Supporting students in making the transition from arithmetic to algebra

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Overview

- Why is the transition from arithmetic difficult? Why is it important?
- What is the difference between arithmetic and algebraic approaches to solving problems?
- Reasoning is the engine of mathematical learning.
- Algebra is about reasoning with symbolic expressions.
- But reasoning about symbolic expressions must precede reasoning with symbolic expressions.
- Supporting students with conceptual tools to reason about symbolic expressions: this will help in the transition to algebra

From arithmetic to algebra

- The transition can indeed be hard for many students.
- This is due to the fact that symbols are handled differently in arithmetic and algebra.
- Example: the conjoining error
  - $5 + 2x = 7x$ *
  - $5x + 2y = 7xy$ *
- Why is this error so common?

Explaining the conjoining error

- Reason 1: Interpretation of the "=" sign
  - Students think that it means calculate and write the answer, a habit carried over from arithmetic.
  - Young students find these sentences very odd:
    - 8 = 5 + 3 or 8 = 8
- Reason 2: Answers must have a closed form
  - $5 + 2x = 5 + 2x$ is not an acceptable response.
  - In arithmetic, operation signs are triggers for calculation and arriving at a closed form.
  - In algebra, the same symbolic expression can represent both a calculation procedure and the result of the procedure: the **process-product duality** (Sfard, Dubinsky, Tall).

Subramaniam, K. (2018, March). The conjoining error in school algebra, *At Right Angles*, Vol.7(1), 44-47

- Why are symbols handled differently in algebra?
- Because the arithmetic and algebraic approaches to problem solving are different.
- Let us try to understand this difference.

Here is a problem for you.

Rules:

- Do not type the answer in the chat box!
- Do not speak out the answer!
- Try not to use algebra, i.e., variables.

A farmer had hens and goats. He counted 50 heads altogether and 144 legs. How many hens and how many goats did he have?

How did you solve the problem?

- Guessed the answer and checked.
- Used algebra.
- Found a way of solving with just arithmetic.

The algebraic approach

- Let the number of hens be x and the number of goats be y.
- Step 1: Set up the equations for heads and legs

$$x + y = 50$$

$$2x + 4y = 144$$

- Step 2: Solve the simultaneous equations to get x and y.

Reasoning using just arithmetic

Suppose the farmer had only hens.

- Step 1: The total number of legs is 100. You have 44 legs less than required.
- Step 2: If you exchange a hen for a goat, you are exchanging 2 legs for 4 legs. And you get 2 extra legs!
- Step 3: If you exchange 22 hens for 22 goats, you get 44 extra legs!

Example of a student’s reasoning

- Hens and goats together — there are 50.
- Step 1: Let us count the hen’s two legs and only the two front legs of the goat. This gives us 100 legs.
- Step 2: So the remaining 44 legs are hind legs (or “back” legs).
- Step 3: So there are 22 goats. The rest of the 50 are hens.

What was different between the arithmetic and algebraic approaches?

- The arithmetic approach was based on reasoning with/ manipulating concrete (mental) objects in each step.

- In contrast the algebraic approach involved concrete reasoning only in the first step of setting up the equation and in the last step of interpreting the result.

- The actual calculation (manipulation of equations) was entirely abstract.

- The algebraic approach was mechanical/ routine – did not need any cleverness.

Another problem

- Cellphone company A charges Rs 100 for rent and Rs 0.20 per minute of talktime. Company B charges Rs 75 rent and Rs 0.25 per minute. What is the minimum talktime I should use for A to become cheaper?
- Observation: A is cheaper if the number of calls is very large. Otherwise B is cheaper.
- So, for how many calls are they equally expensive?
- How can we solve this problem?
  - By guessing and checking
  - Using algebra
  - Using just arithmetic

Algebraic and arithmetic solutions

- Algebraic solution: Assume x is the talkative in minutes.

$$0.2 \times x + 100 = 0.25 \times x + 75$$

$$0.05 \times x = 25$$

$$x = 500$$

- Arithmetic solution:
  - Company B charges Rs 25 less for rent.
  - But Rs 0.05 more for each minute of airtime.
  - So for 20 minutes it charges Re 1 more.
  - So for 20 × 25 = 500 minutes it charges Rs 25 more.
  - For 500 minutes of airtime, both companies are equal. Then A becomes cheaper.

Differences between algebra and arithmetic

Algebraic solution	**Arithmetic solution**
Unknown is fixed at the beginning – remains unchanged through solution process.	Unknowns change at each step.
Represent entire sequence of operations at once.	Proceed in steps of one arithmetic operation at a time.
Set up equation; "=" sign has a different meaning: equal value on both sides.	Perform calculations and give the result after the "=" sign.
No interpretation of quantities in intermediate steps.	Intermediate quantities are meaningful.
Operations are done on symbols (letters representing variables).	Operations are done on numbers.

Expressions containing more than one binary operation

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- This is the reason why we need to write expressions with several operations.
- When we substitute a value for a variable, the value of the expression should be unambiguous: $0.2 \times x + 40$. When $x = 60$, we have $0.2 \times 60 + 40$.
- So we need a convention for order of operations like BODMAS. (Using brackets would make it very hard to read.)

Changed meaning of the "=" sign

Algebraic solution	Arithmetic solution
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Set up equation; "=" sign has a different meaning: equal value on both sides.	Perform calculations and give the result after the "=" sign.
No interpretation of quantities in intermediate steps.	Intermediate quantities are meaningful.
Operations are done on symbols (letters representing variables).	Operations are done on numbers.

- Explains arithmetic errors such as for the question: $15 + 3 = \_ + 5$
- A typical response: $15 + 3 = \underline{18} + 5 = 23$ *
- Explains (partly) the conjoining error: $5x + 2 = 7x$ *

But the formal symbols of algebra are what cause difficulties for students!

- Conjoining error: $5x + 2y = 7xy$ *
- Detachment error: 50 – 10 + 10 + 10 = 20 *
- Write an equation for: There are six times as many students as professors: 6s = p *
- There are 5 white cars and 4 red cars in the parking lot. Write an expression for the total number of cars: 5w + 4r *
- A number of such errors have been documented.
- How then do we support students?

Reasoning is the engine of mathematical learning.

Have students work on tasks that call for reasoning, not memorising and applying rules.

Learning outcomes and processes

- Learning outcomes: Capabilities
- Learning processes: Among the most important is “Reasoning”.
- It’s not an exaggeration to say that reasoning is the engine of learning mathematics.
- Note: No hard separation between outcomes and processes. Capability to reason in certain ways is itself an important learning outcome.
- Reasoning involved in learning mathematics
  - Formal (deductive/ symbolic) reasoning
  - Model based reasoning
  - Context based reasoning

Some examples of reasoning

- Take two fractions: say, $\frac{36}{60}$ and $\frac{32}{40}$. Suppose I add the numerators and denominators, I get $\frac{68}{100}$.
  Is this fraction bigger than, smaller than or in-between the two fractions that I started with?

- **Symbolic reasoning**: Implement a procedure to compare fractions.
  - More general question: Given $\frac{a}{b} < \frac{c}{d}$, Is $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$?
- **Context based reasoning**: Think of the example as marks scored in history and geography. Can you then answer the question? And the more general question?

The mathematics of coffee

- The price of a cup of coffee has gone up by 10%. So a mathematician decides to reduce her consumption by the same ratio, i.e., by 10%. Will her expense decrease, increase or remain the same?
- $(p \times 1.1) \times (c \times 0.9) \;=\; pc \times 1.1 \times 0.9 \;=\; pc \times (1 + 0.1) \times (1 - 0.1)$
- Her expense decreases! (Will always decrease whatever the percent increase!)
- On the other hand if a cup of coffee increases by 5 rupees from Rs 30 to Rs 35 and the mathematician reduced the number of cups in a month by 5 cups from 100 to 95, will her expense decrease?
- Earlier expense was Rs 3000. Now her expense is 35×95, which is…
- Her expense has gone up by more than 10%! How do we understand what is happening?
- Lesson: Context based reasoning is not always helpful! Which is why we need algebra.

Algebraic capability has to do with being able to reason with expressions

- Expressing the relationship between quantities in a general way (writing formulas, manipulating formulas; formulas are also called functions)
- Recognising similar functional relationships, Identifying them and reasoning about them
- Solving equations (finding unknown quantities using functional relationships)
- Proving that certain general relationships are true

But before students reason with expressions, they must learn to reason about expressions.

Subramaniam, K. (2004). Naming practices that support reasoning about and with expressions. ICME regular lecture. Available at https://www.researchgate.net/profile/K_Ravi_Subramaniam/publication/…

Reasoning about expressions

- Compare these two expressions:

    27 + 32 ☐ 29 + 30

- Students reasoned in an interesting way: Take 2 (away) from 32 and "give it to" 27. You will get the expression on the right. So both are equal.
- We decided to build on this intuitive sense and developed a whole approach based on reasoning about expressions:
    - Comparing expressions without calculation
    - Changing an expression without changing its value
    - Finding easy ways of calculating, etc.

Banerjee, R., & Subramaniam, K. (2012). Evolution of a teaching approach for beginning algebra. Educational Studies in Mathematics, 80(3), 351-367.

A key question: What resources do students need in order to reason while learning mathematics?

Resources for reasoning about expressions

- What resources do students draw on in order to reason about expressions?
- Their underlying understanding of arithmetic: their knowledge of numbers and operations or number sense.
- However they need to
  - Overcome their arithmetic habits
  - Grasp/notice the structure of expressions

Key idea: expressions encode the operational composition a number

- Key to the structure of a numerical expression is the "operational composition" that it encodes.
- "5 + 2 × 4" and "3 × 5 – 2" represent the same number "13", but encode different ways in which the number 13 can be composed from other numbers.
- Operational composition refers to how the number or quantity denoted is built up from other numbers or quantities by operating on them.
    1. 540 – 540 × 20/100
    2. 4 × 100 + 3 × 10 + 2
- Expressions can be read via their structure to uncover meanings associated with the encoded operational composition.

Subramaniam, K., & Banerjee, R. (2011). The arithmetic-algebra connection: A historical-pedagogical perspective. In *Early algebraization* (pp. 87-107). Springer, Berlin, Heidelberg.

Understanding Structure: Parsing the expression

- The BODMAS rule is redundant!
- To see the structure of expressions, students must learn to clearly identify the additive units (called “terms”).
- These are the units that can be moved around without changing the value of the expression.
- They are also “homogenous” (in the language of physics one would say, “they have the same units”)
- Thus, parsing into additive units facilitates reading the expression.

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Parsing the expression into additive units or "terms"

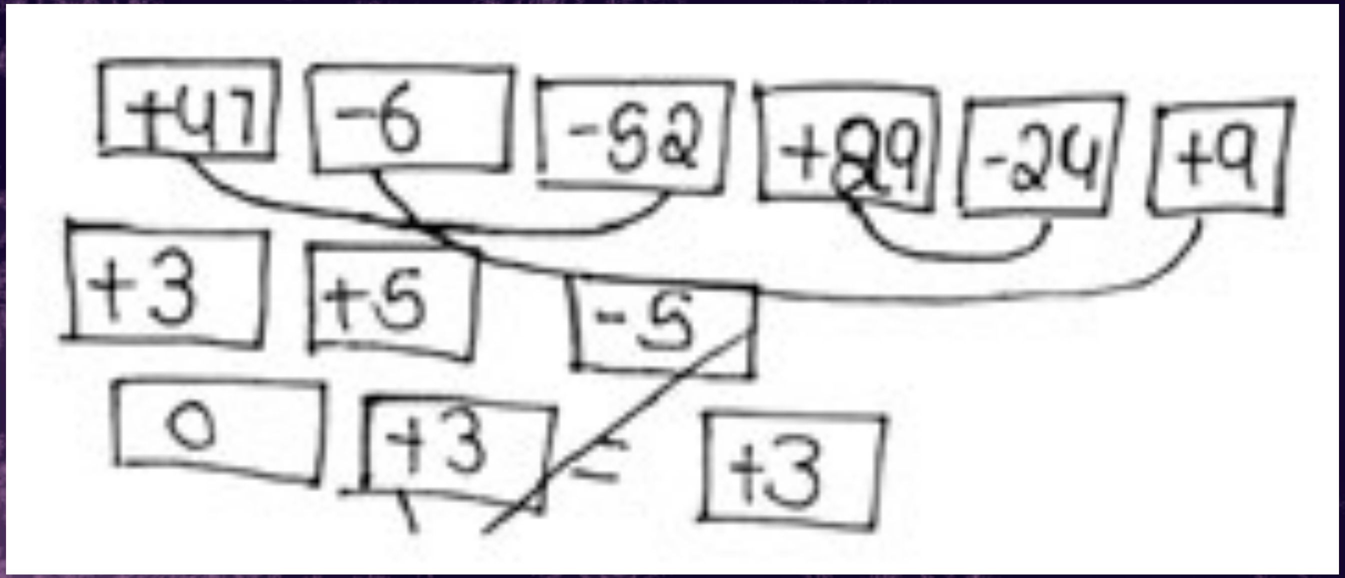
- The concept of "term" is central in the traditional curriculum. However, this concept is used very differently in our approach.
- In the traditional curriculum, concepts of "like terms", "unlike terms" are used to present rules on how to simplify **algebraic** expressions.
- However in our approach, the concept of term is introduced in the context of working with **numerical** expressions.
- The concept is used to guide the visual parsing of numerical expressions, to grasp their structure and to align students' intuitive number sense with this structure.
- Like and unlike terms are not emphasised. In contrast, we use the concept of term to do away with mnemonics like "BODMAS". The order of operations is absorbed into the manner in which visual parsing is done through the introduction of concepts of "simple term", "product term" and "bracket term".

Parsing the expression into additive units or "terms"

- Terms are additive units and they can be moved around in the expression without changing the value of the expression. This is the key idea.
- For this idea to work, all subtraction operations must be converted into (or interpreted as) addition using the additive inverse. This is a crucial insight for students and is gained from working with signed numbers.

Identifying additive units or terms

Evaluating the expression:
47 – 6 – 52 + 29 – 24 +9



- Reading these expressions by parsing into additive units or terms:

    1. $540 - 540 \times 20/100$
    2. $4 \times 100 + 3 \times 10 + 2$

Product terms and factors

- Product terms are composed of "factors", bracket terms contain expressions within brackets and can be parsed as an additive unit on its own.
- Within a product term, factors can be moved around without changing the value of the product term.
- For this idea to work, all division operations must be converted into (or interpreted as) multiplication using the multiplicative inverse. This is a crucial insight for students and is gained from working with rational numbers.
- With these conceptual tools, students work with arithmetic or numerical expressions, before working with variables (introduced through the notion of a "variable term").

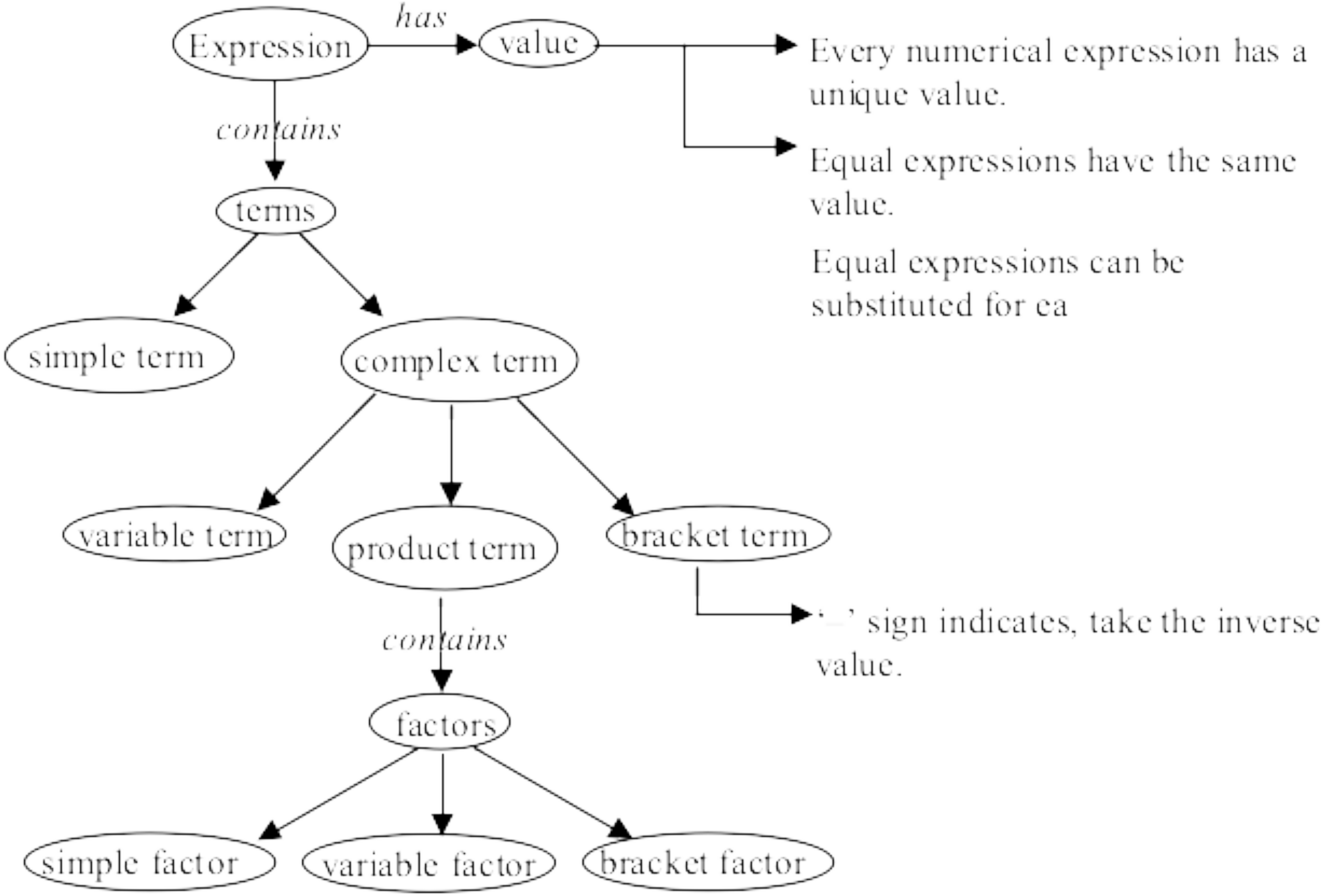
Understanding Structure 2: transformations into equivalent expressions

- Another aspect of understanding the structure of an expression is to understand which transformations give rise to equivalent expressions.
- That is, understanding (and expressing) how the same number can be composed or decomposed in different ways.

Writing equivalent expressions for $11 \times 4 - 21 + 7 \times 4$

$11 \times 4 - 21 + 7 \times 4$

1) $44 - 21 + 28$
2) $40 + 4 - 21 + 20 + 8$
3) $4 \times (7 + 11) - 21$
4) $25 + 3 + 40 + 4 - 25 - 4$*
5) $26 + 46 - 21$
6) $30 - 2 + 50 - 6 - 20 + 1$*



Rules

1. Only simple terms can be combined.
2. Exceptionally, product terms can be combined if they have a common factor.

Procedures

1. Combining terms
2. Writing inverse expression
3. Applying distributive property

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