan (2006) for an exposition of the uddiṣṭa rule.

It is well known that Fibonacci numbers occur widely in many natural contexts. The mātrā prastāra provides a context for grasping the recurrence relation among Fibonacci numbers that is accessible. The idea that Fibonacci numbers form a "base" in which all positive integers can be expressed may be a surprising and interesting fact for many students. One can use a string of zeros and ones to represent a number in this "base", where the position of the ones indicate the Fibonacci numbers which appear in the sum. There will, of course, be no consecutive ones in this number representation.

EXTENSIONS AND OTHER PRASTĀRAS

For the two prastāras discussed earlier (Tables 79.2 and 79.3), we write down the recursive relation and note the similarity in the two relations.

$S_n = S_{n\text{-}1} + S_{n\text{-}1} = 2 \times S_{n\text{-}1}$ (varṇa prastāras, Table 79.1)

$S_n = S_{n\text{-}1} + S_{n\text{-}2}$ (mātrā prastāras, Table 79.2)

The recursive relations suggest different kinds of extensions. One possible extension is of the form

$S_n = S_{n\text{-}1} + S_{n\text{-}1} + S_{n\text{-}1} = 3 \times S_{n\text{-}1}$

This gives rise to the ternary sequence or powers of 3: 3^0, 3^1, 3^2... One can obtain unique representations of positive integers using powers of three, which would correspond to the canonical base 3 representation. Such prastāras are discussed in the work of Nārāyaṇa Paṇḍiṭa in the fourteenth century. In fact, Nārāyaṇa Paṇḍiṭa discussed such relations in their general form (i.e., corresponding to base n representation) (Singh, 2001). However, we do not discuss these any further in this chapter.

An extension of the recursive relation for mātrā prastāras, that is, the Fibonacci relation, could be

$S_n = S_{n\text{-}1} + S_{n\text{-}2} + S_{n\text{-}3}$

The numbers obtained through this recursive relation and the associated mathematics were again discussed by Nārāyaṇa Paṇḍiṭa. Here too, he analysed the most general form of this relation ($S_n = S_{n\text{-}1} + S_{n\text{-}2} + \ldots + S_{n\text{-}q}$), where q is an arbitrary number less than n. Oddly enough, another recurrence relation was analysed by the musicologist Śārṇgadeva before Nārāyaṇa Paṇḍiṭa, who studied the problem for its connection not to prosody, but to rhythm or tāla patterns. In prosody, we considered time units with values of 1 and 2, the laghu and the guru respectively. In the context of tāla patterns, Śārṇgadeva considers four time units: *druta*, laghu, guru and *pluta*. The druta is half the duration of a laghu and a pluta is thrice the duration of a laghu. Re-adjusting to whole number values, we get the following values for the four units: druta – 1, laghu – 2, guru – 4, and pluta – 6. The question that may be asked is what combinations are possible for a sequence of units that has a total duration of say, seven time units. We note that the complete list of such combinations can be derived in the following manner:

- Append a pluta (P) at the end of all strings of duration 1 unit.
- Append a guru (G) at the end of all strings of duration 3 units
- Append a laghu (L) at the end of all strings of duration 5 units
- Append a druta (D) at the end of all strings of duration 6 units

The tāla sequences obtained by applying the algorithm are shown in Table 79.4. The recursive relation for the prastāra of 7 units can be written therefore as $S_7 = S_6 + S_5 + S_3 + S_1$. Generalizing, we get

$$S_n = S_{n-1} + S_{n-2} + S_{n-4} + S_{n-6}$$

The numbers S_n obtained using this recursive relation have been called Śārṅgadeva numbers in analogy with the Fibonacci numbers (Sridharan, Sridharan & Srinivas, 2010). The above recursive relation allows one to solve the saṅkhyā problem, namely, to find the number of strings for a given total duration. Śārṅgadeva also provides solutions to the naṣṭa and uddiṣṭa problems. As one may guess, these depend on the fact that every positive integer can be uniquely expressed as a sum of Śārṅgadeva numbers. We do not discuss the mathematical aspects of the Śārṅgadeva number representation in this chapter and the interested reader is referred to Sridharan, Sridharan and Srinivas (2010). As mentioned, Nārāyaṅa Paṅdiṭa in his *Gaṅitakaumudī* of 1356 CE discusses general recurrence relations of this from a purely mathematical point of view unconnected to applications in prosody or music. Nārāyaṅa Paṅdiṭa's work brings this tradition to its culmination.

Table 79.4. Tāla Combinations of a Total Duration of 7 Units

1	DP	9	GDL	17	DDGD	25	DLLDD
2	DLG	10	LLDL	18	GLD	26	LDLDD
3	LDG	11	DDLDL	19	LLLD	27	DDDLDD
4	DDDG	12	DLDDL	20	DDLLD	28	GDDD
5	DGL	13	LDDDL	21	DLDLD	29	LLDDD
6	DLLL	14	DDDDDL	22	LDDLD	30	DDLDDD
7	LDLL	15	PD	23	DDDDLD	31	DLDDDD
8	DDDLL	16	LGD	24	DGDD	32	LDDDDD
						33	DDDDDDD

The prastāras discussed by Śārṅgadeva which are related to rhythm forms are called tāla prastāras. Śārṅgadeva also considers tāna prastāras, or combinations of musical notes. An example of tāna prastāras considered by Śārṅgadeva is the enumeration of all phrases containing the svaras S, R, G, M (the first four notes of the seven-note musical scale of Indian classical music discussed earlier), where each svara occurs only once. Śārṅgadeva describes a rule for constructing the rows of the prastāra, the number of rows being given by 4 factorial (4!). The prastāra is shown in Table 79.5.

Śārṅgadeva discusses the saṅkhyā, naṣṭa and uddiṣṭa problems for the tāna prastāras. The solution to the latter two problems are based on fact that any positive integer m less than or equal to n! can be uniquely represented as follows:

$$m = d_0 0! + d_1 1! + d_2 2! + \ldots + d_{n\text{-}1}(n\text{-}1)!$$

Where d_i are integers such that $d_0 = 1$ and each d_i lies between 0 and i both inclusive. This is a variant of the general form for the factorial representation of integers (Sridharan, Sridharan, & Srinivas, 2010).

CONCLUDING REMARKS

In the preceding sections, we have discussed the generation of string sequences or prastāras in the context of prosody and music. We considered four kinds of prastāras. Two of these were discussed in greater detail: the varna prastāras, which are the combinations for verses having a fixed length of syllables, and the mātrā prastāras, which are the combinations where the moraic length is fixed. Two more prastāras were discussed briefly, those associated with rhythm forms (tāla prastāras) and those associated with combinations of notes (tāna prastāras). For each of these prastāras, four problems can be considered:

- Prastāra: the rule for generating the prastāra itself,

Table 79.5. Tāna Prastāra for the First Four Notes

1. S R G M	7. S R M G	13. S G M R	19. R G M S
2. R S G M	8. R S M G	14. G S M R	20. G R M S
3. S G R M	9. S M R G	15. S M G R	21. R M G S
4. G S R M	10. M S R G	16. M S G R	22. M R G S
5. R G S M	11. R M S G	17. G M S R	23. G M R S
6. G R S M	12. M R S G	18. M G S R	24. M G R S

- Saṅkhyā: the total number of combinations or strings in the prastāra
- Uddiṣṭa: obtaining the sequence number of a given string
- Naṣṭa: recovering the string when the sequence number is given.

As discussed, the last two problems are of particular importance, since their solution involves decomposing any given positive integer uniquely into numbers of a particular form. In the case of the varna prastāras, the numbers were powers of two. For the mātrā prastāras, these were Fibonacci numbers. For the tāla and the tāna prastāras, these were the Śārṅgadeva and the factorial numbers respectively. All of these lead to different kinds of unique representations for the positive integers. The idea that the base 10 system of number representation is only one among many different kinds of representations is a powerful idea that is made accessible by the consideration of the combinatorial problems such as those discussed by Indian mathematicians. The fact that these problems are associated with cultural forms—music, dance and prosody—that are still a living part of our experience can bring these domains closer to mathematics. The historical perspective on Indian mathematical traditions suggests the mathematics "embedded" in these cultural forms did not remain merely implicit, but were explored explicitly by mathematicians, and led to the development of a productive tradition of combinatorial problems within mathematics. We believe that the discussion of the mathematics associated with such problems holds promise in mathematics education and in the popularization of mathematical ideas. The details of how connections can be made between school mathematics and historical traditions, such as the one that we have discussed in this chapter, requires both further research and more work with learners of mathematics.

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Note: For a more exhaustive set of references, see Sridharan, Sridharan and Srinivas (2010).

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